New invariants for entangled states

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arXiv:1009.2217, 1012.2630
[J. Phys. A: Math. Theor. 45, 182001, 185304 (2012)]

August 16, 2012

Outline

- Introduction
- Method
- Results
- Summary
- Outlook

Definition

- EPR paradox
- fundamental and counterintuitive
- superposition principle
- tensor product postulate
- linear combinations of tensor products
- factorizable vectors and disentangled states

2 spaces

spaces: V_1, V_2

$V = V_1 \otimes V_2$

transformations: $G_1 : V_1 \to V_1$

$G_2 : V_2 \to V_2$

$G_1 \times G_2 : V \to V$

states: $v_{1,j} \in V_1$

$v_{2,j} \in V_2$

$v \in V$

$v = \sum_{j=1}^{l} v_{1,j} \otimes v_{2,j}, \quad \operatorname{rank} v = l, \quad 0 \leq l \leq \min\{\dim V_1, \dim V_2\}$

$l = 2$: the EPR state

Properties

- quantifying entanglement
- smallest number of terms in a linear combination
- not an invariant for more than two systems
- invariant properties of composite systems
- variety of arrangements of terms in linear combinations

Classes

- changes of bases
- equivalent arrangements
- equivalence classes
- invariants as measures

Components

$I = \{1, \ldots, n\}$	a set
F	a field ($\mathbb{R}$ or $\mathbb{C}$)
S	a composite quantum system
$\{S_i\}_{i \in I}$	subsystems of S
$\{V_i\}_{i \in I}$	quantum states of $\{S_i\}_{i \in I}$
	(finite-dimensional vector spaces over F)

Tensor product postulate

$V \subseteq \otimes_{i \in I} V_i$ a space of quantum states of S

Which subspace?

- identical subsystems
- bosonic and fermionic
- symmetric and antisymmetric parts of $\otimes_{i \in I} V_i$
- quotient sets from equivalence relations for V
- for simplicity here: $V = \otimes_{i \in I} V_i$

Transformation group

From the tensor structure of V:

$\mathrm{GL}(V)$	no
$\times_{i\in I}\mathrm{GL}(V_i)$	yes

More generally:

$G_i \subseteq \mathrm{GL}(V_i)$

$G = \times_{i\in I} G_i$

Equivalence classes

$\sim_V$	equivalence relation on V induced by G
$v' \sim_V v$	iff $\exists g \in G : v' = gv$

$C(v) = \{v' \in V : v' \sim_V v\}$	the equivalence class of v
$\tilde{v} \in C(v)$	a representative element of $C(v)$
$C = \cup_{v \in V} \{C(v)\}$	the set of equivalence classes
$\tilde{V} = \{\tilde{v} \in C(v) : C(v) \in C\}$	the quotient set $\tilde{V} = V / \sim_V$

Quotient set

Ultimate goal: understanding the structure of $\tilde{V}$

Types of vectors in $\tilde{V}$:

- the zero vector
- decomposable vectors ($v = \otimes_{i \in I} v_i$, where $v_i \in V_i$)
- nondecomposable vectors

Types of quantum states in S:

- the vacuum state
- disentangled states
- entangled states

one-to-one correspondence

Quotient set

How large is $\tilde{V}$?

From $\tilde{V} = V / \sim_V$:

$\dim \tilde{V} \geq \dim V - \dim G$ (linear dependence)

Two cases:

$\dim V - \dim G \leq 0$ no information on the size of $\tilde{V}$

$\dim V - \dim G > 0$ $|\tilde{V}| = \infty$ (typically)

For $n \to \infty$:

$\dim V = O(\exp(cn)),\ c = \text{const}$

$\dim G = O(n^2)$

3 qubits

spaces: $\{V_i\}_{1\leq i\leq 3}, \quad (\dim V_i = 2)$

$V = \otimes_{1\leq i\leq 3} V_i$

groups: $G_i = \mathrm{GL}(V_i)$

$G = \times_{1\leq i\leq 3} G_i$

transformations: $v \mapsto g_1 g_2 g_3 v, \quad g_i \in G_i$

3 qubits

bases: $\{e_{i,j_i}\}_{1\le i\le 3,\ 1\le j_i\le 2}$

$\{e_{1,j_1}\otimes e_{2,j_2}\otimes e_{3,j_3}\}_{1\le j_1\le 2,\ 1\le j_2\le 2,\ 1\le j_3\le 2}$

coordinates: $\{v_{j_1,j_2,j_3}\}_{1\le j_1\le 2,\ 1\le j_2\le 2,\ 1\le j_3\le 2}$

$$v=\sum_{1\le j_1\le 2}\sum_{1\le j_2\le 2}\sum_{1\le j_3\le 2} v_{j_1,j_2,j_3} e_{1,j_1}\otimes e_{2,j_2}\otimes e_{3,j_3}$$

$$(g_1g_2g_3v)_{j_1,j_2,j_3}=\sum_{1\le k_1\le 2}\sum_{1\le k_2\le 2}\sum_{1\le k_3\le 2} v_{k_1,k_2,k_3}(g_1)_{k_1,j_1}(g_2)_{k_2,j_2}(g_3)_{k_3,j_3}$$

3 qubits

vacuum state: $v = 0$

disentangled state: $v = e_{1,1} \otimes e_{2,1} \otimes e_{3,1}$

entangled states:

type 1:

$$v = (e_{1,1} \otimes e_{2,1} + e_{1,2} \otimes e_{2,2}) \otimes e_{3,1}$$
$$v = (e_{1,1} \otimes e_{3,1} + e_{1,2} \otimes e_{3,2}) \otimes e_{2,1}$$
$$v = (e_{2,1} \otimes e_{3,1} + e_{2,2} \otimes e_{3,2}) \otimes e_{1,1}$$

type 2 (W state):

$$v = e_{1,1} \otimes e_{2,1} \otimes e_{3,2} + e_{1,1} \otimes e_{2,2} \otimes e_{3,1} + e_{1,2} \otimes e_{2,1} \otimes e_{3,1}$$

type 3 (GHZ state):

$$v = e_{1,1} \otimes e_{2,1} \otimes e_{3,1} + e_{1,2} \otimes e_{2,2} \otimes e_{3,2}$$

complete solution with 7 equivalence classes

Methods

How to find $\tilde{V}$?

