Complexity Results for Dependence Logic

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Background

- Dependence logic (Väänänen; 2007) and its variants are new logical tools for formalizing and studying notions of dependence and independence in various contexts (e.g., social choice theory, databases, and quantum foundations).
- Historically dependence logic was preceded by branching quantifiers of Henkin and Independence Friendly Logic of Jaakko Hintikka and Gabriel Sandu (80’s).
- In FO, dependencies between quantifiers are determined by the syntactic notion of scope, e.g., in
  \[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 \psi(x_1, x_2, y_1, y_2) \]
  \[ y_1 \text{ depends on } x_1 \text{ and } y_2 \text{ depends on both } x_1 \text{ and } x_2. \]
- In dependence logic we have new atomic dependence formulas allowing us to express dependencies not expressible in FO, e.g.,
  \[ \forall x_1 \exists y_1 \forall x_2 \exists y_2 (\equiv(x_2, y_2) \land \psi(x_1, x_2, y_1, y_2)). \]
Outline of the talk

1. We first give a short introduction to dependence logic,
2. Discuss the difference between open formulas/sentences,
3. Show how certain syntactic parameters of a sentence reflect on its complexity,
4. Discuss certain extensions of dependence logic.
The syntax of dependence logic ($\mathcal{D}$) extends the syntax of FO, defined in terms of $\lor$, $\land$, $\neg$, $\exists$ and $\forall$, by new atomic (dependence) formulas of the form

$$= (t_1, \ldots, t_n),$$

where $t_1, \ldots, t_n$ are terms.

In (1), $n$ is called the width of the dependence atom.
Semantics of $\mathcal{D}$

The semantics of $\mathcal{D}$ is defined in terms of *teams* (sets of assignments): 

**Definition**

Let $A$ be a set and $\{x_1, \ldots, x_k\}$ variables. A *team* $X$ of $A$ with domain $\{x_1, \ldots, x_k\}$ is a set of assignments $s$, 

$$s : \{x_1, \ldots, x_k\} \rightarrow A.$$
The following operations are used to interpret quantifiers in $\mathcal{D}$. Below, $s(a/x_n)$ is the assignment that agrees otherwise with $s$, but maps $x_n$ to $a$.

**Definition**

Suppose $A$ is a set, $X$ is a team of $A$, and $F: X \to A$.

- **supplementation**: $X(F/x) = \{ s(F(s)/x) : s \in X \}$.
- **Duplication**: $X(A/x) = \{ s(a/x) : s \in X \text{ and } a \in A \}$. 
Satisfaction for NNF-formulas

Definition

Below $\phi(t_1, \ldots, t_n)$ is atomic or negated atomic FO-formula:

- $\mathcal{A} \models_X \phi(t_1, \ldots, t_n) \iff$ for all $s \in X$: $\mathcal{A} \models_s \phi(t_1, \ldots, t_n)$
- $\mathcal{A} \models_X \neg(t_1, \ldots, t_n) \iff$ for all $s, s' \in X$: if $t_i^{\mathcal{A}}(s) = t_i^{\mathcal{A}}(s')$ for $1 \leq i \leq n - 1$, then $t_n^{\mathcal{A}}(s) = t_n^{\mathcal{A}}(s')$.
- $\mathcal{A} \models_X \psi \land \phi \iff \mathcal{A} \models_X \psi$ and $\mathcal{A} \models_X \phi$.
- $\mathcal{A} \models_X \psi \lor \phi \iff X = Y \cup Z$ such that $\mathcal{A} \models_Y \psi$ and $\mathcal{A} \models_Z \phi$.
- $\mathcal{A} \models_X \exists x \psi \iff \mathcal{A} \models_{X(F/x)} \psi$ for some $F: X \to A$.
- $\mathcal{A} \models_X \forall x \psi \iff \mathcal{A} \models_{X(A/x)} \psi$.

Finally, a sentence $\varphi$ of $\mathcal{D}$ is true in $\mathcal{A}$ if $\mathcal{A} \models_{\emptyset} \varphi$. 
Example

Not all familiar propositional equivalences of connectives hold for $\mathcal{D}$, e.g., idempotence of disjunction, and the distributivity laws of disjunction and conjunction fail.

