Using Fluid Variational Variables to Obtain New Analytic Solutions with Non-Zero Helicity

Asher Yahalom

Isaac Newton Institute for Mathematical Sciences, Cambridge, UK
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Eulerian Fluid Dynamics
Basic Quantities

1. The velocity field $\vec{V}$ 3 functions.
2. The density field $\rho$ 1 function.

Total of 4 functions.
Derived Quantities

Pressure: $p(\rho)$

For Barotropic flows.
Basic Equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0
\]

\[
\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{\nabla p(\rho)}{\rho} - \nabla \Psi
\]

Total of 4 equations.

\(\Psi\) is a potential of some specific force (gravity).
Content of the equations

The continuity equation – mass is conserved.
Content of the equations

The Euler equations: Newton’s second law for continuous matter moving under the influence of pressure forces and other potential forces.

\[
\frac{d\vec{v}}{dt} \equiv \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{\vec{\nabla} p(\rho)}{\rho} - \vec{\nabla} \Psi
\]
Vorticity

\[ \vec{\omega} = \vec{\nabla} \times \vec{v} \]
The vortex lines are co-moving that is they move with the fluid.

Hence the topology of the vortex lines is conserved and one can create a coordinate system based on the vortex lines.
Co-moving Coordinate System

\((\alpha, \beta, \mu)\)
(\alpha, \beta) \quad \text{Surfaces who's intersection is a vorticity line.}
For a suitable choice of co-moving surfaces and for some topologies we can write:

\[
\vec{\omega} = \vec{\nabla} \times \vec{v} = \vec{\nabla} \alpha \times \vec{\nabla} \beta
\]

Clebsch Form

\[
\vec{v} = \alpha \vec{\nabla} \beta + \vec{\nabla} \nu
\]
Consider an arbitrary comoving point on the vortex line and denote it by \( i \), and consider an additional comoving point on the vortex line and denote it by \( r \), the quantity:

\[
\mu(r) = \int_{i}^{r} \frac{\rho}{\omega} dl + \mu(i)
\]

\[ \mu(r) = \oint_{i}^{r} \frac{\rho}{\omega} dl + \mu(i) \]

\[ \frac{d\mu}{dt} = 0 \]

\[ \rho = \nabla \mu \cdot \vec{\omega} = \nabla \mu \cdot (\nabla \alpha \times \nabla \beta) = \frac{\partial (\alpha, \beta, \mu)}{\partial (x, y, z)} \]
Helicity


\[ H \equiv \int \omega \cdot \vec{v} \, d^3x, \]
\[ \bar{\omega} \cdot \bar{v} = (\nabla \alpha \times \nabla \beta) \cdot \nabla \nu. \]

\[ \nabla \nu = \frac{\partial \nu}{\partial \alpha} \bar{\nabla} \alpha + \frac{\partial \nu}{\partial \beta} \bar{\nabla} \beta + \frac{\partial \nu}{\partial \mu} \bar{\nabla} \mu. \]

\[ \bar{\omega} \cdot \bar{v} = \frac{\partial \nu}{\partial \mu} (\nabla \alpha \times \nabla \beta) \cdot \nabla \mu = \frac{\partial \nu}{\partial \mu} \frac{\partial \nu}{\partial (\alpha, \beta, \mu)} \frac{\partial (x, y, z)}{\partial \mu} \]
Only vortex lines on which $\nu$ is discontinuous contribute to the Helicity.
One can show that this is also a conserved quantity:

\[
[\nu] = \frac{d\mathcal{H}}{d\Phi},
\]

\[
\frac{d[\nu]}{dt} = 0.
\]
Stationary Flows

\[ \frac{\partial \rho}{\partial t} = \frac{\partial \mathbf{v}}{\partial t} = 0 \]
\[
\frac{d\alpha}{dt} = 0
\]

\[\nabla \alpha + \vec{v} \cdot \nabla \alpha = 0\]

\[\vec{v} = \frac{\nabla \alpha \times \vec{K}}{\rho}\]
The continuity equation – mass is conserved.
\[ \frac{\partial \vec{\omega}}{\partial t} = \nabla \times (\vec{v} \times \vec{\omega}) \]
$\nabla \times (\vec{v} \times \vec{\omega}) = 0$
\[ \vec{v} = \frac{\vec{\nabla} \mu \times \vec{\nabla} \alpha}{\rho} \]

