On the regularity of Lagrangian trajectories for suitable weak solutions of the Navier–Stokes equations

Topological Fluid Dynamics (IUTAM Symposium)

July 25, 2012

Setting of the problem

- The Navier-Stokes equations

$$\begin{cases} u_t - \triangle u + u \cdot \nabla u + \nabla p = 0 \\ \nabla \cdot u = 0 \end{cases}$$

- Standard regularity of weak solutions

$$u \in L^2(0,T;V) \cap L^\infty(0,T;H)$$

- Regularity in $L^w L^s$ spaces: $u \in L^w(0,T;L^s(\Omega))$ for

$$\frac{2}{w} + \frac{3}{s} \leq \frac{3}{2},\ 2 \leq s \leq 6$$

- Serrin's condition

$$\frac{2}{w} + \frac{3}{s} \leq 1, s \geq 3$$

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- Lagrange trajectories

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- Foias *et al.* 1981: $u \in L^{2/3}(0, T; D(A)) \cap L^2(0, T; V)$

$$\Rightarrow u \in L^1(0, T; L^\infty(\Omega))$$

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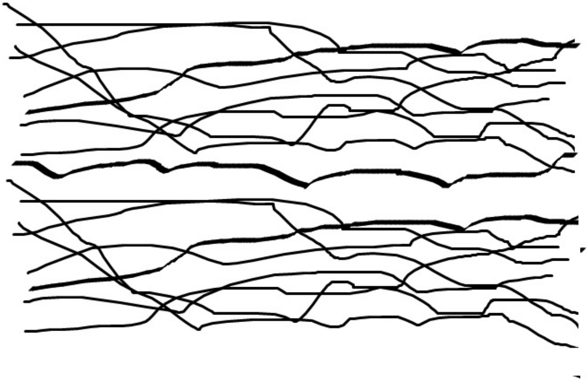


Foias *et al* 1985

There exists at least one function $\Phi : \Omega \times [0; T] \to \Omega$ such that

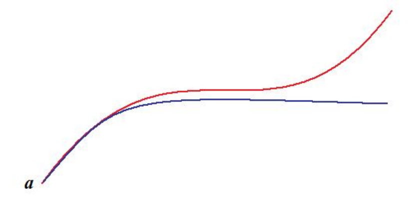
1. the function $\phi_a(\cdot) = \Phi(a, \cdot)$ satisfies

$$\phi(t) = a + \int_0^t u(\phi(s), s) ds$$

2. $\varphi_a(\cdot) \in W^{1,1}(0, T)$
3. the mapping $a \to \Phi(a, \cdot)$ belongs to $L^\infty(\Omega; C([0, T], \bar{\Omega}))$
4. Φ is volume-preserving: for any Borel set $B \subset \Omega$,

$$\mu(\Phi(\cdot, t)^{-1}(B)) = \mu(B)$$

1. Question: Are the particle trajectories unique for almost all $a \in \Omega$?



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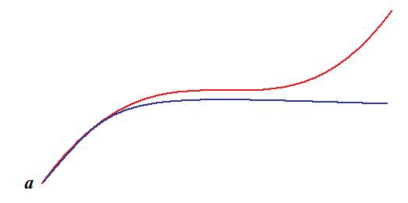

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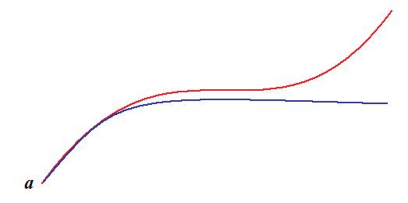

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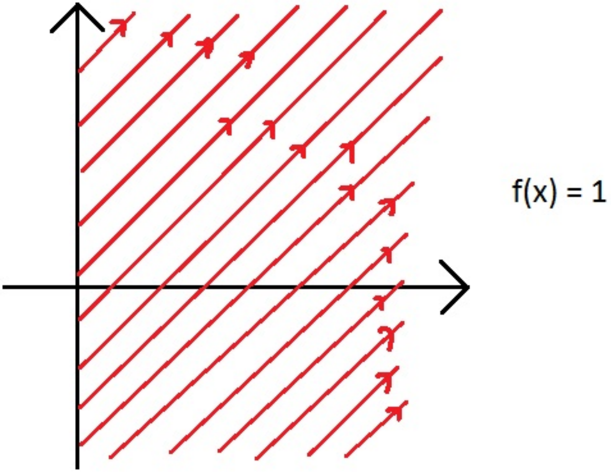
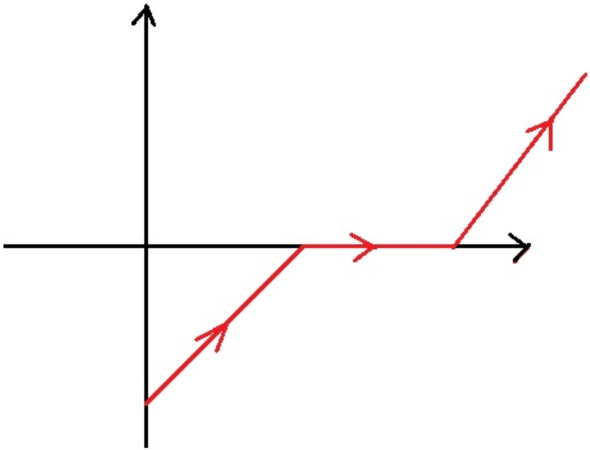