Example

Let $A = \{0, 1, 2\}$ and $X$ be

\[
\begin{array}{c|c|c|c}
 & x_0 & x_1 & x_2 \\
 s_0 & 1 & 2 & 2 \\
 s_1 & 2 & 1 & 2 \\
 s_2 & 2 & 0 & 2 \\
\end{array}
\]

Now $\mathcal{A} \not\models_X x_0 = x_2$ and $\mathcal{A} \not\models_X \neg x_0 = x_2$. Also $\mathcal{A} \not\models_X = (x_2, x_0)$, but $\mathcal{A} \models_X (=(x_2, x_0) \lor = (x_2, x_0))$. 
The following sentence $\phi$ expresses "$|A|$ even" for a finite $A$:

$$
\phi : \forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2,x_3) \land \neg (x_0 = x_1) \land (x_0 = x_2 \rightarrow x_1 = x_3) \land (x_1 = x_2 \rightarrow x_3 = x_0))
$$

$A \models \phi$ iff there is $f : A \rightarrow A$ s.t. for all $x$: $f(f(x)) = x$ and $f(x) \neq x$. 
Basic properties of $\mathcal{D}$

Proposition

Let $\phi$ be a formula of $\mathcal{D}$ without dependence atoms. Then for all $\mathcal{A}$ and $X$:

$$\mathcal{A} \models_X \phi \iff \text{for all } s \in X: \mathcal{A} \models_s \phi$$

Also the following fragments collapse to FO:

Proposition (Case 2 by Galliani 2011)

Suppose that a sentence $\phi \in \mathcal{D}$ satisfies either of the following

1. $\phi$ is in NNF and does not contain universal quantifiers,
2. $\phi$ contains only dependence atoms of width 1 as subformulas.

Then $\phi$ is equivalent to a first-order sentence.

Theorem

$\mathcal{D} \equiv \text{ESO}$.
Formulas with free variables

Proposition (Downward closure)

Let $Y \subseteq X$ teams. Then $\mathcal{A} \models_X \phi$ implies $\mathcal{A} \models_Y \phi$.

Let $A$ be a set and $X$ a team with domain $\{x_1, \ldots, x_k\}$. Define $rel(X) \subseteq A^k$ by

$$rel(X) = \{(s(x_1), \ldots, s(x_k)) : s \in X\}.$$ 

An upperbound for the complexity of formulas of $\mathcal{D}$ is given by:

Theorem

For every $\phi(x_1, \ldots, x_k) \in \mathcal{D}[\tau]$ there is a ESO[$\tau \cup \{R\}$]-sentence $\psi$, where $R$ is $k$-ary and appears only negatively, s.t. for all $\mathcal{A}$ and $X$ with domain $\{x_1, \ldots, x_k\}$:

$$\mathcal{A} \models_X \phi \iff (\mathcal{A}, rel(X)) \models \psi.$$
Theorem (K. and Väänänen 2009)

For every sentence $\psi \in \text{ESO}[\tau \cup \{R\}]$, in which $R$ appears only negatively, there is $\phi(y_1, \ldots, y_k) \in \mathcal{D}[\tau]$ s.t. for all $\mathcal{A}$ and $X$:

$$\mathcal{A} \models_X \phi \iff (\mathcal{A}, \text{rel}(X)) \models \psi \lor R = \emptyset.$$ 

Theorem (Jarmo Kontinen 2010)

Define

- $\varphi \equiv = (x, y) \lor = (z, u)$
- $\psi \equiv = (x, y) \lor = (z, u) \lor = (z, u)$

Deciding whether $X$ satisfies $\varphi$ is NL-complete and, for $\psi$, NP-complete.
Definition (Satisfiability Problem)

Let $L$ be a logic. The satisfiability problem $\text{Sat}[L]$ of $L$ is the following problem: \textbf{Input:} a sentence $\phi \in L$.