\[ \vec{v} \cdot \vec{\nabla} \mu = 0, \quad \vec{v} \cdot \vec{\nabla} \alpha = 0 \]
\[ \vec{v} = \frac{\nabla \mu \times \nabla \alpha}{\rho} \]

\[ \vec{\omega} = \nabla \alpha \times \nabla \beta \]

\[ \vec{v} \times \vec{\omega} = \nabla \alpha \]
Summary

- Both the velocity and vorticity vectors lie on alpha surfaces.
Variational Principle for Stationary Flows
Lagrangian and Lagrangian Density

\[ L \equiv \int \mathcal{L} d^3 x \]

\[ \mathcal{L} \equiv \rho \left( \frac{1}{2} \bar{v}^2 - \varepsilon(\rho) - \Psi \right) \]

\[ \varepsilon(\rho) \text{ is the specific internal energy} \]
A Field theory of three functions (less than 4)
Calculating the variational derivatives of the Lagrangian one arrives at the stationary Euler equation of the form:

\[ \rho \vec{v} \times \vec{\omega} - \rho \nabla (\frac{1}{2} \vec{v}^2 + w + \Psi) = 0 \]

\( w \) is the specific enthalpy.
Bernoullian Surfaces

\[ \rho \mathbf{v} \times \mathbf{\omega} - \rho \nabla \left( \frac{1}{2} \mathbf{v}^2 + w + \Psi \right) = 0 \]

\[ \mathbf{v} \times \mathbf{\omega} = \nabla \alpha \]

\[ \alpha = \frac{1}{2} \mathbf{v}^2 + w(\rho) + \Psi \]
Analytical Solutions
The flow equations are notoriously difficult to solve analytically. (non-linear partial differential equations), let us see how can we solve them.
First Step: Choose a Surface
Torus
Choose a Coordinate System on the Torus
\[ \bar{r} \equiv \sqrt{z^2 + (R - 1)^2} \]

\((R, \phi, z)\) are standard cylindrical coordinates

\[ \bar{r} = \text{constant} \]
Short way angle

\[ \eta \equiv \arctan \frac{z}{R - 1} \]

Hence we have an orthogonal toroidal coordinate system:

\[ \bar{r}, \phi, \eta \]
Vector Analysis in Toroidal Coordinates

Unit vectors:

\[ \hat{r} = \frac{\vec{\nabla}r}{|\vec{\nabla}r|}, \quad \hat{\phi} = \frac{\vec{\nabla}\phi}{|\vec{\nabla}\phi|}, \quad \hat{\eta} = \frac{\vec{\nabla}\eta}{|\vec{\nabla}\eta|} \]

Orthogonality:

\[ \hat{r} \cdot \hat{\phi} = \hat{r} \cdot \hat{\eta} = \hat{\phi} \cdot \hat{\eta} = 0 \]
Right handed coordinate system:

\[ \hat{\mathbf{r}} \times \hat{\mathbf{\phi}} = \hat{\mathbf{\eta}}, \quad \hat{\mathbf{\eta}} \times \hat{\mathbf{r}} = \hat{\mathbf{\phi}}, \quad \hat{\mathbf{\phi}} \times \hat{\mathbf{\eta}} = \hat{\mathbf{r}}. \]
Line element

\[ ds^2 = h_r^2 d\bar{r}^2 + h_\phi^2 d\phi^2 + h_\eta^2 d\eta^2 \]

\[ h_r = 1, \quad h_\phi = 1 + \bar{r} \cos \eta = R, \quad h_\eta = r. \]
Vector Operators

\[ \vec{\nabla} f = \hat{r} \frac{\partial f}{\partial r} + \hat{\phi} \frac{1}{R} \frac{\partial f}{\partial \phi} + \hat{\eta} \frac{1}{r} \frac{\partial f}{\partial \eta} \]

