Velocity, Energy and Helicity of Vortex Knots and Unknots

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Fluid flow evolution

We consider a *homogeneous, incompressible fluid* in $\mathbb{R}^3$:

\[
\begin{aligned}
\text{velocity field } & \quad u = u(X, t) : \\
& \begin{cases}
\nabla \times u = \omega \\
\nabla \cdot u = 0 \\
\quad u = 0 \text{ as } X \to \infty
\end{cases}
\end{aligned}
\]

Fluid motion is governed by:

**Euler equations**

\[
\frac{\partial u}{\partial t} + (u \cdot \nabla) u = -\frac{1}{\rho} \nabla p
\]

Under Euler equations **topology is conserved**.

- **Vortex circulation**
  \[ \Gamma = \text{constant}; \]

- **Kinetic energy**
  \[ E = \frac{1}{2} \rho \int_V |u|^2 dV = \text{constant}; \]

- **Kinetic Helicity**
  \[ H = \int_V u \cdot \omega dV = \text{constant}; \]

- **topological quantities**
  knot type, linking number, crossing number . . .
Biot-Savart Law

- Let $\gamma_t$ be a smooth space curve given by:

  $\gamma_t : \begin{cases} 
  \mathbf{X}(s, t) := \mathbf{X}_t(s) \\
  s \in [0, L] \to \mathbb{R}^3
  \end{cases}$

- The intrinsic reference on $\gamma_t$ is given by the Frenet frame $\{\hat{t}, \hat{n}, \hat{b}\}$:

  $\begin{cases} 
  \hat{t} = \mathbf{X}'(s, t) = \frac{\partial \mathbf{X}}{\partial s} \\
  \hat{n} = \frac{\hat{t}'}{c} \\
  \hat{b} = \hat{t}' \times \hat{n}
  \end{cases}$

- Let identify $\gamma_t$ with a thin vortex filament of circulation $\Gamma$.

  The vortex line moves with a self induced velocity:

  **Biot-Savart (BS) law**

  \[ \mathbf{u}(\mathbf{x}, t) = \frac{\Gamma}{4\pi} \oint_{\gamma_t} \hat{t} \times \frac{(\mathbf{x} - \mathbf{X}(s))}{|\mathbf{x} - \mathbf{X}(s)|^3} \, ds \]
Vortex Filament Motion Under the Localized Induction Approximation (LIA)

The Biot-Savart law can be simplified according to the following:

Asymptotic theory: 
\[
\begin{align*}
\omega &= \omega_0 \hat{t} \\
\frac{R}{a} &= \delta \gg 1 \\
\left| \frac{X(s_i) - X(s_j)}{|s_i - s_j|} \right| &= o(1)
\end{align*}
\]

\[\Downarrow\]

LIA law 
\[
\mathbf{u}_{LIA} = \dot{\mathbf{X}}(s, t) \equiv \frac{\partial \mathbf{X}}{\partial t} = \frac{\Gamma}{4\pi} \ln \delta \ \mathbf{X}' \times \mathbf{X}'' = \frac{\Gamma}{4\pi} \ln \delta \ c \hat{b}
\]

Under LIA the following quantities are conserved:

\[
E \propto \int_{\mathcal{C}} c^2 \ ds, \quad H \propto \int_{\mathcal{C}} c^2 \tau \ ds, \quad Wr \propto \int_{\mathcal{C}} c^2 \ ds, \quad Tw \propto \int_{\mathcal{C}} \tau \ ds
\]
Torus Knots

Theorem (Massey, 1967)

A closed, non-self intersecting curve embedded in a torus $\Pi$, that cuts a meridian at $p > 1$ points and a longitude at $q > 1$ points ($p$ and $q$ relatively prime integers), is a non-trivial knot $\mathcal{T}_{p,q}$, with winding number $w = q/p$.