\textbf{Output:} Yes, if there is a model $\mathfrak{A}$ such that $\mathfrak{A} \models \phi$, and No otherwise.

The finite satisfiability problem $\text{FinSat}(L)$ is the version of the above question in which $\mathfrak{A}$ must be finite.
Denote by $\mathcal{D}^2$ the sentences of $\mathcal{D}$ in which only variables $x$ and $y$ appear.

**Theorem (K., Kuusisto, Lohmann, and Virtema; 2011)**

1. The Satisfiability (and Finite Satisfiability) problem of $\mathcal{D}^2$ is $\text{NEXPTIME}$-complete.
2. The logic $\mathcal{D}^2$ is quite expressive being able to express, e.g., "$\forall$ infinite" and "$|P| = |Q|".
3. In contrast, the satisfiability (and finite satisfiability) problem of $\text{IF}^2$ is undecidable.

**Remark**

Jonni Virtema (Univ. Tampere) recently observed that $\mathcal{D}^2$ can express $\text{NP}$-complete problems.
Restricting the number of universal quantifiers/the width of dependence atoms
joint work with Arnaud Durand

Definition
Let $k \in \mathbb{N}^*$.

- $\mathcal{D}(k\forall)$ consists of those sentences $\phi$ of $\mathcal{D}$ having at most $k$ occurrences of $\forall$ (no reusing of variables).
- $\mathcal{D}(k-\text{dep})$ consists of sentences $\phi$ of $\mathcal{D}$ in which dependence atoms of width at most $k + 1$ appear.
The case of $\mathcal{D}(k – \text{dep})$

**Definition**

Denote by $\text{ESO}_f(k\text{-ary})$ the class of ESO-sentences

$$\exists f_1 \ldots \exists f_n \psi,$$

in which the function symbols $f_i$ are at most $k$-ary and $\psi$ is a FO-formula.

**Theorem**

Let $k \in \mathbb{N}^*$. $\mathcal{D}(k – \text{dep}) = \text{ESO}_f(k\text{-ary})$. 
The case of $\mathcal{D}(k\forall)$

Definition

Denote by $\text{ESO}_f(k\forall)$ the class of ESO-sentences in Skolem Normal Form

$$\exists f_1 \ldots \exists f_n \forall x_1 \ldots \forall x_r \psi,$$

where $r \leq k$ and $\psi$ is quantifier-free.

Theorem

Let $k \in \mathbb{N}^*$. Then

$$\text{NTIME}_{\text{RAM}}(n^k) = \text{ESO}_f(k\forall) \leq \mathcal{D}(2k\forall) \leq \text{ESO}_f(2k\forall) = \text{NTIME}_{\text{RAM}}(n^{2k}).$$

The equality $\text{NTIME}_{\text{RAM}}(n^k) = \text{ESO}_f(k\forall)$ is due to Grandjean and Olive (2004).
Hierarchy theorems

**Theorem**

*If* $\tau$ *has a* $k + 1$-ary $R$, then* $\mathcal{D}(k - \text{dep})[\tau] \subseteq \mathcal{D}(k + 1 - \text{dep})[\tau]$.

**Theorem**

*For* $k \geq 1$ *and any vocabulary:*

1. $\mathcal{D}(k \forall) \subseteq \mathcal{D}(k - \text{dep})$,
2. $\mathcal{D}(k \forall) \subsetneq \mathcal{D}((k + 1) - \text{dep})$,
3. $\mathcal{D}(k \forall) \subsetneq \mathcal{D}((2k + 2) \forall)$. 
\[ \mathcal{D}((2k + 2)\forall) \hspace{1cm} \mathcal{D}((k + 1)\text{-dep}) \]

\[ \mathcal{D}((k\forall) \equiv \mathcal{D}(k\text{-dep}) \equiv \mathcal{D}(k\text{-ary}) \]

Figure: Summary of inclusions (for all signatures and all \( k \geq 1 \))
Extensions of dependence logic

Definition (Abramsky and Väänänen, 2009)