f(x) = 1



Suitable weak solutions

- $p \in L^{5/3}$
- local energy inequality

$\Rightarrow$

- small singular set: the Hasdorff dimension no greater than 1
- small singular set: the box-counting dimension no greater than $5/3$

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Definition

1. A point (x, t) is called regular if u is Hölder continuous in some neighbourhood of (x, t). A point is singular if it is not regular. The set of singular points in $(0, T) \times \Omega$ we denote by S.

CKN - 1982

1. The set S of singular points of a suitable weak solution u has 1-dimensional parabolic Hausdorff measure zero: for any $n \in N$ it can be covered by cylinders $Q_k = (t_k, t_k + r_k^2) \times B(x_k, r_k)$ such that

$$\sum_{k=1}^{\infty} r_k < \frac{1}{n}$$

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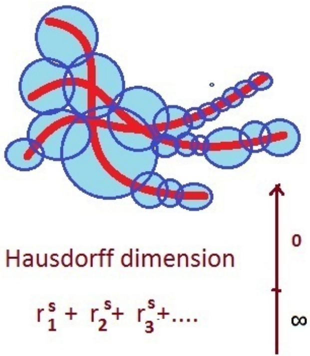
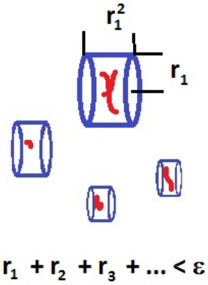

Hausdorff dimension
$r_1^s + r_2^s + r_3^s + \dots$
0
∞


r_1^2
r_1
$r_1 + r_2 + r_3 + \dots < \varepsilon$

Conditional result

Theorem 1. James Robinson and WS 2008

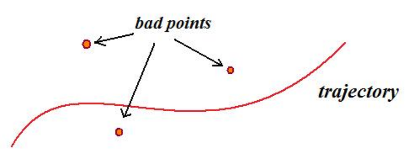
If u is a suitable weak solution with

$$u \in L^{6/5}(0, T; L^\infty)$$

corresponding to $u_0 \in H \cap H^{1/2}$, then almost every initial condition $a \in \Omega$ gives rise to a unique particle trajectory, which is C^1 function of time.

Idea of proof

Idea of the proof: show that particle trajectories "usually" avoid points where u behaves badly



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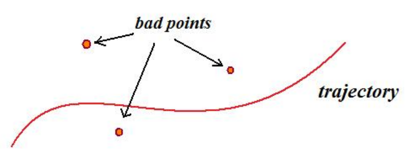


Avoiding "bad points"

Theorem 2

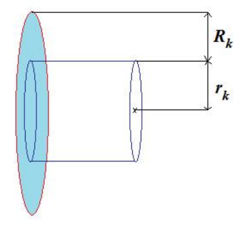
If u is a suitable weak solution with $u \in L^{6/5}(0, T; L^\infty)$ then the set of initial conditions $a \in \Omega$ that give rise to trajectories intersecting the singular set S is of Lebesgue measure zero.

Proof of Theorem 2 - part one

1. We cover the singular set by cylinders Q_k
2. We define the numbers R_k:

$$R_k = \int_{t_k}^{t_k + r_k^2} ||u||_\infty$$

3. Then we consider balls $\hat{B}_k = (x_k, r_k + R_k)$



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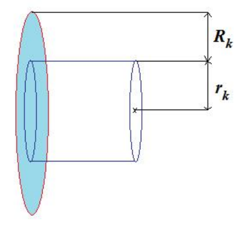

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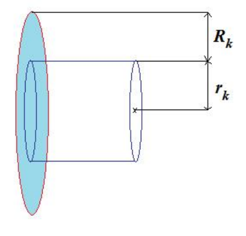

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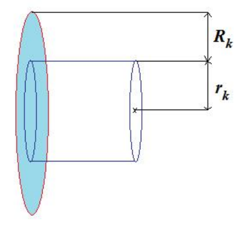

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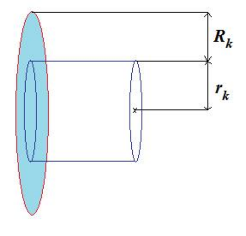