Two methods:

direct:	general form of $\tilde{v} \in \tilde{V}$ from definition	(inefficient)
indirect:	invariants of $\tilde{v} \in \tilde{V}$ under G	(efficient)

Invariants

Standard method

classical theory of invariants and covariants

$A(v)$: set of algebraically independent continuous invariants

$$n < \infty$$
$$|A(v)| < \infty$$

computationally challenging:

$|\tilde{V}| = \infty \quad \Rightarrow \quad |\{A(v)\}_{v \in V}| = \infty$

overwhelming and impractical amount of information

Equivalence classes

projective spaces in quantum mechanics:
$v \sim f v, v \in V, f \in F, f \neq 0$ ($F = \mathbb{R}$ or $F = \mathbb{C}$)

invariants are homogeneous polynomials
$A(v) = 0$ is the most important value

equivalence for invariants:
$a' \sim_F a$ iff $\exists f \in F, f \neq 0 \colon a' = f a$
$(a'_k)_{k \in K} \sim_F (a_k)_{k \in K}$ iff $\exists f_k \in F, f_k \neq 0 \colon a'_k = f_k a_k$

Equivalence classes of equivalence classes

additional equivalences on V:
$v' \sim'_V v$ iff $A(v') \sim_F A(v)$ (for old invariants)
$v' \sim''_V v$ iff $\tilde{N}(v') = \tilde{N}(v)$ (for new invariants)

$\sim_V, C(v), \tilde{v}, C, \tilde{V}$ complete classification

$\sim'_V, C'(v), \tilde{v}', C', \tilde{V}'$ restricted old classification

$\sim''_V, C''(v), \tilde{v}'', C'', \tilde{V}''$ restricted new classification

Properties

- $\tilde{N}(v)$ depends only on the equivalence class $C(v)$
- $\tilde{N}(v)$ depends only on properties of linear subspaces of V
- the set of such subspaces $L(v)$ depends linearly on v
- $L(v)$ describes properties of v associated with all partitions of the system S into subsystems build from $\{S_i\}_{i\in I}$

$\tilde{N}(v)$ is uniquely determined by these conditions.

$$n = 2, 3, 4, 5, \ldots$$
$$|\tilde{N}(v)| = 1, 4, 19, 167(?), \ldots$$

Maps

a partition:

$S = T \cup T'$
$V = W \otimes W'$

a linear map:

$f(v) \colon W \to W'$
$f(v)(w) = v \otimes w^*$

fundamental spaces of $f(v)$:

$\ker f(v) = \{w \in W : f(v)(w) = 0\} \subseteq W$
$\operatorname{im} f(v) = \{w' \in W' : w' = f(v)(w), w \in W\} \subseteq W'$

Maps

$$v = \sum_{i=1}^{\dim W} \sum_{j=1}^{\dim W'} v_{i,j} e_i \otimes e'_j$$

$$f(v)(w) = \sum_{i=1}^{\dim W} \sum_{j=1}^{\dim W'} v_{i,j} w_i e'_j$$

$$\ker f(v) = \left\{ w \in W : \sum_{i=1}^{\dim W} v_{i,j} w_i = 0,\ j \in \{1, \ldots, \dim W'\} \right\}$$

All partitions

$\{K_J(v)\}_{J\in P(I)}$: properties of v related to partitions into any two subsystems

We need partitions into any number of subsystems.

the power set: $P(I)$

$$P'(I) = P(I) \setminus \{\varnothing, I\}$$

$$T = S_J, \quad T' = S_{I\setminus J}, \quad S_H = \cup_{h\in H} S_h$$
$$W = V_J, \quad W' = V_{I\setminus J}, \quad V_H = \otimes_{h\in H} V_h$$

$f_J(v)$

$K_J(v) = \ker f_J(v)$

$k_J(v) = \dim K_J(v)$

All partitions

$$v = \sum_{i=1}^{\dim V_J} \sum_{j=1}^{\dim V_{I\setminus J}} v_{i,j} e_i \otimes e'_j,$$

$$f_J(v)(w) = \sum_{i=1}^{\dim V_J} \sum_{j=1}^{\dim V_{I\setminus J}} v_{i,j} w_i e'_j,$$

$$K_J(v) = \left\{ w \in V_J : \sum_{i=1}^{\dim V_J} v_{i,j} w_i = 0,\ j \in \{1, \ldots, \dim V_{I\setminus J}\} \right\}.$$

Tensor products of maps

old maps: $\{f_J(v)\}_{J\in P(I)}$

new maps: $\{\tilde{f}_J(v)\}_{J\in P(I)}$

(linearity in v): the only additional maps must be identities

(comparisons): common domains; V is a natural choice

$$\tilde{f}_J(v)\colon V \to V_{I\setminus J} \otimes V_{I\setminus J}$$
$$\tilde{f}_J(v) = f_J(v) \otimes \mathrm{id}_{I\setminus J}$$

$$\tilde{K}_J(v) = \ker \tilde{f}_J(v), \quad \tilde{n}_J(v) = \dim \tilde{K}_J(v)$$

3 qubits

new maps:

$$\tilde{f}_{\{1\}}(v) \colon V \to V_2 \otimes V_3 \otimes V_2 \otimes V_3$$

$$\tilde{f}_{\{1\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \sum_{k_2=1}^{2} \sum_{k_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1,k_2,k_3} e_{2,j_2} \otimes e_{3,j_3} \otimes e_{2,k_2} \otimes e_{3,k_3}$$

$$\tilde{f}_{\{2\}}(v) \colon V \to V_1 \otimes V_3 \otimes V_1 \otimes V_3$$

$$\tilde{f}_{\{2\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \sum_{k_1=1}^{2} \sum_{k_3=1}^{2} v_{j_1,j_2,j_3} w_{k_1,j_2,k_3} e_{1,j_1} \otimes e_{3,j_3} \otimes e_{1,k_1} \otimes e_{3,k_3}$$

$$\tilde{f}_{\{3\}}(v) \colon V \to V_1 \otimes V_2 \otimes V_1 \otimes V_2$$

$$\tilde{f}_{\{3\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} \sum_{k_1=1}^{2} \sum_{k_2=1}^{2} v_{j_1,j_2,j_3} w_{k_1,k_2,j_3} e_{1,j_1} \otimes e_{2,j_2} \otimes e_{1,k_1} \otimes e_{2,k_2}$$

3 qubits

new maps:

$$\tilde{f}_{\{1,2\}}(v)\colon V \to V_3 \otimes V_3$$

$$\tilde{f}_{\{1,2\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1,j_2,k_3} e_{3,j_3} \otimes e_{3,k_3}$$

$$\tilde{f}_{\{1,3\}}(v)\colon V \to V_2 \otimes V_2$$

$$\tilde{f}_{\{1,3\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_2=1}^{2} v_{j_1,j_2,j_3} w_{j_1,k_2,j_3} e_{2,j_2} \otimes e_{2,k_2}$$

$$\tilde{f}_{\{2,3\}}(v)\colon V \to V_1 \otimes V_1$$

$$\tilde{f}_{\{2,3\}}(v)(w) = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2}\sum_{k_1=1}^{2} v_{j_1,j_2,j_3} w_{k_1,j_2,j_3} e_{1,j_1} \otimes e_{1,k_1}$$

Spaces and their intersections

$$K_J(v) \otimes V_{I\setminus J} \subseteq \tilde{K}_J(v) \quad \text{(general property)}$$
$$K_J(v) \otimes V_{I\setminus J} \subset \tilde{K}_J(v) \quad \text{(nontrivial invariants)}$$

The full information about a set of linear subspaces is given by the dimensions of the subspaces and of all their intersections.