\[ \vec{\nabla} \cdot \vec{A} = \frac{1}{r R} \left( \frac{\partial (r R A_r)}{\partial r} + \frac{\partial (r A_\phi)}{\partial \phi} + \frac{\partial (R A_\eta)}{\partial \eta} \right) \]

\[ \vec{\nabla}^2 f = \frac{1}{r R} \left( \frac{\partial}{\partial r} (r R \frac{\partial f}{\partial r}) + \frac{\partial}{\partial \phi} \left( \frac{\partial f}{\partial \phi} \right) + \frac{\partial}{\partial \eta} \left( \frac{R \partial f}{\partial \eta} \right) \right) \]
Rotor

\[
\mathbf{\nabla} \times \mathbf{A} = \frac{1}{\hat{r} R} \left[ \frac{\partial (\hat{r} A_\eta)}{\partial \phi} - \frac{\partial (RA_\phi)}{\partial \eta} \right] \hat{r} + \frac{1}{\hat{r}} \left[ \frac{\partial (A_{\bar{r}})}{\partial \eta} - \frac{\partial (\hat{r} A_\eta)}{\partial \bar{r}} \right] \hat{\phi} + \frac{1}{R} \left[ \frac{\partial (RA_\phi)}{\partial \bar{r}} - \frac{\partial (A_{\bar{r}})}{\partial \phi} \right] \hat{\eta}
\]
Bernoullian Surfaces

\[ \alpha = \alpha(\bar{r}) \]

Flow is confined between the tori \( 0 \leq \bar{r} \leq a \)

\[ 0 < a < 1 \]

Vorticity and Velocity fields lay on those surfaces:

\[ \vec{\nu} \times \vec{\omega} = \vec{\nabla} \alpha \]
Vorticity

\[ \vec{\omega} = \nabla \times \vec{v} = \nabla \alpha \times \nabla \beta \]

Clebsch Form

\[ \vec{v} = \alpha \nabla \beta + \nabla \nu = \nabla (\alpha \beta + \nu) - \beta \nabla \alpha \]
\[ \tilde{\nu} = \alpha \beta + \nu \]

\[ \tilde{\nu} = \nabla \tilde{\nu} - \beta \nabla \alpha \]

\[ \tilde{\nu} \cdot \hat{r} = 0 \quad (\hat{\eta} \cdot \nabla \alpha = \hat{\phi} \cdot \nabla \alpha = 0) \]

\[ \partial_{\tilde{r}} \tilde{\nu} = \beta \partial_{\tilde{r}} \alpha, \quad \nu_{\eta} = \frac{1}{\tilde{r}} \partial_{\eta} \tilde{\nu}, \quad \nu_{\phi} = \frac{1}{R} \partial_{\phi} \tilde{\nu} \]
\( v_\eta = \frac{1}{r} \partial_\eta \tilde{\nu} , \quad v_\phi = \frac{1}{R} \partial_\phi \tilde{\nu} \)

\[ \partial^2_{\eta \phi} \tilde{\nu} = \partial^2_{\phi \eta} \tilde{\nu} \]

\[ \partial_\phi (r v_\eta) = \partial_\eta (R v_\phi) \]
A possible solution:

\[ v_\eta = \frac{\Gamma_S(\bar{r})}{2\pi \bar{r}} \], \quad \quad v_\phi = \frac{\Gamma_L(\bar{r}, \phi)}{2\pi R} \]
Velocity

\[ \vec{v} = \frac{\nabla \mu \times \nabla \alpha}{\rho} \]

\[ v_\eta = -\frac{\partial_{\vec{r}} \alpha}{\rho R} \partial_\phi \mu, \quad v_\phi = \frac{\partial_{\vec{r}} \alpha}{\rho \vec{r}} \partial_\eta \mu, \]
\[ \partial_{\eta\phi}^2 \mu = \partial_{\phi\eta}^2 \mu \]

\[ \partial_{\phi} \left( \frac{\rho \bar{r} v_{\phi}}{\partial_{\bar{r}} \alpha} \right) + \partial_{\eta} \left( \frac{\rho R v_{\eta}}{\partial_{\bar{r}} \alpha} \right) = 0 \]

