Velocity, Energy and Helicity of Vortex Knots and Unknots

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Fluid flow evolution

We consider a *homogeneous, incompressible fluid* in $\mathbb{R}^3$:

velocity field $\mathbf{u} = \mathbf{u}(\mathbf{X}, t) : \begin{cases} \nabla \times \mathbf{u} = \boldsymbol{\omega} \\ \nabla \cdot \mathbf{u} = 0 \\ \mathbf{u} = 0 \text{ as } \mathbf{X} \to \infty \end{cases}$ in $\mathbb{R}^3$

- Fluid motion is governed by:

Euler equations $$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\frac{1}{\rho}\nabla p$$

- Under Euler equations **topology is conserved.**
  - **Vortex circulation** $\Gamma = constant$;
  - **Kinetic energy** $E = \dfrac{1}{2}\rho \displaystyle\int_V |\mathbf{u}|^2 \mathrm{d}V = constant$;
  - **Kinetic Helicity** $H = \displaystyle\int_V \mathbf{u}\cdot\boldsymbol{\omega}\,\mathrm{d}V = constant$;
  - **topological quantities** knot type, linking number, crossing number . . .

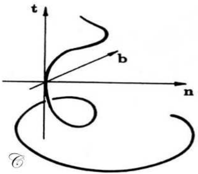
Biot-Savart Law

- Let $\mathscr{C}_t$ be a smooth **space curve** given by:

$$\mathscr{C}_t : \begin{cases} \mathbf{X}(s,t) := \mathbf{X}_t(s) \\ s \in [0, L] \to \mathbb{R}^3 \end{cases}$$

- The intrinsic reference on $\mathscr{C}_t$ is given by the **Frenet frame** $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$:

$$\begin{cases} \hat{\mathbf{t}} = \mathbf{X}'(s,t) = \dfrac{\partial \mathbf{X}}{\partial s} \\ \hat{\mathbf{n}} = \dfrac{\hat{\mathbf{t}}'}{c} \\ \hat{\mathbf{b}} = \hat{\mathbf{t}}' \times \hat{\mathbf{n}} \end{cases}$$



- Let identify $\mathscr{C}_t$ with a thin vortex filament of circulation Γ.
  The vortex line moves with a **self induced velocity**:

Biot-Savart (BS) law

$$\mathbf{u}(\mathbf{x},t) = \frac{\Gamma}{4\pi} \oint_{\mathscr{C}_t} \frac{\hat{\mathbf{t}} \times (\mathbf{x} - \mathbf{X}(s))}{|\mathbf{x} - \mathbf{X}(s)|^3} \mathrm{d}s$$

Vortex Filament Motion Under the Localized Induction Approximation (LIA)

The Biot-Savart law can be simplified according to the following:

Asymptotic theory :
$$\begin{cases} \boldsymbol{\omega} = \omega_0 \hat{\mathbf{t}} & \omega_0 = \text{constant} \\ \dfrac{R}{a} = \delta \gg 1 & \text{thin tube approximation} \\ \dfrac{|X(s_i) - X(s_j)|}{|s_i - s_j|} = o(1) & \text{no self-intersection} \end{cases}$$

$$\Downarrow$$

LIA law $$\mathbf{u}_{LIA} = \dot{\mathbf{X}}(s,t) \equiv \frac{\partial \mathbf{X}}{\partial t} = \frac{\Gamma}{4\pi} \ln \delta \; \mathbf{X}' \times \mathbf{X}'' = \frac{\Gamma}{4\pi} \ln \delta \; c\hat{\mathbf{b}}$$

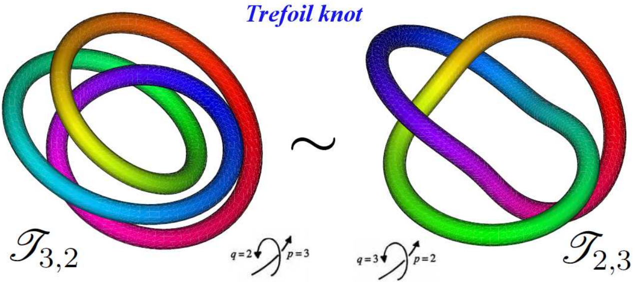
Under **LIA** the following quantities are conserved:

$$E \propto \oint_{\mathscr{C}} c^2 \, \mathrm{ds} \,, \qquad \mathrm{H} \propto \oint_{\mathscr{C}} \mathrm{c}^2 \tau \, \mathrm{ds} \,, \qquad \mathrm{Wr} \propto \oint_{\mathscr{C}} \mathrm{c}^2 \, \mathrm{ds} \,, \qquad \mathrm{Tw} \propto \oint_{\mathscr{C}} \tau \, \mathrm{ds}$$

