Intermittency in solutions of the 3D Navier-Stokes equations

J. D. Gibbon: Imperial College London

IUTAM-INI
Consider the **3D Navier-Stokes equations** in the domain \( [0, L]_\text{per}^3 \)

\[
    u_t + u \cdot \nabla u = \nu \Delta u - \nabla p + f(x)
\]

\[
    \text{div} \; u = \text{div} \; f = 0
\]

**Is intermittency the key to understanding the NSE?**

(a) Intermittent events are manifest as violent spiky surges away from space-time averages in both vorticity \( \omega \) & strain \( S_{ij} \). Spectra have non-Gaussian characteristics – Batchelor & Townsend 1949; Kuo & Corrsin (1971), Sreenivasan & Meneveau (1988, 1991); Frisch (1995).

(b) Are the spikes smooth down to some small scale, or does vorticity cascade down to length scales where the NSE are invalid? Is there a connection with the 3D-NS regularity problem?

**Aims of this talk**

(i) Estimate **resolution lengths** for weak solutions. Their meaning?

(ii) **3D-NS intermittency**: an unusual conditional regularity result.
Illustration: typical intermittency in wind tunnel data

Typical dissipation-range intermittency from wind tunnel turbulence where hot wire anemometry has been used to measure the longitudinal velocity derivative at a single point (D. Hurst & J. C. Vassilicos). The horizontal axis spans 8 integral time scales with $Re_\lambda \sim 200$. 
Kolmogorov & intermittency

Kolmogorov’s arguments: (Frisch 1995)

(i) The energy dissipation rate $\varepsilon$ defines the inverse Kolmogorov length

$$L\lambda_k^{-1} = \left(\frac{\varepsilon}{\nu^3}\right)^4 \sim Re^{3/4}.$$ 

(ii) Velocity structure functions are used to study intermittency

$$\left\langle |u(x + r) - u(x)|^p \right\rangle_{\text{ens. av.}} \sim r^{\zeta_p}$$

$\zeta_p$ is a concave curve below linear ($p > 3$): anomalous scaling.

Navier-Stokes equations?

Theoretically, structure functions are hard to handle for a PDE but we can still define a time-averaged energy dissipation rate $\varepsilon$ as

$$\varepsilon = \nu L^{-3} \left\langle \int_V |\omega|^2 dV \right\rangle_T$$

where

$$\left\langle \cdot \right\rangle_T = \frac{1}{T} \int_0^T \cdot d\tau$$
Higher moments of vorticity for the NS equations

**How might we pick up intermittent behaviour?** Higher moments of vorticity \((m \geq 1)\) are considered as “frequencies”

\[
\Omega_m(t) = \left( L^{-3} \int_{V} |\omega|^{2m} dV \right)^{1/2m} + \varpi_0
\]

The basic frequency associated with the domain is \(\varpi_0 = \nu L^{-2}\).

\[
\Omega_1^2 = L^{-3} \int_{V} |\omega|^2 dV + \varpi_0 \quad \text{div} \, u = 0
\]

\[
= L^{-3} \int_{V} |\nabla u|^2 dV + \varpi_0 \quad H_1\text{-norm}
\]

is the enstrophy/unit volume – related to the energy dissipation rate \(\varepsilon\).

**Higher moments will naturally “see” events at smaller scales**

\(\varpi_0 \leq \Omega_1(t) \leq \Omega_2(t) \leq \ldots \leq \Omega_m(t) \leq \Omega_{m+1}(t) \leq \ldots\)
3D-NS estimates in terms of $Re$

$$u_t + u \cdot \nabla u = \nu \Delta u - \nabla p + f(x) \quad \text{div } u = \text{div } f = 0$$

Traditionally, NS-estimates have been found in terms of $Gr$ (Grashof no) associated with the RMS-forcing ($f_{\text{rms}}^2 = L^{-3} \| f \|_2^2$) but it would be better to express these in terms of the Reynolds no $Re$

$$Gr = L^3 f_{\text{rms}} \nu^{-2}, \quad Re = U_0 L \nu^{-1}.$$ 

Doering and Foias (JFM 2002) used the idea of defining $U_0$ as

$$U_0^2 = L^{-3} \langle \| u \|_2^2 \rangle_T$$

where the time average $\langle \cdot \rangle_T$ over an interval $[0, T]$ is defined by

$$\langle g(\cdot) \rangle_T = \lim \sup_{g(0)} \frac{1}{T} \int_0^T g(\tau) \, d\tau.$$ 