$$Q \in P(P'(I))$$
$$\tilde{K}_Q(v) = \cap_{J\in Q} \tilde{K}_J(v)$$
$$\tilde{n}_Q(v) = \dim \tilde{K}_Q(v)$$

new invariants: $\tilde{N}(v) = \{\tilde{n}_Q(v)\}_{Q\in P(P'(I))}$

3 qubits

sets:

$$I = \{1,2,3\}, \quad P'(I) = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$$

maps:

$$f_{\{1\}}(v) \colon V_1 \to V_2 \otimes V_3, \quad f_{\{1\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1} e_{2,j_2} \otimes e_{3,j_3}$$

$$f_{\{2\}}(v) \colon V_2 \to V_1 \otimes V_3, \quad f_{\{2\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_2} e_{1,j_1} \otimes e_{3,j_3}$$

$$f_{\{3\}}(v) \colon V_3 \to V_1 \otimes V_2, \quad f_{\{3\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_3} e_{1,j_1} \otimes e_{2,j_2}$$

3 qubits

maps:

$$f_{\{1,2\}}: V_1 \otimes V_2 \to V_3, \quad f_{\{1,2\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1,j_2} e_{3,j_3}$$

$$f_{\{1,3\}}: V_1 \otimes V_3 \to V_2, \quad f_{\{1,3\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_1,j_3} e_{2,j_2}$$

$$f_{\{2,3\}}: V_2 \otimes V_3 \to V_1, \quad f_{\{2,3\}}(v)(w) = \sum_{j_1=1}^{2} \sum_{j_2=1}^{2} \sum_{j_3=1}^{2} v_{j_1,j_2,j_3} w_{j_2,j_3} e_{1,j_1}$$

3 qubits

kernels:

$$K_1(v) = \{w \in V_1 : \sum_{i=1}^{2} v_{ijk} w_i = 0,\ j,k \in \{1,2\}\}$$

$$K_2(v) = \{w \in V_2 : \sum_{j=1}^{2} v_{ijk} w_j = 0,\ i,k \in \{1,2\}\}$$

$$K_3(v) = \{w \in V_3 : \sum_{k=1}^{2} v_{ijk} w_k = 0,\ i,j \in \{1,2\}\}$$

3 qubits

kernels:

$$K_{1,2}(v) = \{wf\,inV_1 \otimes V_2 : \sum_{i=1}^{2}\sum_{j=1}^{2} v_{ijk}w_{ij} = 0,\ k \in \{1,2\}\}$$

$$K_{1,3}(v) = \{w \in V_1 \otimes V_3 : \sum_{i=1}^{2}\sum_{k=1}^{2} v_{ijk}w_{ik} = 0,\ j \in \{1,2\}\}$$

$$K_{2,3}(v) = \{w \in V_2 \otimes V_3 : \sum_{j=1}^{2}\sum_{k=1}^{2} v_{ijk}w_{jk} = 0,\ i \in \{1,2\}\}$$

3 qubits

kernels:

$$K_{1,2,3}(v) = \Big\{ w \in V_1 \otimes V_2 \otimes V_3 : \sum_{i=1}^{2} \sum_{j=1}^{2} v_{ijk} w_{ijl} = 0,\ k,l \in \{1,2\}; \sum_{i=1}^{2} \sum_{k=1}^{2} v_{ijk} w_{ilk} = 0,\ j,l \in \{1,2\};\ \sum_{j=1}^{2} \sum_{k=1}^{2} v_{ijk} w_{ljk} = 0,\ i,l \in \{1,2\} \Big\}$$

3 qubits

possible values:

$$0 \leq \dim K_i(v) \leq 2,\ i \in \{1,2,3\}$$
$$0 \leq \dim K_{j,k}(v) \leq 4,\ (j,k) \in \{(1,2),(1,3),(2,3)\}$$
$$0 \leq \dim K_{1,2,3}(v) \leq 8$$

3 qubits

Case 1: Let $\dim K_1(v) = 2$. This means that w in the definition of $K_1(v)$ is arbitrary. Since w satisfies 4 constraint equations, we find that the equations are over-constrained unless $v = 0$. This leads to

$$
\begin{aligned}
\dim K_1(v) = \dim K_2(v) = \dim K_3(v) &= 2, \\
\dim K_{1,2}(v) = \dim K_{1,3}(v) = \dim K_{2,3}(v) &= 4, \\
\dim K_{1,2,3}(v) &= 8
\end{aligned}
$$

and gives the class C_0.

3 qubits

Case 2: Let $\dim K_1(v) = 1$, which implies $v \neq 0$. This means that w in the definition of $K_1(v)$ satisfies

$$c_1 w_1 + c_2 w_2 = 0$$

for some constants c_1 and c_2. Solving for w_2 and substituting into the $K_1(v)$ equations leads to

$$\frac{v_{211}}{v_{111}} = \frac{v_{212}}{v_{112}} = \frac{v_{221}}{v_{121}} = \frac{v_{222}}{v_{122}}.$$

Using these results in the $K_2(v)$ equations, we find

$$\begin{aligned} v_{111} w_1 + v_{121} w_2 &= 0, \\ v_{112} w_1 + v_{122} w_2 &= 0. \end{aligned}$$

This allows three possibilities:

3 qubits

Case 2.1: Let $\dim K_2(v) = 2$. This implies $v_{111} = v_{121} = v_{112} = v_{122} = 0$, which contradicts the requirement $v \neq 0$.

3 qubits

Case 2.2: Let $\dim K_2(v) = 1$. This implies

$$c_1' w_1 + c_2' w_2 = 0$$

for some constants c_1' and c_2'. Using this equation in the remaining two $K_2(v)$ equations, we find

$$\frac{v_{121}}{v_{111}} = \frac{v_{122}}{v_{112}}.$$

Using these results in the $K_3(v)$ equations, we find they reduce to a single independent equation

$$v_{111} w_1 + v_{112} w_2 = 0.$$

Hence $\dim K_3(v) = 1$.

3 qubits

Now using the above results in the $K_{1,2}(v), K_{1,3}(v), K_{2,3}(v)$, we find that there is only one constraint for each of these kernels,

$$v_{111}^2 w_{11} + v_{111}v_{121}w_{12} + v_{111}v_{211}w_{21} + v_{121}v_{211}w_{22} = 0, \; w \in K_{1,2}(v),$$
$$v_{111}^2 w_{11} + v_{111}v_{112}w_{12} + v_{111}v_{211}w_{21} + v_{112}v_{211}w_{22} = 0, \; w \in K_{1,3}(v),$$
$$v_{111}^2 w_{11} + v_{111}v_{112}w_{12} + v_{111}v_{121}w_{21} + v_{112}v_{121}w_{22} = 0, \; w \in K_{2,3}(v).$$

Since there is one constraint for each 4-dimensional vector w, we conclude $\dim K_{1,2}(v) = \dim K_{1,3}(v) = \dim K_{2,3}(v) = 3$.

3 qubits

Finally we turn to $K_{1,2,3}(v)$. After some straightforward algebra, 12 equations in the definition of $K_{1,2,3}(v)$ reduce to

$$w_{112} = -\frac{1}{v_{111}^2}(v_{111}v_{121}w_{122} + v_{111}v_{211}w_{212} + v_{121}v_{211}w_{222}),$$
$$w_{121} = -\frac{1}{v_{111}^2}(v_{111}v_{112}w_{122} + v_{111}v_{211}w_{221} + v_{112}v_{211}w_{222}),$$
$$w_{211} = -\frac{1}{v_{111}^2}(v_{111}v_{112}w_{212} + v_{111}v_{121}w_{221} + v_{112}v_{121}w_{222}),$$
$$v_{111}^2 w_{111} - v_{112}v_{121}w_{122} - v_{112}v_{211}w_{212} - v_{121}v_{211}w_{221} - 2\frac{v_{112}v_{121}v_{211}}{v_{111}}w_{222} = 0,$$

from which we read off $\dim K_{1,2,3}(v) = 4$. This is the class C_1.

