Knots and links in fluid mechanics

Alberto Enciso

Instituto de Ciencias Matemáticas, Madrid

July 2012

Outline of the talk

1. Context
2. Realization Theorem
3. Sketch of Proof
4. Vortex lines vs. Vortex tubes

Context

We will be playing with the Euler equation in $\mathbb{R}^3$ for ideal fluids:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\nabla P \,, \qquad \operatorname{div} u = 0$$

The trajectories of the velocity field u are called stream lines, and those of its curl $\omega := \operatorname{curl} u$ (the *vorticity*) are the vortex lines.

Question

Which knots and links can be vortex lines of a steady solution of the Euler equation? (Conjectural answer: "all").

This question is obviously related to the problem on the existence of knotted and linked vortex tubes that we have just considered. As in the case of vortex tubes, the steady solutions we'll consider are *Beltrami fields*:

$$\operatorname{curl} u = \lambda u \quad \text{in } \mathbb{R}^3, \qquad \lambda \in \mathbb{R} \backslash \{0\} \,.$$

Obviously they are not in $L^2(\mathbb{R}^3)$, but we'll get optimal decay in this class.

Realization Theorem

Realization Theorem (Enciso, Peralta-Salas 2012)

Let $L \subset \mathbb{R}^3$ be a finite link. For any $\lambda \neq 0$, we can transform L with a diffeomorphism Φ of $\mathbb{R}^3$, close to the identity in any C^p norm, such that $\Phi(L)$ is a set of periodic vortex lines of a Beltrami field satisfying curl $u = \lambda u$ in $\mathbb{R}^3$. Moreover, the Beltrami field falls off as $|D^j u(x)| < C_j/|x|$ for all j.

["Knots and links in steady solutions of the Euler equation", ArXiv:1003.3122 (Ann. of Math., 2012)]

Realization Theorem

Realization Theorem (Enciso, Peralta-Salas 2012)

Let $L \subset \mathbb{R}^3$ be a finite link. For any $\lambda \neq 0$, we can transform L with a diffeomorphism Φ of $\mathbb{R}^3$, close to the identity in any C^p norm, such that $\Phi(L)$ is a set of periodic vortex lines of a Beltrami field satisfying curl $u = \lambda u$ in $\mathbb{R}^3$. Moreover, the Beltrami field falls off as $|D^j u(x)| < C_j/|x|$ for all j.

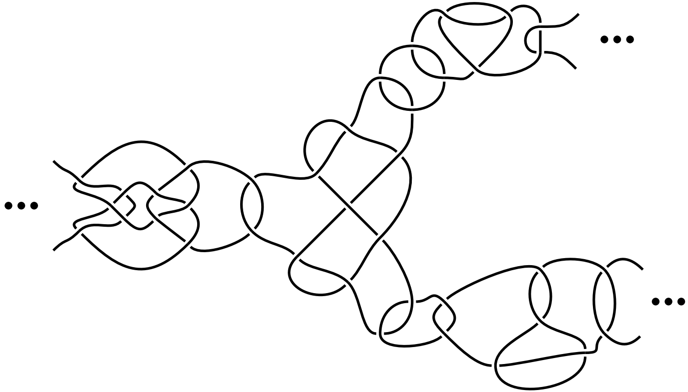
["Knots and links in steady solutions of the Euler equation", ArXiv:1003.3122 (Ann. of Math., 2012)]

Locally finite links can be dealt with too at the expense of losing the decay at infinity:

Corollary (Question by Etnyre & Ghrist)

There exists a steady solution of the Euler equation having periodic orbits of all knot types at the same time.

Realization Theorem



A locally finite (but scary enough) link. The meaning of the Realization Theorem is that, up to a small diffeomorphism, there is a steady solution of the Euler equation having all these periodic orbits!

Sketch of the proof

For simplicity, we'll prove the theorem for connected links, so that L is simply a knot.

An is the case of vortex tubes, the proof also consist in constructing "robust" local solutions and approximating them by a global Beltrami field. To see how this is done, we'll divide the proof in four steps.

Sketch of the proof

Step 1 (Local existence for the Cauchy problem): A natural way of obtaining local solutions of the Beltrami equation with partially prescribed behavior is to consider the Cauchy problem

$$\operatorname{curl} v = \lambda v\,, \qquad v|_\Sigma = w\,,$$

where Σ is an embedded oriented surface in $\mathbb{R}^3$ of class C^ω:

Theorem

Let Σ be an embedded oriented analytic surface in $\mathbb{R}^3$. Let w be a C^ω vector field tangent to Σ. Suppose that the pullback of its associated 1-form γ to the surface Σ is a closed form (i.e., $dj_\Sigma^*(\gamma) = 0$ with $j_\Sigma : \Sigma \to \mathbb{R}^3$ the inclusion map). Then the equation $\operatorname{curl} v = \lambda v$ with Cauchy datum $v|_\Sigma = w$ has a unique solution in a neighborhood of the surface Σ. This solution is analytic.