Torus Knots

Theorem (Massey, 1967)

A closed, non-self intersecting curve embedded in a torus Π, that cuts a meridian at $p > 1$ points and a longitude at $q > 1$ points (p and q relatively prime integers), is a **non-trivial knot** $\mathscr{T}_{p,q}$, *with winding number $w = q/p$.*

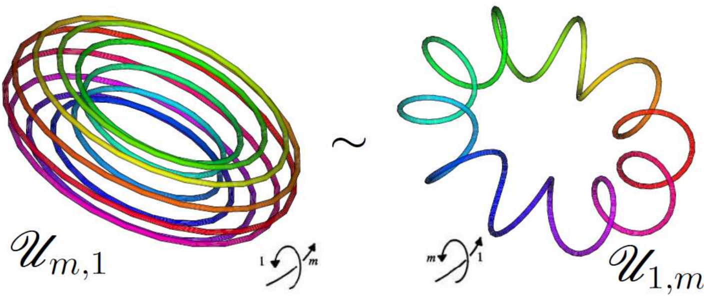


- $p > 1$ *longitudinal* wraps and $q > 1$ *meridional* wraps;
- For given p, q (p, q coprimes) $\mathscr{T}_{p,q} \sim \mathscr{T}_{q,p}$ **topologically equivalent**.

Torus Unknots

- Case $p = 1$ or $q = 1$:

Unknots $\mathscr{U}_{1,m} \sim \mathscr{U}_{m,1} \sim \mathscr{U}_0$ $(circle)$



Vortex Torus Knot Solution Under LIA

Theorem (Kida, 1981)

Let $\mathscr{K}_v$ denote the embedding of a knotted vortex filament in an ideal fluid. If $\mathscr{K}_v$ evolves under LIA, then there exist a class of **steady solutions** *in the shape of* **torus knots** $\mathscr{K}_v \equiv \mathscr{T}_{p,q}$ *in terms of incomplete elliptic integrals.*

- Solution $\mathscr{T}_{p,q}$ in explicit **analytic closed form**, based on **linear perturbation** from the circular solution $\mathscr{U}_0$ of NLSE:

$$
\text{(Ricca, 1993)}: \begin{cases} r = r_0 + \epsilon r_1 \\ \alpha = \dfrac{s}{r_0} + \epsilon \alpha_1 \\ z = \dfrac{\hat{t}}{r_0} + \epsilon z_1 \end{cases} \Rightarrow \begin{cases} r = r_0 + \epsilon \sin(w\phi) \\ \alpha = \dfrac{s}{r_0} + \epsilon \dfrac{1}{w r_0} \cos(w\phi) \\ z = \dfrac{\hat{t}}{r_0} + \epsilon \left(1 + \dfrac{1}{w^2}\right)^{1/2} \cos(w\phi) \end{cases}
$$

Theorem (Ricca, 1993)

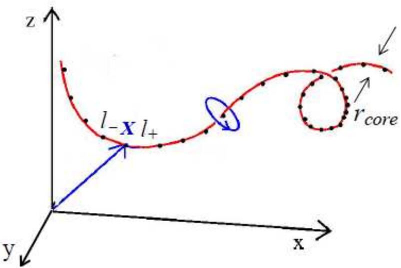
Let $\mathscr{T}_{p,q}$ denote the embedding of a small-amplitude vortex torus knot $\mathscr{K}_v$ evolving under LIA. $\mathscr{T}_{p,q}$ is steady and stable under linear perturbation iff $q > p$ $(w > 1)$.

Numerical Study of Evolution of Vortex Knots

- The **evolution** is obtained by integrating the equation of motion forward in time at each point from the initial vortex configuration (fourth order Runge Kutta algorithm);
- **De-singularization of the BS intergral**:

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 = \frac{\Gamma}{4\pi}(\mathbf{X}' \times \mathbf{X}'') ln \frac{(2l_+ l_-)^{1/2}}{C r_{core}} + \mathbf{u}_1$$

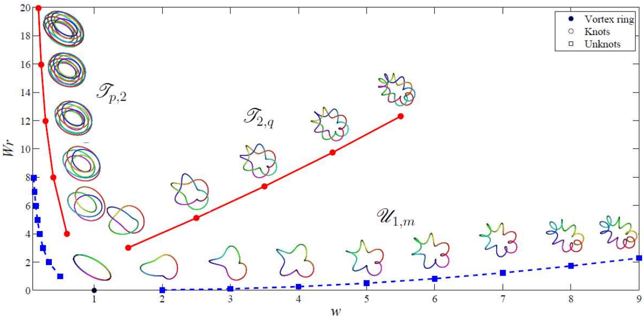
where l_+ and l_- are the distances to the nearest neighbouring point.