3 qubits

Case 2.3: Let $\dim K_2(v) = 0$. Algebra similar to what we have encountered above leads to the result $\dim K_3(v) = 0$, $\dim K_{1,2}(v) = 2$, $\dim K_{1,3}(v) = 2$, $\dim K_{2,3}(v) = 3$, and $\dim K_{1,2,3}(v) = 3$. There are also two more Cases $2.3'$ and $2.3''$, where we cyclically permute indices 1, 2 and 3. We obtain the classes C_2, C_3, C_4.

Case 3: Here a similar analysis shows that we can have $\dim K_1(v) = \dim K_2(v) = \dim K_3(v) = 0$ and $\dim K_{1,2}(v) = \dim K_{1,3}(v) = \dim K_{2,3}(v) = 2$ with two subcases.

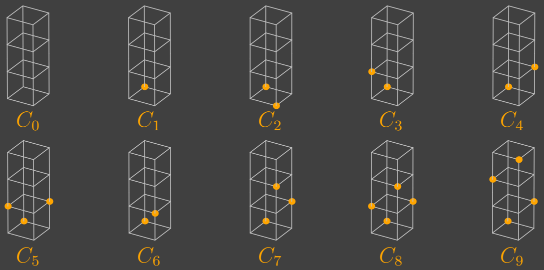
Case 3.1: $\dim K_{1,2,3}(v) = 1$, which is the class C_5.

Case 3.2: $\dim K_{1,2,3}(v) = 0$, which is the class C_6.

3 qubits

	$k_1(v)$	$k_2(v)$	$k_3(v)$	$k_{1,2,3}(v)$	v
C_0	2	2	2	8	0
C_1	1	1	1	4	$[1,1,1]$
C_2	0	0	1	3	$[1,1,1]+[2,2,1]$
C_3	0	1	0	3	$[1,1,1]+[2,1,2]$
C_4	1	0	0	3	$[1,1,1]+[1,2,2]$
C_5	0	0	0	1	$[1,1,1]+[1,2,2]+[2,1,2]$
C_6	0	0	0	0	$[1,1,1]+[2,2,2]$

Results for $n = 3,\ D = (2, 2, d)$



Theorem

$\sim'_V$ and $\sim''_V$ are identical

There is a one-to-one correspondence between the quotient set C' and the set of algebraically independent invariants $\{\tilde{N}(v)\}_{v \in V}$.

Proof by construction.

Independent invariants

$$\dim V_J - \frac{\tilde{n}_J(v)}{\dim V_{I\setminus J}} = \dim V_{I\setminus J} - \frac{\tilde{n}_{I\setminus J}(v)}{\dim V_J}$$

$\tilde{N}'(v) = (\tilde{n}_Q(v))_{Q\in R}, \quad R \subseteq P(P'(I)),$

sequence of sets $(R_1, R_2, \dots)$, $R_k \supseteq R_{k+1}$, $k \in \mathbb{N}$

set $R_1 = P(P'(I))$ and iterate:

1. If there exist $X \in R'_k$ and $X_1, X_2 \in X$ such that $X_1 \subseteq X_2$, then $R''_k = (R'_k \setminus \{X\}) \cup (X \setminus \{X_1\})$ for any such X, X_1; otherwise, $R''_k = R'_k$.
2. If there exists $X \in R''_k$ such that $X_1 \cap X_2 = \varnothing$ for any $X_1, X_2 \in X$, then $R'''_k = R''_k \setminus \{X\}$ for any such X; otherwise, $R'''_k = R''_k$.
3. If there exist $X, Y \in R'''_k$ such that $X_1 \in Y$ for any $X_1 \in X$, then $R_{k+1} = R'''_k \setminus \{X\}$ for any such X; otherwise, $R_{k+1} = R'''_k$.

$R = \lim_{k\to\infty} R_k$

Results for $n = 3$

generating invariants: $\tilde{N}(v) = (n_{Q_1}(v), \ldots, n_{Q_4}(v))$

$Q_1 = \{\{1\}\}$, $Q_2 = \{\{2\}\}$, $Q_3 = \{\{3\}\}$
$Q_4 = \{\{1,2\},\{1,3\},\{2,3\}\}$

$C'' = \{C_0\} \cup \{C_{k_1,k_2,k_3,j}\}$

$$\begin{aligned}
v \in C_0: \quad & n_{Q_1}(v) = d_1, n_{Q_2}(v) = d_2, n_{Q_3}(v) = d_3 \\
& n_{Q_4}(v) = d_1 d_2 d_3 \\
v \in C_{k_1,k_2,k_3,j}: \quad & n_{Q_1}(v) = d_1 - k_1, n_{Q_2}(v) = d_2 - k_2, n_{Q_3}(v) = d_3 - k_3 \\
& n_{Q_4}(v) = d_1 d_2 d_3 - k_1 d_1 - k_2 d_2 - k_3 d_3 + (M_{k_1,k_2,k_3})_j
\end{aligned}$$

$M_{k_1,k_2,k_1 k_2} = (k_1^2 + k_2^2)$
$M_{k_1,k_2,k_1 k_2 - 1} = (\ldots, k_1^2 + k_2^2 - (k_1 + k_2) + 2, k_1^2 + k_2^2 - 2(k_1 + k_2) + 5)$

Results for $n = 3$

(k_1, k_2, k_3)	M_{k_1,k_2,k_3}
$(1,1,1)$	(2)
$(1,2,2)$	(5)
$(1,3,3)$	(10)
$(1,4,4)$	(17)
$(1,5,5)$	(26)
$(2,2,2)$	$(5,4)$
$(2,2,3)$	$(6,5)$
$(2,2,4)$	(8)
$(2,3,3)$	$(8,\ldots,4)$
$(2,3,4)$	$(10,8,\ldots,5)$
$(2,3,5)$	$(10,8)$
$(2,3,6)$	(13)
$(2,4,4)$	$(13,\ldots,5)$
$(2,4,5)$	$(16,13,\ldots,5)$
$(2,4,6)$	$(15,13,12,10,9,8)$
$(2,4,7)$	$(16,13)$
$(2,4,8)$	(20)
$(2,5,5)$	$(20,19,16,\ldots,6)$
$(2,5,6)$	$(24,18,\ldots,5)$
$(2,5,7)$	$(22,20,18,\ldots,10,8)$
$(2,5,8)$	$(22,20,19,16,15,13)$
$(2,5,9)$	$(24,20)$
$(2,5,10)$	(29)