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$$|\varphi_a(t) - \varphi_a(t_k)| \leq R_k$$

we prove that trajectories that do not meet $\hat{B}_k$ at time t_k cannot enter cylinder Q_k

2. we estimate the volume of the family of balls $\hat{B}_k$:

$$\sum_{k=1}^{\infty} \mu(\hat{B}_k) \leq \sum_{k=1}^{\infty} c(r_k + R_k)^3 \leq C[\sum_{k=1}^{\infty} r_k^3 + \sum_{k=1}^{\infty} R_k^3]$$

3. Proof

$$R_k \leq \left(\int_{t_k}^{t_k+r_k^2} ds\right)^{1/6} \left(\int_{t_k}^{t_k+r_k^2} ||u||_{\infty}^{6/5}\right)^{5/6} \leq C r_k^{1/3}$$

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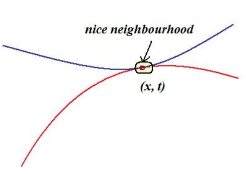
Theorem 2 implies Theorem 1

Proof

1. since $u_0 \in H^{1/2}$ trajectories are unique on some interval $[0, \varepsilon)$
2. assume that for some $t > 0$ and $x \in \Omega$ (x, t) is not singular and there are two trajectories passing through (x, t)



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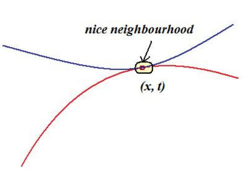

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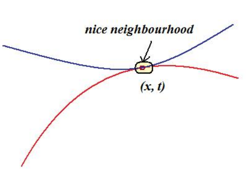

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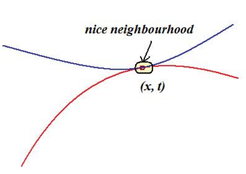

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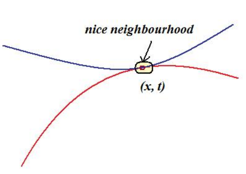

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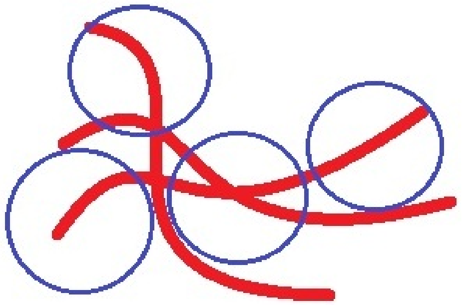
Theorem 3. (James Robinson and WS 2009)

If u is a suitable weak solution with $p \in L^{5/3}((0,T)\times\Omega)$ corresponding to $u_0 \in H \cap H^{1/2}(\Omega)$ then almost every initial condition $a \in \Omega$ gives rise to a unique particle trajectory, which is a C^1 function of time.



box-counting dimension

$$N \sim cr^{-d}$$

Fractal dimension

Definition.

The upper box-counting dimension of a set X is given by

$$\limsup_{\epsilon \to 0} \frac{log\ N(X,\epsilon)}{-log\epsilon},$$

where $N(X,\epsilon)$ is

1. the minimum number of balls of radius ϵ required to cover X, or
2. the maximum number of ϵ-separated points in X.

Remark. We always have

$$d_H(X) \leq d_F(X)$$

and for some X $d_H(X) < d_F(X)$.

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$$d_H(X) \leq d_F(X)$$

and for some X $d_H(X) < d_F(X)$.

Fractal dimension

Definition.

The upper box-counting dimension of a set X is given by

$$\limsup_{\epsilon \to 0} \frac{log\ N(X,\epsilon)}{-log\epsilon},$$

where $N(X,\epsilon)$ is

1. the minimum number of balls of radius ϵ required to cover X, or
2. the maximum number of ϵ-separated points in X.

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and for some X $d_H(X) < d_F(X)$.

Fractal dimension of a singular set

Theorem

The upper box-counting dimension of a putative singular set S is less or equal $5/3$.

Proof is based on Ladyzhenskaya-Seregin condition:

$$\frac{1}{r^2}\int_{Q_r(x,t)} |u|^3 + |p|^{3/2} \leq \varepsilon$$

from which we deduce that at a singular point we must have

$$\int_{Q_r(x,t)} |u|^{10/3} + |p|^{5/3} > cr^{5/3}.$$

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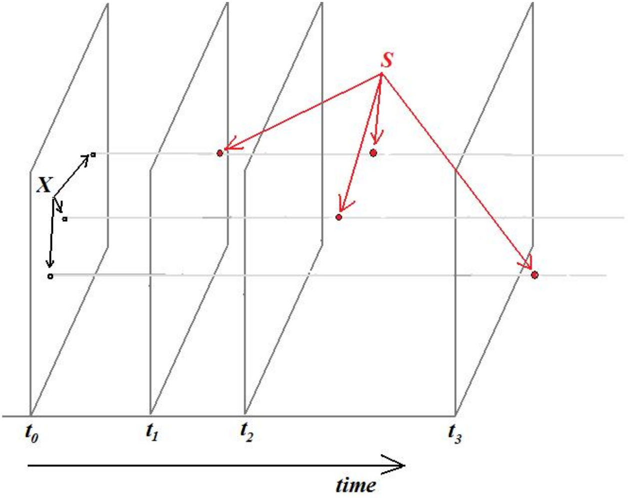
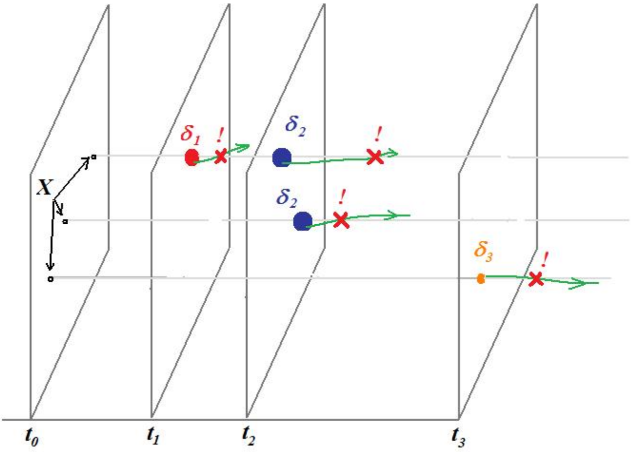
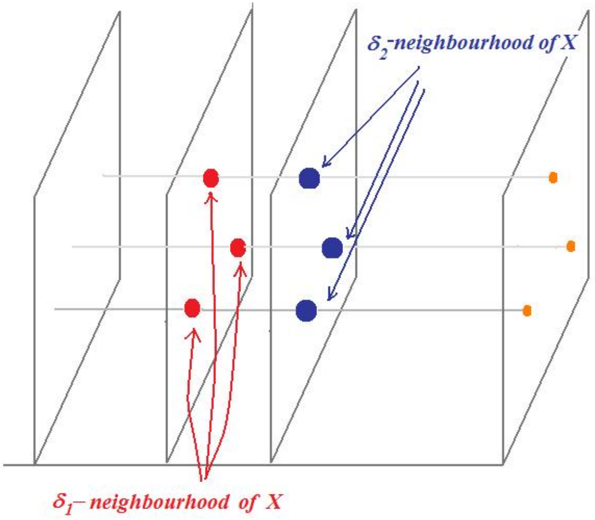

S
X
t_0
t_1
t_2
t_3
time


X
δ1
!
δ2
!
δ2
!
δ3
!
t0
t1
t2
t3

Definition of deltas

$$\delta_k = \int_{t_k}^{t_{k+1}} ||u(s)||_\infty ds$$


δ_2-neighbourhood of X
δ_1– neighbourhood of X

Proposition

If $X \in R^n$ has a box-counting dimension d, then for any $d' > d$ there exists an $\epsilon_0 > 0$ such that

$$\mu(O(X,\epsilon)) \leq c_n \epsilon^{n-d'} \quad \textit{for all} \quad 0 < \epsilon < \epsilon_0$$

Corollary

1. $V_1 =$ total volume of δ_1-neighbourhood of $X \leq c\delta_1^{6/5}$
2. $V_2 =$ total volume of δ_2-neighbourhood of $X \leq c\delta_2^{6/5}$
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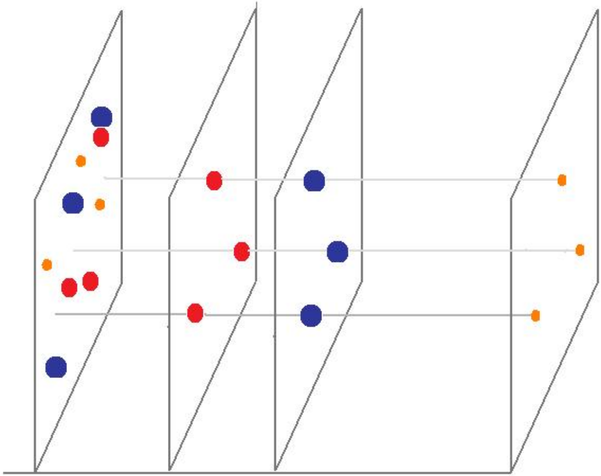

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$$\leq c[\epsilon^{1/5}\int_{t_0}^{t_1}||u||_\infty + \epsilon^{1/5}\int_{t_1}^{t_2}||u||_\infty + \ldots + \epsilon^{1/5}\int_{t_{N-1}}^{t_N}||u||_\infty] \leq$$

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Open problems

1. In what class $C^{1,\gamma}$ are the trajectories?
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