\[ \partial_{\phi} (\rho \bar{r} v_{\phi}) + \partial_{\eta} (\rho R v_{\eta}) = 0 \]
A possible solution:

\[ \rho = \frac{g(\bar{r})}{R}, \quad \Gamma_L = \Gamma_L(\bar{r}) \]
Hence we can deduce that $\Gamma_L$ is the circulation along the ”long path”:

$$\oint_{\eta=\text{const.},\, \tilde{r}=\text{const.}} \vec{v} \cdot d\vec{s} = \int_0^{2\pi} v_\phi R d\phi = 2\pi R v_\phi = \Gamma_L,$$

and $\Gamma_S$ is the circulation along the ”short path”:

$$\oint_{\phi=\text{const.},\, \tilde{r}=\text{const.}} \vec{v} \cdot d\vec{s} = \int_0^{2\pi} v_\eta \tilde{r} d\eta = 2\pi \tilde{r} v_\eta = \Gamma_S,$$
The components of the velocity field take the form:

\[ v_\eta = \frac{\Gamma_S(\bar{r})}{2\pi \bar{r}}, \quad v_\phi = \frac{\Gamma_L(\bar{r})}{2\pi \bar{R}}, \quad v_\bar{r} = 0. \]
\[ \mathbf{\nabla} \mu \cdot \mathbf{\omega} = \rho \]

\[
\begin{align*}
\omega_{\bar{r}} &= 0 \\
\omega_{\eta} &= \frac{1}{R} \partial_{\bar{r}}(Rv_\phi) = +\frac{1}{2\pi R} \partial_{\bar{r}} \Gamma_L(\bar{r}) \\
\omega_{\phi} &= -\frac{1}{\bar{r}} \partial_{\bar{r}}(\bar{r}v_\eta) = -\frac{1}{2\pi \bar{r}} \partial_{\bar{r}} \Gamma_S(\bar{r}).
\end{align*}
\]
\[ v_\eta = -\frac{\partial r}{\rho R} \partial_\phi \mu, \quad v_\phi = \frac{\partial r}{\rho r} \partial_\eta \mu, \]

\[ \mu = -\frac{g \Gamma_S \phi}{2\pi r \partial_r \alpha} + \frac{g r \Gamma_L}{2\pi \partial_r \alpha} \int_0^\eta \frac{d\eta'}{(1 + \bar{r} \cos \eta')^2} + \mu_2(\bar{r}), \]

Non Single Valued
\[ II(\bar{r}, \eta) \equiv \int_{0}^{\eta} \frac{d\eta'}{(1 + \bar{r} \cos \eta')^2} = \frac{I(\bar{r}, \eta)}{1 - \bar{r}^2} - \frac{\bar{r} \sin \eta}{(1 - \bar{r}^2)(1 + \bar{r} \cos \eta)}. \]

\[ I(\bar{r}, \eta) \equiv \int_{0}^{\eta} \frac{d\eta'}{1 + \bar{r} \cos \eta'} = \frac{2}{\sqrt{1 - \bar{r}^2}} \left[ \arctan\left( \sqrt{\frac{1 - \bar{r}}{1 + \bar{r}}} \tan\left( \frac{\eta}{2} \right) \right) + \begin{cases} 0, & 0 \leq \eta < \pi \\ \pi, & \pi \leq \eta < 2\pi. \end{cases} \right] \]
Multi-valued functions.
\[ \vec{\nabla}_\mu \cdot \vec{\omega} = \rho \]

\[ \rho = \frac{\omega_\phi}{R} \left( -\frac{\rho R \Gamma_S}{2\pi \bar{r} \partial_{\bar{r}} \alpha} \right) + \frac{\omega_\eta}{\bar{r}} \left( \frac{\rho \bar{r} \Gamma_L}{2\pi R \partial_{\bar{r}} \alpha} \right) \]

\[ \partial_{\bar{r}} \alpha = \frac{\omega_\eta \Gamma_L}{2\pi R} - \frac{\omega_\phi \Gamma_S}{2\pi \bar{r}} \]
\[ \partial_{\tilde{r}} \alpha = \frac{\Gamma_L(\tilde{r}) \partial_{\tilde{r}} \Gamma_L(\tilde{r})}{(2\pi R)^2} + \frac{\Gamma_S(\tilde{r}) \partial_{\tilde{r}} \Gamma_S(\tilde{r})}{(2\pi \tilde{r})^2} = \frac{\Gamma_L(\tilde{r}) \partial_{\tilde{r}} \Gamma_L(\tilde{r})}{(2\pi(1 + \tilde{r} \cos \eta))^2} + \frac{\Gamma_S(\tilde{r}) \partial_{\tilde{r}} \Gamma_S(\tilde{r})}{(2\pi \tilde{r})^2} \]