(k_1, k_2, k_3)	M_{k_1,k_2,k_3}
$(3,3,3)$	$(10,8,\ldots,2)$
$(3,3,4)$	$(11,\ldots,2)$
$(3,3,5)$	$(14,11,\ldots,2)$
$(3,3,6)$	$(12,\ldots,2)$
$(3,3,7)$	$(14,11,\ldots,4)$
$(3,3,8)$	$(14,11,10)$
$(3,3,9)$	(18)
$(3,4,4)$	$(14,12,\ldots,2)$
$(3,4,5)$	$(16,\ldots,2)$
$(3,4,6)$	$(20,16,\ldots,2)$
$(3,4,7)$	$(17,\ldots,2)$
$(3,4,8)$	$(19,17,\ldots,2)$
$(3,4,9)$	$(20,17,\ldots,2)$
$(3,4,10)$	$(20,19,16,\ldots,5)$
$(3,4,11)$	$(20,16,14)$
$(3,4,12)$	(25)

Results for $n = 3$

Numbers of classes

D	m
$\{2,2,2\}$	7
$\{2,2,3\}$	9
$\{2,2,d\}, d \geq 4$	10
$\{2,3,3\}$	17
$\{2,3,4\}$	23
$\{2,3,5\}$	25
$\{2,3,d\}, d \geq 6$	26
$\{2,4,4\}$	39
$\{2,4,5\}$	51
$\{2,4,6\}$	58
$\{2,4,7\}$	60
$\{2,4,d\}, d \geq 8$	61
$\{2,5,5\}$	77
$\{2,5,6\}$	99
$\{2,5,7\}$	113
$\{2,5,8\}$	120
$\{2,5,9\}$	122
$\{2,5,d\}, d \geq 10$	123

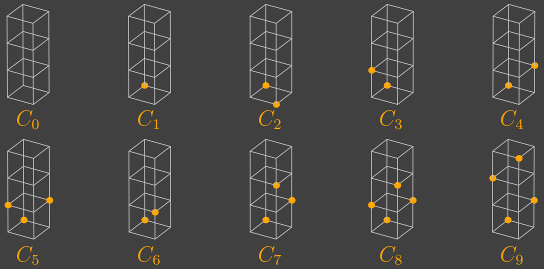
D	m
$\{3,3,3\}$	39
$\{3,3,4\}$	60
$\{3,3,5\}$	75
$\{3,3,6\}$	88
$\{3,3,7\}$	97
$\{3,3,8\}$	100
$\{3,3,d\}, d \geq 9$	101
$\{3,4,4\}$	103
$\{3,4,5\}$	143
$\{3,4,6\}$	178
$\{3,4,7\}$	205
$\{3,4,8\}$	226
$\{3,4,9\}$	244
$\{3,4,10\}$	258
$\{3,4,11\}$	261
$\{3,4,d\}, d \geq 12$	262

Results for $n = 3$, $D = (2, 2, d)$

canonical form: $v = \sum_{(j_1,j_2,j_3)\in J(v)} e_{1,j_1} \otimes e_{2,j_2} \otimes e_{3,j_3}$

	$\tilde{N}(v)$	$J(v)$
C_0	$(2, 2, d, 4d)$	$\varnothing$
C_1	$(1, 1, d-1, 3d-2)$	$\{(1,1,1)\}$
C_2	$(0, 0, d-1, 3d-3)$	$\{(1,1,1), (2,2,1)\}$
C_3	$(0, 1, d-2, 2d-1)$	$\{(1,1,1), (2,1,2)\}$
C_4	$(1, 0, d-2, 2d-1)$	$\{(1,1,1), (1,2,2)\}$
C_5	$(0, 0, d-2, 2d-3)$	$\{(1,1,1), (1,2,2), (2,1,2)\}$
C_6	$(0, 0, d-2, 2d-4)$	$\{(1,1,1), (2,2,2)\}$
C_7	$(0, 0, d-3, d-2)$	$\{(1,1,1), (1,2,2), (2,2,3)\}$
C_8	$(0, 0, d-3, d-3)$	$\{(1,1,1), (1,2,2), (2,1,2), (2,2,3)\}$
C_9	$(0, 0, d-4, 0)$	$\{(1,1,1), (1,2,2), (2,1,3), (2,2,4)\}$

Results for $n = 3$, $D = (2, 2, d)$



Classical invariants for $n = 3$, $D = (2,2,2)$

$$v = \sum_{j_1=1}^{2}\sum_{j_2=1}^{2}\sum_{j_3=1}^{2} v_{j_1,j_2,j_3} e_{1,j_1} \otimes e_{2,j_2} \otimes e_{3,j_3}$$

From the Hilbert series:

$$h_1 = v_{1,1,1}v_{1,2,2} - v_{1,1,2}v_{1,2,1} + v_{2,1,1}v_{2,2,2} - v_{2,1,2}v_{2,2,1}$$
$$h_2 = v_{1,1,1}v_{2,1,2} - v_{1,1,2}v_{2,1,1} + v_{1,2,1}v_{2,2,2} - v_{1,2,2}v_{2,2,1}$$
$$h_3 = v_{1,1,1}v_{2,2,1} - v_{1,2,1}v_{2,1,1} + v_{1,1,2}v_{2,2,2} - v_{1,2,2}v_{2,1,2}$$
$$\begin{aligned} h_4 = {} & v_{1,1,1}^2 v_{2,2,2}^2 + v_{1,1,2}^2 v_{2,2,1}^2 + v_{1,2,1}^2 v_{2,1,2}^2 + v_{2,1,1}^2 v_{1,2,2}^2 \\ & - 2(v_{1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2} + v_{1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2} + v_{1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2} \\ & + v_{1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2} + v_{1,1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2} + v_{1,1,1}v_{1,1,2}v_{2,2,1}v_{2,2,2}) \\ & + 4(v_{1,1,1}v_{1,2,2}v_{2,1,2}v_{2,2,1} + v_{1,1,2}v_{1,2,1}v_{2,1,1}v_{2,2,2}) \end{aligned}$$

Classical invariants for new classes for $n = 3$, $D = (2, 2, 2)$

	h_1	h_2	h_3	h_4
C_0	0	0	0	0
C_1	0	0	0	0
C_2	0	0	$\neq 0$	0
C_3	0	$\neq 0$	0	0
C_4	$\neq 0$	0	0	0
C_5	$\neq 0$	$\neq 0$	$\neq 0$	0
C_6	0	0	0	$\neq 0$