$Gr$ is fixed provided $f$ is $L^2$-bounded, while $Re$ is the system response.
Leray’s energy inequality

Leray’s energy inequality shows that

\[ \frac{1}{2} \frac{d}{dt} \int_V |u|^2 \, dV \leq -\nu \int_V |\omega|^2 \, dV + \| f \|_2 \| u \|_2 , \]

and so, with \( L^{-3} \Omega_1^2 = \int_V |\omega|^2 \, dV \)

\[ \langle \Omega_1^2 \rangle_T \leq \omega_0^2 \text{Gr \, Re} + O \left( T^{-1} \right) , \quad (H_1\text{-norm}). \]

Doering & Foias (2002) showed that NS-solutions obey

\[ \text{Gr} \leq c \, \text{Re}^2 \]

provided the spectrum of \( f(x) \) is concentrated in a narrow band around a single frequency or is bounded above & below \( (\omega_0 = \nu L^{-2}) \).

\[ \varepsilon = \nu \langle \Omega_1^2 \rangle_T \leq c \, L^2 \omega_0^3 \text{Re}^3 + O \left( T^{-1} \right) . \]
Estimates for inverse Kolmogorov & Taylor micro-scales

Inverse Kolmogorov length

The time-averaged energy dissipation rate $\varepsilon$ & the inverse Kolmogorov length $\lambda_k^{-1}$ are related as (see Doering & Foias JFM 2002)

$$\lambda_k^{-4} = \frac{\varepsilon}{\nu^3}$$

and so

$$L\lambda_k^{-1} \leq c \, \text{Re}^{3/4} + O\left( T^{-1/4} \right).$$

Inverse Taylor micro-scale

Define

$$\lambda_{TMS}^{-2} = \frac{\left\langle \int_V |\omega|^2 \, dV \right\rangle_T}{\left\langle \int_V |u|^2 \, dV \right\rangle_T},$$

then

$$L\lambda_{TMS}^{-1} \leq c \, \text{Re}^{1/2} + O\left( T^{-1/2} \right).$$
Weak solution result

Invariance property

The NSE have the invariance property under the transformations $x' = \epsilon x$, $t' = \epsilon^2 t$, $u = \epsilon u'$ and $p = \epsilon^2 p'$. The $\Omega_m$ scale as

$$\Omega_m^{\alpha_m} = \epsilon \Omega_m'^{\alpha_m} \quad \alpha_m = \frac{2m}{4m - 3}.$$

Definition ($D_m = \left( \frac{\omega_0^{-1} \Omega_m}{\alpha_m} \right)^{\alpha_m}$ where $\alpha_m = \frac{2m}{4m - 3}$)

Note that while $\Omega_m \leq \Omega_{m+1}$, $\alpha_m \geq \alpha_{m+1}$: thus $D_m \leq D_{m+1}$?


Weak solutions of the 3D-Navier-Stokes equations satisfy

$$\langle D_m \rangle_T \leq c \text{Re}^3 + O(T^{-1}), \quad 1 \leq m \leq \infty,$$

where $\omega_0 = \nu L^{-2}$ and $c$ is a uniform constant.
A hierarchy of length scales

Based on the definition of the inverse Kolmogorov length $\lambda_k^{-1}$, a generalization of this to a hierarchy of inverse lengths $\lambda_m^{-1}$ is:

$$ (L\lambda_m^{-1})^{2\alpha_m} := \langle D_m \rangle_T, \quad \alpha_m = \frac{2m}{4m - 3}. $$

Proposition

For $1 \leq m \leq \infty$, members of the hierarchy of $L\lambda_m^{-1}$ are estimated as

$$ L\lambda_m^{-1} \leq c \ Re^{3/2\alpha_m} + O \left( T^{-1/2\alpha_m} \right). $$

Point-wise estimates not required: weak solutions exist but not unique.