This is a variant of the Cauchy–Kowalewski theorem, which is nontrivial because curl does not have any non-characteristic surfaces. The assumption in the statement is not only sufficient, but also necessary. It is proved using the auxiliary Dirac-type operator $d + d^*$ on $\Omega^\bullet(\mathbb{R}^3)$.

Sketch of the proof

Step 2 (Construction of appropriate Cauchy data):
In view of the local existence theorem proved in Step 1, we need vector fields tangent to certain analytic surfaces that we can feed this theorem as "Cauchy datum" and, in addition, have certain convenient dynamical properties.

As a preliminary technical result, we notice that we can assume that the knot L is analytic. Let's embed it in a strip (or ribbon) Σ. That is, Σ is an analytic surface diffeomorphic to a cylinder that is an orientable line bundle over the link component L.

Lemma

Given a knot L and a strip Σ around it, we can construct an analytic vector field w in a neighborhood of L, tangent to the strip and whose pullback to the strip has the knot L as a stable hyperbolic limit cycle. Furthermore, one can assume that the pullback to the strip Σ of the 1-form associated to the field w is a closed form.

(Roughly speaking, the dual 1-form is $\gamma = d\theta - z\,dz$.)

Sketch of the proof

Step 3 (Stability of periodic stream lines):
Since we have constructed a field w with the properties need to be plugged into de local existence theorem of Step 1 as Cauchy datum, it stems that there is a unique local Beltrami field v in a neighborhood of the knot L whose restriction to the strip is $v|_\Sigma = w$.

Since the field v is divergence-free and the periodic trajectory L is a stable hyperbolic limit cycle of $v|_\Sigma$ by construction, one can show that there is an exponentially repelling direction. Hence the periodic trajectory L is hyperbolic and from the hyperbolic stability theorem we get that the periodic trajectory L is "robust" under perturbations of the field v:

Theorem

Let v be the local Beltrami field corresponding to the "Cauchy datum" w. Then if u is any vector field C^k-close to the field v in a neighborhood of the knot L, there is a diffeomorphism Φ of $\mathbb{R}^3$, C^k-close to the identity, which transforms this component into a periodic orbit $\Phi(L)$ of the perturbed vector field u.

Sketch of the proof

Step 4 (Approximation by global solutions):
We have constructed a Beltrami field v in a neighborhood N_L of the knot L that has robust periodic trajectory diffeomorphic to L (that is to say, any C^k-small perturbation of the field v has a periodic trajectory diffeomorphic to L).
To complete the proof we approximate the local Beltrami field v by a global Beltrami field u as in the case of vortex tubes:

Theorem

The local Beltrami field v can be approximated in the neighborhood N_L by a global Beltrami field u, satisfying the equation $\operatorname{curl} u = \lambda u$ in $\mathbb{R}^3$ and falling off at infinity as $|D^j u(x)| < C_j/|x|$, as

$$\|u - v\|_{C^k(N_L)} < \epsilon\,.$$

The field u is the steady solution we were looking for.

Vortex lines vs. Vortex tubes

The basic idea of the proof for vortex lines is the following:

- Periodic orbits are "often" robust.
- We "know" how to construct local solutions.
- We "can" approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

The basic idea of the proof for vortex lines is the following:

- Periodic orbits are "often" robust.
  - Hyperbolic periodic orbits are robust.
- We "know" how to construct local solutions.

- We "can" approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

The basic idea of the proof for vortex lines is the following:

- Periodic orbits are "often" robust.
  - Hyperbolic periodic orbits are robust.
- We "know" how to construct local solutions.
  - Cauchy–Kowalewski-type theorem. *Compatibility:* The periodic orbit of the local solution can be made hyperbolic with a suitable choice of the field used as "Cauchy datum"
- We "can" approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

How do things go in the case of vortex tubes? (at the level of wishful thinking)

- Invariant tori are "often" robust.

- We "know" how to construct local solutions.

- We "can" approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

How do things go in the case of vortex tubes? (at the level of wishful thinking)

- Invariant tori are “often” robust.
  - By KAM theory, robust if we can guarantee that the local solution satisfies certain (sophisticated) nondegeneracy conditions on the boundary of the tube.
- We “know” how to construct local solutions.

- We “can” approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

How do things go in the case of vortex tubes? (at the level of wishful thinking)

- Invariant tori are “often” robust.
  - By KAM theory, robust if we can guarantee that the local solution satisfies certain (sophisticated) nondegeneracy conditions on the boundary of the tube.
- We “know” how to construct local solutions.
  - A Cauchy–Kowalewski-type theorem won’t do it. But we can resort to an indirect argument using a boundary value problem in which you cannot prescribe the tangent part of the vector field on the boundary of the tube.
- We “can” approximate local solutions by global solutions.

Vortex lines vs. Vortex tubes

"Easy prescription vs. Hard extraction"

We can't prescribe local solutions with good robustness properties; we have to extract conditions that ensure robustness from the guts of the equation.

Besides, while hyperbolicity is an easy condition to deal with, KAM-type nondegeneracy conditions are much more subtle.

This requires new ideas (and is technically involved), using in an essential way the thinness of the vortex tubes!

Thanks for your attention!