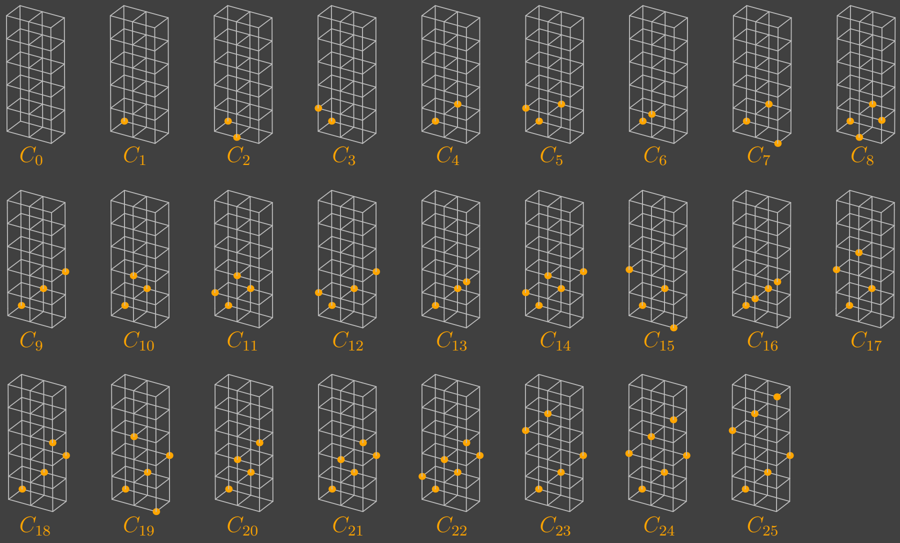
Results for $n = 3$, $D = (2, 3, d)$

	$\tilde{N}(v)$	$J(v)$
C_0	$(2, 3, d, 6d)$	$\varnothing$
C_1	$(1, 2, d-1, 5d-3)$	$\{(1,1,1)\}$
C_2	$(0, 1, d-1, 5d-5)$	$\{(1,1,1),(2,2,1)\}$
C_3	$(0, 2, d-2, 4d-2)$	$\{(1,1,1),(2,1,2)\}$
C_4	$(1, 1, d-2, 4d-3)$	$\{(1,1,1),(1,2,2)\}$
C_5	$(0, 1, d-2, 4d-5)$	$\{(1,1,1),(1,2,2),(2,1,2)\}$
C_6	$(0, 1, d-2, 4d-6)$	$\{(1,1,1),(2,2,2)\}$
C_7	$(0, 0, d-2, 4d-7)$	$\{(1,1,1),(1,2,2),(2,3,1)\}$
C_8	$(0, 0, d-2, 4d-8)$	$\{(1,1,1),(1,2,2),(2,2,1),(2,3,2)\}$
C_9	$(1, 0, d-3, 3d-1)$	$\{(1,1,1),(1,2,2),(1,3,3)\}$
C_{10}	$(0, 1, d-3, 3d-4)$	$\{(1,1,1),(1,2,2),(2,1,3)\}$
C_{11}	$(0, 1, d-3, 3d-5)$	$\{(1,1,1),(1,2,2),(2,1,2),(2,2,3)\}$
C_{12}	$(0, 0, d-3, 3d-5)$	$\{(1,1,1),(1,2,2),(1,3,3),(2,1,2)\}$
C_{13}	$(0, 0, d-3, 3d-6)$	$\{(1,1,1),(1,2,2),(2,3,3)\}$
C_{14}	$(0, 0, d-3, 3d-7)$	$\{(1,1,1),(1,2,2),(1,3,3),(2,1,2),$ $(2,2,3)\}$
C_{15}	$(0, 0, d-3, 3d-8)$	$\{(1,1,1),(1,2,2),(2,1,3),(2,3,1)\}$
C_{16}	$(0, 0, d-3, 3d-9)$	$\{(1,1,1),(1,2,2),(2,2,2),(2,3,3)\}$

Results for $n = 3$, $D = (2, 3, d)$

	$\tilde{N}(v)$	$J(v)$
C_{17}	$(0, 1, d-4, 2d-2)$	$\{(1,1,1), (1,2,2), (2,1,3), (2,2,4)\}$
C_{18}	$(0, 0, d-4, 2d-3)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,3,4)\}$
C_{19}	$(0, 0, d-4, 2d-5)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,2,4), (2,3,1)\}$
C_{20}	$(0, 0, d-4, 2d-6)$	$\{(1,1,1), (1,2,2), (2,2,3), (2,3,4)\}$
C_{21}	$(0, 0, d-4, 2d-7)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,2,3), (2,3,4)\}$
C_{22}	$(0, 0, d-4, 2d-8)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,1,2), (2,2,3), (2,3,4)\}$
C_{23}	$(0, 0, d-5, d-3)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,1,4), (2,2,5)\}$
C_{24}	$(0, 0, d-5, d-5)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,1,3), (2,2,4), (2,3,5)\}$
C_{25}	$(0, 0, d-6, 0)$	$\{(1,1,1), (1,2,2), (1,3,3), (2,1,4), (2,2,5), (2,3,6)\}$

Results for $n = 3,\ D = (2, 3, d)$



Results for $n = 4$

generating invariants: $\tilde{N}(v) = (n_{Q_1}(v), \ldots, n_{Q_{19}}(v))$

$Q_1 = \{\{1\}\}$
$Q_2 = \{\{2\}\}$
$Q_3 = \{\{3\}\}$
$Q_4 = \{\{4\}\}$
$Q_5 = \{\{1,2\},\{1,3\},\{2,3\}\}$
$Q_6 = \{\{1,2\},\{1,4\},\{2,4\}\}$
$Q_7 = \{\{1,3\},\{1,4\},\{3,4\}\}$
$Q_8 = \{\{2,3\},\{2,4\},\{3,4\}\}$

$Q_9 = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$
$Q_{10} = \{\{1,3\},\{1,2,4\},\{2,3,4\}\}$
$Q_{11} = \{\{1,4\},\{1,2,3\},\{2,3,4\}\}$
$Q_{12} = \{\{2,3\},\{1,2,4\},\{1,3,4\}\}$
$Q_{13} = \{\{2,4\},\{1,2,3\},\{1,3,4\}\}$
$Q_{14} = \{\{3,4\},\{1,2,3\},\{1,2,4\}\}$
$Q_{15} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$
$Q_{16} = \{\{1,2\},\{2,3\},\{2,4\},\{1,3,4\}\}$
$Q_{17} = \{\{1,3\},\{2,3\},\{3,4\},\{1,2,4\}\}$
$Q_{18} = \{\{1,4\},\{2,4\},\{3,4\},\{1,2,3\}\}$
$Q_{19} = \{\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\}\}$

Results for $n = 4$, $D = (2, 2, 2, 2)$

	$\tilde{N}(v)$
C_0	$(2, 2, 2, 2, 8, 8, 8, 8, 16, 16, 16, 16, 16, 16, 16, 16, 16, 16)$
C_1	$(1, 1, 1, 1, 4, 4, 4, 4, 10, 10, 10, 10, 10, 10, 8, 8, 8, 8, 11)$
C_2	$(0, 0, 1, 1, 3, 3, 2, 2, 9, 7, 7, 7, 7, 10, 3, 3, 7, 7, 10)$
C_3	$(0, 1, 0, 1, 3, 2, 3, 2, 7, 9, 7, 7, 10, 7, 3, 7, 3, 7, 10)$
C_4	$(0, 1, 1, 0, 2, 3, 3, 2, 7, 7, 9, 10, 7, 7, 3, 7, 7, 3, 10)$
C_5	$(1, 0, 0, 1, 3, 2, 2, 3, 7, 7, 10, 9, 7, 7, 7, 3, 3, 7, 10)$
C_6	$(1, 0, 1, 0, 2, 3, 2, 3, 7, 10, 7, 7, 9, 7, 7, 3, 7, 3, 10)$
C_7	$(1, 1, 0, 0, 2, 2, 3, 3, 10, 7, 7, 7, 7, 9, 7, 7, 3, 3, 10)$
C_8	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 9, 0, 0, 0, 0, 0, 0, 9)$
C_9	$(0, 0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 9, 0, 0, 0, 0, 0, 9)$
C_{10}	$(0, 0, 0, 0, 0, 0, 0, 0, 9, 0, 0, 0, 0, 9, 0, 0, 0, 0, 9)$
C_{11}	$(0, 0, 0, 1, 1, 2, 2, 2, 5, 5, 7, 5, 7, 7, 2, 2, 2, 7, 8)$
C_{12}	$(0, 0, 1, 0, 2, 1, 2, 2, 5, 7, 5, 7, 5, 7, 2, 2, 7, 2, 8)$
C_{13}	$(0, 1, 0, 0, 2, 2, 1, 2, 7, 5, 5, 7, 7, 5, 2, 7, 2, 2, 8)$
C_{14}	$(1, 0, 0, 0, 2, 2, 2, 1, 7, 7, 7, 5, 5, 5, 7, 2, 2, 2, 8)$