<table>
<thead>
<tr>
<th>$m$</th>
<th>1</th>
<th>9/8</th>
<th>3/2</th>
<th>2</th>
<th>3</th>
<th>...</th>
<th>\infty</th>
</tr>
</thead>
<tbody>
<tr>
<td>$3/2\alpha_m$</td>
<td>3/4</td>
<td>1</td>
<td>3/2</td>
<td>15/8</td>
<td>9/4</td>
<td>...</td>
<td>3</td>
</tr>
</tbody>
</table>

Computationally hard to get beyond $m = 1: m = 9/8 (Re^1)$ is close to modern resolutions.
Conditional 3D-NS regularity: a very brief history

**Older history:** Leray (1934); Prodi (1959), Foias & Prodi (1962), Serrin (1963) & Ladyzhenskaya (1964): every Leray-Hopf solution $u$ of the 3D-NSE with $u \in L^r((0, T); L^s)$ is regular on $(0, T]$ provided $2/r + 3/s = 1$ with $s \in (3, \infty]$ or if $u \in L^{\infty}((0, T); L^p)$ with $p > 3$. When $s = 3$: von Wahl (1983) & Giga (1986) first proved the regularity in the space $C((0, T]; L^3)$: see also Kozono & Sohr (1997) & Escauriaza, Seregin & Sverák (2003).

**Books:** Constantin & Foias 1988; Foias, Manley, Rosa, Temam (2001).

**Modern developments:**

(i) Assumptions on the pressure or 1 velocity derivative: Kukavica & Ziane (2006, 2007), Zhou (2002), Cao & Titi (2008, 2010), Cao (2010), Cao, Qin & Titi (2008), & Chen & Gala (2011);


(iii) Biswas and Foias (2012) study analyticity properties associated with Gevrey-class norms.
A new look at conditional regularity: unforced NS

Lemma

A differential inequality for $\Omega_m$ in the range $0 \leq m < \infty$ is

$$\dot{\Omega}_m \leq \varpi_0 \Omega_m \left\{ - \frac{1}{c_{1,m}} \left( \frac{\Omega_{m+1}}{\Omega_m} \right)^{\beta_m} + c_{2,m} \left( \varpi^{-1}_0 \Omega_m \right)^{2\alpha_m} \right\}$$

where $\beta_m = \frac{4}{3} m(m + 1)$.

Note that

$$\left( \frac{1}{\alpha_{m+1}} - \frac{1}{\alpha_m} \right) \beta_m = 2.$$ 

For $1 \leq m < \infty$ the $D_m(t) = \left(\varpi^{-1}_0 \Omega_m\right)^{\alpha_m}$ satisfy

$$\dot{D}_m \leq D_m^3 \left\{ -\varpi_{1,m} \left(\frac{D_{m+1}}{D_m}\right)^{\rho_{m}} + \varpi_{2,m} \right\},$$

where $\rho_{m} = \frac{2}{3} m(4m + 1)$ & $c_{n,m}$ are algebraically increasing with $m$.

$$\varpi_{1,m} = \varpi_0 \alpha_m c_{1,m}^{-1} \quad \text{and} \quad \varpi_{2,m} = \varpi_0 \alpha_m c_{2,m}.$$
2nd conditional regularity result

Inclusion of the forcing modifies the differential inequality for $D_m$ to

$$
\dot{D}_m \leq D_m^3 \left\{ -\frac{1}{\omega_{1,m}} \left( \frac{D_{m+1}}{D_m} \right)^{\rho_m} + \omega_{2,m} \right\} + \omega_{3,m} \text{Gr} \ D_m ,
$$

Integrate :

$$
[D_m(t)]^2 \leq \frac{\exp \left\{ -2 \int_0^t X_m \ d\tau \right\}}{[D_m(0)]^{-2} - 2 \omega_{2,m} \int_0^t \exp \left\{ -2 \int_{\tau}^t X_m \ d\tau' \right\} \ d\tau} .
$$

$$
X_m = \omega_{1,m} \left( \frac{D_{m+1}}{D_m} \right)^{\rho_m} D_m^2 - \omega_{3,m} \text{Gr} .
$$

$$
\int_0^t X_m \ d\tau \geq \omega_{1,m} \left( \frac{1}{\int_0^t D_1 \ d\tau} \right)^{\rho_m \gamma_m + 2} - \omega_{3,m} t \text{Gr}
$$
**Theorem**