Twist and Writhe Contributions to Helicity

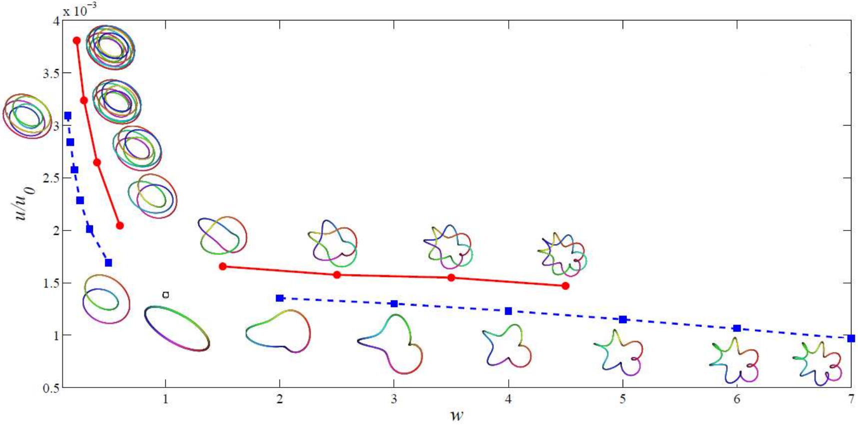
$$H = \int_V \mathbf{u} \cdot \boldsymbol{\omega} \, \mathrm{d}V = Lk\Gamma^2 = pq\Gamma^2 = H_{wr} + H_{tw} = constant$$ **Helicity**

For **Unknots**: $Lk = 0$, hence $H = 0$ and $Tw = -Wr$.



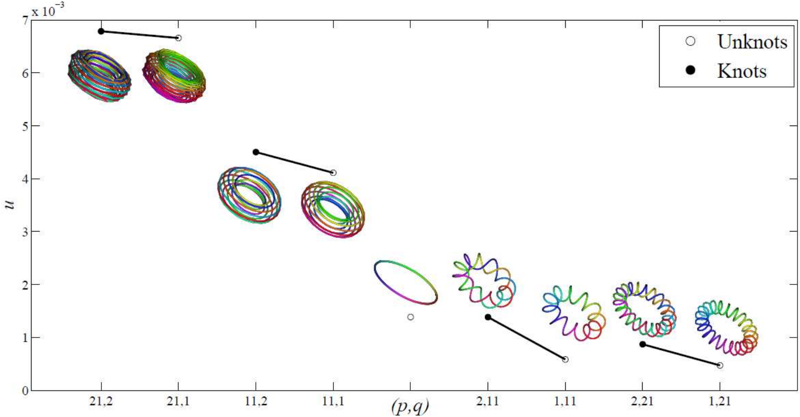
- **Meridian wraps** contribuite modestly to the total writhing number;
- The dominant contribution comes from the **longitudinal wraps**.

Translation Velocity of Vortex Knots and Unknots



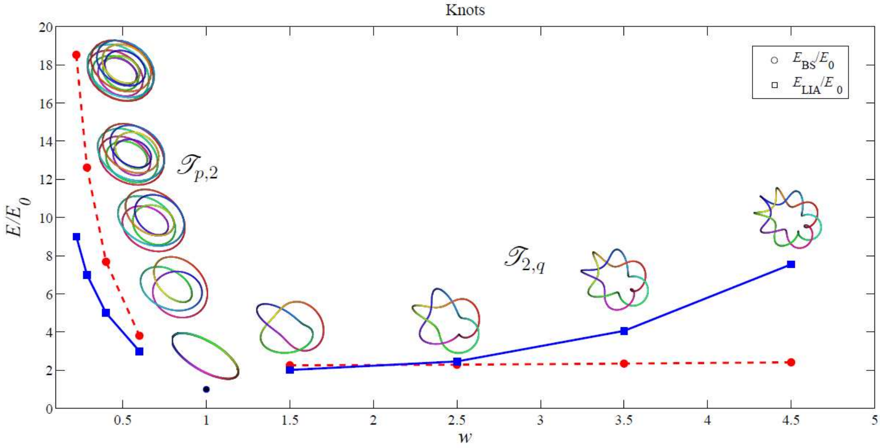
- **Velocity** decreases with increasing **winding number**;
- **Fastest** torus knots: highest number of **longitudinal wraps**;
- At **high winding number** torus knots/unknots **reverse** their velocity.