Results for $n = 4$, $D = (2, 2, 2, 2)$

	$\tilde{N}(v)$
C_{15}	$(0,0,0,1,0,2,2,2,4,4,7,4,7,7,2,2,2,7,7)$
C_{16}	$(0,0,1,0,2,0,2,2,4,7,4,7,4,7,2,2,7,2,7)$
C_{17}	$(0,1,0,0,2,2,0,2,7,4,4,7,7,4,2,7,2,2,7)$
C_{18}	$(1,0,0,0,2,2,2,0,7,7,7,4,4,4,7,2,2,2,7)$
C_{19}	$(0,0,0,0,1,1,1,1,5,5,5,5,5,5,2,2,2,2,7)$
C_{20}	$(0,0,0,0,1,0,0,1,2,2,4,5,2,2,1,1,1,1,6)$
C_{21}	$(0,0,0,0,0,1,1,0,2,2,5,4,2,2,1,1,1,1,6)$
C_{22}	$(0,0,0,0,0,1,0,1,2,4,2,2,5,2,1,1,1,1,6)$
C_{23}	$(0,0,0,0,1,0,1,0,2,5,2,2,4,2,1,1,1,1,6)$
C_{24}	$(0,0,0,0,0,0,1,1,4,2,2,2,2,5,1,1,1,1,6)$
C_{25}	$(0,0,0,0,1,1,0,0,5,2,2,2,2,4,1,1,1,1,6)$
C_{26}	$(0,0,0,0,0,0,0,0,4,4,4,4,4,4,0,0,0,0,6)$
C_{27}	$(0,0,0,0,0,0,0,0,0,0,5,5,0,0,0,0,0,0,6)$
C_{28}	$(0,0,0,0,0,0,0,0,0,5,0,0,5,0,0,0,0,0,6)$
C_{29}	$(0,0,0,0,0,0,0,0,5,0,0,0,0,5,0,0,0,0,6)$

Results for $n = 4$, $D = (2, 2, 2, 2)$

	$\tilde{N}(v)$
C_{30}	$(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,6)$
C_{31}	$(0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,6)$
C_{32}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,6)$
C_{33}	$(0,0,0,0,0,0,0,1,2,2,2,2,2,2,0,1,1,1,5)$
C_{34}	$(0,0,0,0,0,0,1,0,2,2,2,2,2,2,1,0,1,1,5)$
C_{35}	$(0,0,0,0,0,1,0,0,2,2,2,2,2,2,1,1,0,1,5)$
C_{36}	$(0,0,0,0,1,0,0,0,2,2,2,2,2,2,1,1,1,0,5)$
C_{37}	$(0,0,0,0,0,0,0,0,2,2,4,4,2,2,0,0,0,0,5)$
C_{38}	$(0,0,0,0,0,0,0,0,2,4,2,2,4,2,0,0,0,0,5)$
C_{39}	$(0,0,0,0,0,0,0,0,4,2,2,2,2,4,0,0,0,0,5)$
C_{40}	$(0,0,0,0,0,0,0,0,0,0,4,5,0,0,0,0,0,0,5)$
C_{41}	$(0,0,0,0,0,0,0,0,0,0,5,4,0,0,0,0,0,0,5)$
C_{42}	$(0,0,0,0,0,0,0,0,0,4,0,0,5,0,0,0,0,0,5)$
C_{43}	$(0,0,0,0,0,0,0,0,0,5,0,0,4,0,0,0,0,0,5)$
C_{44}	$(0,0,0,0,0,0,0,0,4,0,0,0,0,5,0,0,0,0,5)$
C_{45}	$(0,0,0,0,0,0,0,0,5,0,0,0,0,4,0,0,0,0,5)$

Results for $n = 4$, $D = (2, 2, 2, 2)$

	$\tilde{N}(v)$
C_{46}	$(0,0,0,0,0,0,0,0,0,4,4,0,0,0,0,0,5)$
C_{47}	$(0,0,0,0,0,0,0,0,4,0,0,4,0,0,0,0,5)$
C_{48}	$(0,0,0,0,0,0,0,4,0,0,0,0,4,0,0,0,5)$
C_{49}	$(0,0,0,0,0,0,0,0,1,1,2,1,2,2,0,0,0,1,4)$
C_{50}	$(0,0,0,0,0,0,0,0,1,2,1,2,1,2,0,0,1,0,4)$
C_{51}	$(0,0,0,0,0,0,0,0,2,1,1,2,2,1,0,1,0,0,4)$
C_{52}	$(0,0,0,0,0,0,0,0,2,2,2,1,1,1,1,0,0,0,4)$
C_{53}	$(0,0,0,0,0,0,0,0,1,2,2,2,2,2,0,0,0,0,4)$
C_{54}	$(0,0,0,0,0,0,0,0,2,1,2,2,2,2,0,0,0,0,4)$
C_{55}	$(0,0,0,0,0,0,0,0,2,2,1,2,2,2,0,0,0,0,4)$
C_{56}	$(0,0,0,0,0,0,0,0,2,2,2,1,2,2,0,0,0,0,4)$
C_{57}	$(0,0,0,0,0,0,0,0,2,2,2,2,1,2,0,0,0,0,4)$
C_{58}	$(0,0,0,0,0,0,0,0,2,2,2,2,2,1,0,0,0,0,4)$
C_{59}	$(0,0,0,0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,4)$

Results for $n = 4$, $D = (2,2,2,2)$

	$\tilde{N}(v)$
C_{60}	$(0,0,0,0,0,0,0,0,0,0,2,2,0,0,0,0,0,0,4)$
C_{61}	$(0,0,0,0,0,0,0,0,0,2,0,0,2,0,0,0,0,0,4)$
C_{62}	$(0,0,0,0,0,0,0,0,2,0,0,0,0,2,0,0,0,0,4)$
C_{63}	$(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,4)$
C_{64}	$(0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,4)$
C_{65}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,4)$
C_{66}	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,4)$
C_{67}	$(0,0,0,0,0,0,0,0,1,1,2,1,2,2,0,0,0,0,3)$
C_{68}	$(0,0,0,0,0,0,0,0,1,2,1,2,1,2,0,0,0,0,3)$
C_{69}	$(0,0,0,0,0,0,0,0,2,1,1,2,2,1,0,0,0,0,3)$
C_{70}	$(0,0,0,0,0,0,0,0,2,2,2,1,1,1,0,0,0,0,3)$
C_{71}	$(0,0,0,0,0,0,0,0,1,1,1,1,1,1,0,0,0,0,3)$