On $[0, t]$ if there exists a value of $m$ in the range $1 < m < \infty$, with large initial data bounded by

$$[D_m(0)]^2 < C_m \text{Re}^{3\Delta_m/2} \quad 1 \leq \Delta_m \leq 4$$

for which the integral lies on or above the critical value

$$c_m \left( t \text{Re}^{3\delta_m} + \eta_2 \right) \leq \int_0^t D_m \, d\tau$$

then $D_m(t)$ decays exponentially on $[0, t]$. ($\delta_m \downarrow 11/20$ as $m \to \infty$)

$$\frac{1 + 4\rho_m\gamma_m}{4(1 + \rho_m\gamma_m)} < \delta_m < 1$$

$$\Delta_m = 4 \left\{ \delta_m(1 + \rho_m\gamma_m) - \rho_m\gamma_m \right\}$$

$$\gamma_m = \frac{\alpha_m + 1}{2(m^2 - 1)}$$
Relaxation oscillator behaviour

- $D_m(t)$

$Re^3$ upper bound of $\langle D_m \rangle_t$

$Re^{3\delta_m}$ critical lower-bound of $\langle D_m \rangle_t$

$t_0$ $t_1$ $t_2$ potential singularity
Conclusion:

The feature of this work that is new is the control that comes from assuming a critical lower bound on $\langle D_m \rangle_t$ in Theorem 4.

Singulares are still possible when

$$\int_0^t D_m d\tau < c_m \left( t \text{Re}^{3\delta_m} + \eta_2 \right)$$

- Potential singularities in $D_m(t)$ that raise the value of the integral past critical can be ruled out.

- We cannot rule out those that contribute little or nothing to the integral.
The 3D-Euler equations I

\[ \frac{D\omega}{Dt} = \omega \cdot \nabla \nu \]

\( \text{div } u = 0 \)

\( D_m \) has the same definitions as in N.S. but \( \varpi_0 \) is the circulation

\[ D_m = \left( \varpi_0^{-1} \Omega_m \right)^{\alpha_m}. \]

Then

\[ \dot{\Omega}_m \leq c_{1,m} \left( \frac{\Omega_{m+1}}{\Omega_m} \right)^{m+1} \Omega_m^2 \quad 1 \leq m < \infty \]

With \( \gamma_m = \frac{1}{2}(4m + 1) \) the \( D_m \) formally satisfy

\[ \dot{D}_m \leq c_{2,m} \varpi_0 \left( \frac{D_{m+1}}{D_m} \right)^{\gamma_m} D_m^3. \]
If one suspects a f.t. singularity then divide by $D_m^3$ & integrate

\[
D_m(t)^2 \leq \frac{1}{D_m(t_0)^{-2} - F_1(t)}
\]

\[
F_{1,}(t) = c_{3,m} \omega_0 \int_{t_0}^{t} \left( \frac{D_{m+1}}{D_m} \right)^{\frac{1}{2}(4m+1)} d\tau
\]

Numerical questions:

1. Is $F_1$ linear in $t$?
2. If so, is there a convergence $D_m^2(t)(T_{c,m} - t) \rightarrow c_m$?

with a convergence $T_{c,m} \rightarrow T_c$ uniformly in $m$ for a range of $m$?

C. Cao and E. S. Titi, Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics, Ann. Math., 166, 245 (2007).


I. Kukavica and M. Ziane, One component regularity for the Navier-Stokes equations, Nonlinearity, 19, 453 (2006).


C. Cao and E. S. Titi, Regularity Criteria for the three-dimensional Navier-Stokes Equations, Indiana Univ. Math. J., 57, 2643 (2008).

C. Cao, J. Qin & E. S. Titi, Regularity Criterion for Solutions of 3D Turbulent Channel Flows, Comm. PDEs, 33, 419 (2008).


A. Vasseur, Regularity criterion for 3D Navier-Stokes equations in terms of the direction of the velocity, Applns of Math., 54, No. 1, 47 (2009).