Since \( \alpha \) is assumed to be a function of \( \tilde{r} \) and not of \( \eta \) we arrive at the conclusion:

\[ \partial_{\tilde{r}} \Gamma_L(\tilde{r}) = 0 \Rightarrow \Gamma_L(\tilde{r}) = \Gamma_L = \text{constant}, \]
and hence:

\[ \partial_{\bar{r}} \alpha = \frac{\Gamma_S(\bar{r}) \partial_{\bar{r}} \Gamma_S(\bar{r})}{(2\pi \bar{r})^2}, \quad \omega_\eta = 0. \]

Thus up to a constant factor \( \alpha \) is not an arbitrary function but is dictated by the short way circulation.
The rest of the functions

\[ v_\eta = \frac{1}{r} \partial_\eta \tilde{\nu}, \quad v_\phi = \frac{1}{R} \partial_\phi \tilde{\nu} \]

Non Single Valued

\[ \tilde{\nu} = \Gamma_S(\bar{r}) \frac{\eta}{2\pi} + \Gamma_L \frac{\phi}{2\pi} + \tilde{\nu}_2(\bar{r}). \]
\[ \partial_\bar{r} \tilde{\nu} = \beta \partial_\bar{r} \alpha \]

\[ \beta = \frac{\partial_\bar{r} \Gamma_S}{\partial_\bar{r} \alpha} \frac{\eta}{2\pi} + \beta_2(\bar{r}) = \frac{2\pi \bar{r}^2}{\Gamma_S(\bar{r})} \eta + \beta_2(\bar{r}). \]

\[ \nu = \tilde{\nu} - \alpha \beta = (\Gamma_S - \frac{(2\pi \bar{r})^2 \alpha}{\Gamma_S}) \frac{\eta}{2\pi} + \Gamma_L \frac{\phi}{2\pi} + \nu_2(\bar{r}). \]
Dynamics

\[ \alpha = \frac{1}{2} \bar{v}^2 + w(\rho) + \Psi \]

\[ \Psi = \frac{1}{2} \bar{v}^2 + w(\rho) - \alpha \]

\[ = \frac{1}{2} \left( \left( \frac{\Gamma_S(\bar{r})}{2\pi \bar{r}} \right)^2 + \left( \frac{\Gamma_L}{2\pi R} \right)^2 \right) + w\left( \frac{g(\bar{r})}{R} \right) - \int_0^r d\bar{r}' \frac{\Gamma_S(\bar{r}') \partial_{\bar{r}'} \Gamma_S(\bar{r}')}{(2\pi \bar{r}')^2} \]
Thus the force potential needed to maintain this family of flows is dependent on the equation of state $w(\rho)$ of the material under consideration, the arbitrary functions $\Gamma_S(\vec{r}), g(\vec{r})$ and the arbitrary constant $\Gamma_L$. 
$$[\nu]_\phi = \Gamma_L$$

$$d\Phi = \vec{\omega} \cdot d\vec{S} = \omega_\phi dS_\phi = -\frac{1}{2\pi \bar{r}} \partial_{\bar{r}} \Gamma_S(\bar{r}) \bar{r} d\bar{r} d\eta = -\frac{1}{2\pi} \partial_{\bar{r}} \Gamma_S(\bar{r}) d\bar{r} d\eta.$$
The Helicity is a result of the azimuthal vortex lines surrounding a "virtual" vortex line going along the torus symmetry axis.

The Helicity can also be calculated using the traditional equation yielding the same result.
Conclusion

It was shown that using the variational variables a new class of helical flows was derived.