Translation Velocity of Vortex Knots and Unknots



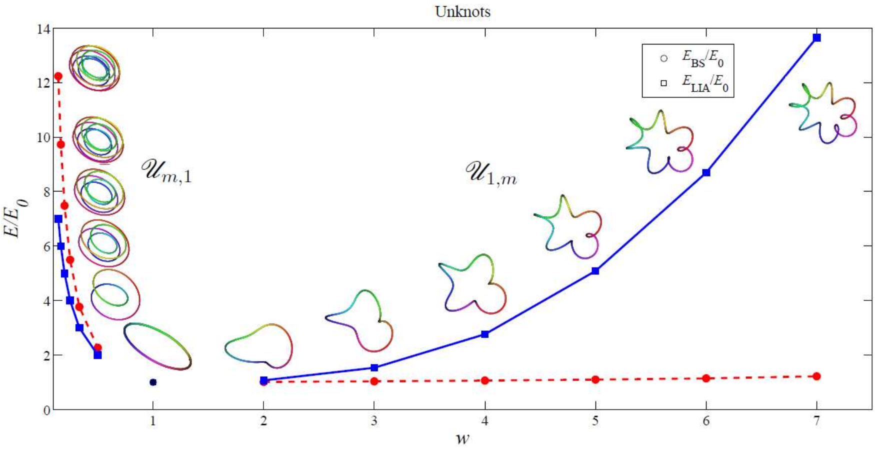
- **Vortex knots** travel **faster** then their corresponding **unknots**;
- The higher is the number of **longitudinal** wraps $\mathbf{p}$, the **faster** is the translational motion;

Kinetic Energy of Vortex Knots



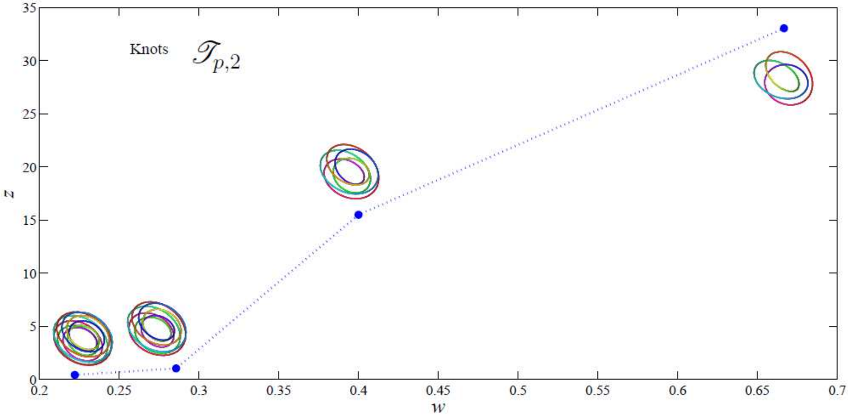
- for $w < 1$ **LIA** law **underestimates** the actual energy of vortex knot;
- for $w > 1$ **LIA** provides much **higher** energy values.

Kinetic Energy of Vortex Unknots



- for $w < 1$ **LIA** law **underestimates** the actual energy of vortex unknot;
- for $w > 1$ **LIA** provides much **higher** energy values.

Structural Stability



- For $w < 1$ the space traveled tends to decrease with increasing knot complexity: the **stabilizing effect** due to the BS is confirmed;
- For $w < 1$ the vortex knots/unknots are **LIA-unstable**.

Conclusions and References

- The effect of **geometric and topological aspects** on the **dynamics and energetics** of vortex torus knots/unknots have been analyzed;
- In general:
  - for $w < 1$ the more complex the vortex structure is, the faster it moves.
  - For $w > 1$ all vortex structures move essentially as fast as U_0, almost independently from their total twist.
- The **LIA law** tend to **under-estimate** the energy of knots with $w < 1$ and to **over-estimate** the energy of knots with $w > 1$.
- The **stabilizing effect of the Biot-Savart** law for knots with $w < 1$ (LIA-unstable) has been confirmed.

References

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