Define the connective called *intuitionistic implication* as follows:

\[ \mathcal{A} \models_X \phi \rightarrow \psi \text{ iff } (\text{for all } Y \subseteq X : \text{if } \mathcal{A} \models_Y \phi \text{ then } \mathcal{A} \models_Y \psi). \]

Let \( \mathcal{D}(\rightarrow) \) be the extension of \( \mathcal{D} \) by \( \rightarrow \). The following now holds:

Theorem (Abramsky and Väänänen, 2009)

1. \( \mathcal{D}(\rightarrow) \) has the downwards closure property,
2. Atoms \( = (t_i) \) are sufficient to expresss all atoms \( = (t_1, \ldots, t_k) \),
3. Closed under classical negation for sentences: \( \sim \phi \equiv \phi \rightarrow \bot \),
4. Any \( \phi(x_1, \ldots, x_k) \in \mathcal{D}(\rightarrow)[\tau] \) translates to a sentence \( \psi(R) \in \text{SO}[\tau \cup \{R\}] \).

Theorem (Yang, 2010)

\( \mathcal{D}(\rightarrow) = \text{SO} \)
Extensions of dependence logic cont.

**Definition (Grädel and Väänänen, 2010)**

Let $\bar{y} \perp_{\bar{x}} \bar{z}$ be a new atomic formula (independence atom) with semantics:

$\mathcal{A} \models \bar{x} \bar{y} \perp_{\bar{x}} \bar{z}$ iff for all $s, s' \in X$ s.t. $s(\bar{x}) = s'(\bar{x})$ there exists $s'' \in X$ s.t.

$s''(\bar{x}) = s(\bar{x})$, $s''(\bar{y}) = s'(\bar{y})$ and $s''(\bar{z}) = s'(\bar{z})$.

The meaning of the independence atom is that $\bar{y}$ are independent of $\bar{z}$ when $\bar{x}$ are fixed. Let $\mathcal{I}$ be the variant of $\mathcal{D}$ with dependence atoms replaced by independence atoms.

**Theorem (Grädel and Väänänen, 2010)**

1. $= (\bar{x}, \bar{y}) \equiv \bar{y} \perp_{\bar{x}} \bar{y}$, hence $\mathcal{D} \subseteq \mathcal{I}$,
2. $\mathcal{I} = \text{ESO}$.
Further observations on $\mathcal{I}$

1. Engström (2011): The independence atom $\bar{y} \perp_{\bar{x}} \bar{z}$ in fact expresses an embedded multivalued dependency $\bar{x} \rightarrow \bar{y} | \bar{z}$.

2. Galliani (2011): $\mathcal{I}$ can be alternatively defined as a logic with certain inclusion and exclusion dependencies (alternative atomic formulas).

3. Galliani (2011): Open $\tau$-formulas of $\mathcal{I}$ correspond to full $\text{ESO}[\tau \cup \{R\}]$ (no downwards closure).
Extensions of dependence logic cont.

Let $\mathcal{D}(M)$ be the extension of $\mathcal{D}$ by the following majority quantifier:

$$\mathfrak{A} \models_{X} M x \phi(x) \text{ iff for at least } |A|^{X}/2 \text{ many function } F: X \to A \text{ we have }$$

$$\mathfrak{A} \models_{X(F/x)} \phi(x).$$

**Theorem (Durand, Ebbing, K., and Vollmer; 2011)**

$\mathcal{D}(M) = \text{SO}(\text{Most}) = \text{CH}.$

Above, CH is the complexity class the *Counting Hierarchy* that contains PH and full second-order logic.
Let $Q$ be a monotone increasing first-order generalized quantifier. Then there is a natural way to define the extension $\mathcal{D}(Q)$ of $\mathcal{D}$ by $Q$.

**Theorem (Engström, K.; 2012)**

$\mathcal{D}(Q) = \text{ESO}(Q)$. 
Dependence logic and its variants are tools for formalizing and studying dependencies and independences in various contexts.

The framework of dependence logic is very flexible allowing various generalizations. There is much work to be done to understand the computational properties of these new logics.