- $p > 1$ longitudinal wraps and $q > 1$ meridional wraps;
- For given $p, q$ ($p, q$ coprimes) $\mathcal{T}_{p,q} \sim \mathcal{T}_{q,p}$ topologically equivalent.
Torus Unknots

- Case $p = 1$ or $q = 1$:

Unknots

$$\mathcal{U}_{1,m} \sim \mathcal{U}_{m,1} \sim \mathcal{U}_0 \ (\text{circle})$$
Vortex Torus Knot Solution Under LIA

Theorem (Kida, 1981)
Let $\mathcal{K}_v$ denote the embedding of a knotted vortex filament in an ideal fluid. If $\mathcal{K}_v$ evolves under LIA, then there exist a class of steady solutions in the shape of torus knots $\mathcal{K}_v \equiv \mathcal{T}_{p,q}$ in terms of incomplete elliptic integrals.

Solution $\mathcal{T}_{p,q}$ in explicit analytic closed form, based on linear perturbation from the circular solution $\mathcal{U}_0$ of NLSE:

\[
\begin{align*}
    r &= r_0 + \epsilon r_1 \\
    \alpha &= \frac{s}{r_0} + \epsilon \alpha_1 \\
    z &= \frac{t}{r_0} + \epsilon z_1
\end{align*}
\]  
(Ricca, 1993)

\[
\begin{align*}
    r &= r_0 + \epsilon \sin (w\phi) \\
    \alpha &= \frac{s}{r_0} + \epsilon \frac{1}{wr_0} \cos (w\phi) \\
    z &= \frac{t}{r_0} + \epsilon \left(1 + \frac{1}{w^2}\right)^{1/2} \cos (w\phi)
\end{align*}
\]

Theorem (Ricca, 1993)
Let $\mathcal{T}_{p,q}$ denote the embedding of a small-amplitude vortex torus knot $\mathcal{K}_v$ evolving under LIA. $\mathcal{T}_{p,q}$ is steady and stable under linear perturbation iff $q > p$ ($w > 1$).
Numerical Study of Evolution of Vortex Knots

- The **evolution** is obtained by integrating the equation of motion forward in time at each point from the initial vortex configuration (fourth order Runge Kutta algorithm);
- **De-singularization of the BS integral:**

\[
\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 = \frac{\Gamma}{4\pi} (\mathbf{X}' \times \mathbf{X}'') \ln \left( \frac{2l_+l_-}{C r_{\text{core}}} \right)^{1/2} + \mathbf{u}_1
\]

where \( l_+ \) and \( l_- \) are the distances to the nearest neighbouring point.
Twist and Writhe Contributions to Helicity

\[ H = \int_V \mathbf{u} \cdot \mathbf{\omega} \, dV = Lk \Gamma^2 = pq \Gamma^2 = H_{wr} + H_{tw} = \text{constant} \]

For Unknots: \( Lk = 0 \), hence \( H = 0 \) and \( Tw = -Wr \).

- **Meridian wraps** contribute modestly to the total writhing number;
- The dominant contribution comes from the **longitudinal wraps**.
Translation Velocity of Vortex Knots and Unknots

- **Velocity** decreases with increasing **winding number**;
- **Fastest** torus knots: highest number of **longitudinal wraps**;
- At **high winding number** torus knots/unknots reverse their velocity.
Translation Velocity of Vortex Knots and Unknots

- Vortex knots travel faster than their corresponding unknots;
- The higher is the number of longitudinal wraps $p$, the faster is the translational motion;
Kinetic Energy of Vortex Knots

- for $w < 1$ LIA law **underestimates** the actual energy of vortex knot;
- for $w > 1$ LIA provides much **higher** energy values.
Kinetic Energy of Vortex Unknots

- For $w < 1$, LIA law underestimates the actual energy of vortex unknot;
- For $w > 1$, LIA provides much higher energy values.
For $w < 1$ the space traveled tends to decrease with increasing knot complexity: the **stabilizing effect** due to the BS is confirmed;

- For $w < 1$ the vortex knots/unknots are **LIA-unstable**.
Conclusions and References

- The effect of **geometric and topological aspects** on the **dynamics and energetics** of vortex torus knots/unknots have been analyzed;
- In general:
  - for $w < 1$ the more complex the vortex structure is, the faster it moves.
  - For $w > 1$ all vortex structures move essentially as fast as $U_0$, almost independently from their total twist.
- The **LIA law** tend to **under-estimate** the energy of knots with $w < 1$ and to **over-estimate** the energy of knots with $w > 1$.
- The **stabilizing effect of the Biot-Savart** law for knots with $w < 1$ (LIA-unstable) has been confirmed.

References