Results for $n = 4$, $D = (2, 2, 2, 2)$

	$\tilde{N}(v)$
C_{72}	$(0,0,0,0,0,0,0,0,0,0,1,2,0,0,0,0,0,0,3)$
C_{73}	$(0,0,0,0,0,0,0,0,0,0,2,1,0,0,0,0,0,0,3)$
C_{74}	$(0,0,0,0,0,0,0,0,0,1,0,0,2,0,0,0,0,0,3)$
C_{75}	$(0,0,0,0,0,0,0,0,0,2,0,0,1,0,0,0,0,0,3)$
C_{76}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,2,0,0,0,0,3)$
C_{77}	$(0,0,0,0,0,0,0,0,2,0,0,0,0,1,0,0,0,0,3)$
C_{78}	$(0,0,0,0,0,0,0,0,0,0,1,1,0,0,0,0,0,0,3)$
C_{79}	$(0,0,0,0,0,0,0,0,0,1,0,0,1,0,0,0,0,0,3)$
C_{80}	$(0,0,0,0,0,0,0,0,1,0,0,0,0,1,0,0,0,0,3)$
C_{81}	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,3)$

Results for $n = 4$, $D = (2, 2, 2, 2)$

$$[j_1, j_2, j_3, j_4] = v_{1,j_1} \otimes v_{2,j_2} \otimes v_{3,j_3} \otimes v_{4,j_4}$$

	v
C_0	0
C_1	$[1,1,1,1]$
C_2	$[1,1,1,1]-[2,2,1,1]$
	$[1,1,1,1]+[2,2,1,1]$
C_3	$[1,1,1,1]-[2,1,2,1]$
	$[1,1,1,1]+[2,1,2,1]$
C_4	$[1,1,1,1]-[2,1,1,2]$
	$[1,1,1,1]+[2,1,1,2]$
C_5	$[1,1,1,1]-[1,2,2,1]$
	$[1,1,1,1]+[1,2,2,1]$
C_6	$[1,1,1,1]-[1,2,1,2]$
	$[1,1,1,1]+[1,2,1,2]$
C_7	$[1,1,1,1]-[1,1,2,2]$
	$[1,1,1,1]+[1,1,2,2]$

Results for $n = 4$, $D = (2, 2, 2, 2)$

$$[j_1, j_2, j_3, j_4] = v_{1,j_1} \otimes v_{2,j_2} \otimes v_{3,j_3} \otimes v_{4,j_4}$$

	v
C_8	$[1,1,1,1] - [1,2,2,1] - [2,1,1,2] + [2,2,2,2]$
	$[1,1,1,1] - [1,2,2,1] + [2,1,1,2] - [2,2,2,2]$
	$[1,1,1,1] + [1,2,2,1] - [2,1,1,2] - [2,2,2,2]$
	$[1,1,1,1] + [1,2,2,1] + [2,1,1,2] + [2,2,2,2]$
C_9	$[1,1,1,1] - [1,2,1,2] - [2,1,2,1] + [2,2,2,2]$
	$[1,1,1,1] - [1,2,1,2] + [2,1,2,1] - [2,2,2,2]$
	$[1,1,1,1] + [1,2,1,2] - [2,1,2,1] - [2,2,2,2]$
	$[1,1,1,1] + [1,2,1,2] + [2,1,2,1] + [2,2,2,2]$
C_{10}	$[1,1,1,1] - [1,1,2,2] - [2,2,1,1] + [2,2,2,2]$
	$[1,1,1,1] - [1,1,2,2] + [2,2,1,1] - [2,2,2,2]$
	$[1,1,1,1] + [1,1,2,2] - [2,2,1,1] - [2,2,2,2]$
	$[1,1,1,1] + [1,1,2,2] + [2,2,1,1] + [2,2,2,2]$

Classical invariants for $n = 4$, $D = (2, 2, 2, 2)$

$$v = \sum_{j_1=1}^{2} \cdots \sum_{j_4=1}^{2} v_{j_1,\ldots,j_4} e_{1,j_1} \otimes \cdots \otimes e_{4,j_4}$$

From the Hilbert series:

$$\begin{aligned} h_1 = {} & v_{1,1,1,1}v_{2,2,2,2} - v_{1,1,1,2}v_{2,2,2,1} - v_{1,1,2,1}v_{2,2,1,2} + v_{1,1,2,2}v_{2,2,1,1} \\ & - v_{1,2,1,1}v_{2,1,2,2} + v_{1,2,1,2}v_{2,1,2,1} + v_{1,2,2,1}v_{2,1,1,2} - v_{1,2,2,2}v_{2,1,1,1} \end{aligned}$$

$$h_2 = \begin{vmatrix} v_{1,1,1,1} & v_{1,2,1,1} & v_{2,1,1,1} & v_{2,2,1,1} \\ v_{1,1,1,2} & v_{1,2,1,2} & v_{2,1,1,2} & v_{2,2,1,2} \\ v_{1,1,2,1} & v_{1,2,2,1} & v_{2,1,2,1} & v_{2,2,2,1} \\ v_{1,1,2,2} & v_{1,2,2,2} & v_{2,1,2,2} & v_{2,2,2,2} \end{vmatrix}$$

$$h_3 = \begin{vmatrix} v_{1,1,1,1} & v_{2,1,1,1} & v_{1,1,2,1} & v_{2,1,2,1} \\ v_{1,1,1,2} & v_{2,1,1,2} & v_{1,1,2,2} & v_{2,1,2,2} \\ v_{1,2,1,1} & v_{2,2,1,1} & v_{1,2,2,1} & v_{2,2,2,1} \\ v_{1,2,1,2} & v_{2,2,1,2} & v_{1,2,2,2} & v_{2,2,2,2} \end{vmatrix}$$

Classical invariants for $n = 4$, $D = (2, 2, 2, 2)$

$$h_4 = \begin{vmatrix} v_{1,1,1,1} & v_{1,1,1,2} & v_{2,1,1,1} & v_{2,1,1,2} \\ v_{1,1,2,1} & v_{1,1,2,2} & v_{2,1,2,1} & v_{2,1,2,2} \\ v_{1,2,1,1} & v_{1,2,1,2} & v_{2,2,1,1} & v_{2,2,1,2} \\ v_{1,2,2,1} & v_{1,2,2,2} & v_{2,2,2,1} & v_{2,2,2,2} \end{vmatrix},$$

$$h_5 = \det(h_{5,i,j})_{1\le i\le 3; 1\le j\le 3}$$

$$h_{5,1,1} = -v_{1,1,1,2}v_{1,1,2,1} + v_{1,1,1,1}v_{1,1,2,2}$$

$$h_{5,1,2} = v_{1,1,2,2}v_{1,2,1,1} - v_{1,1,2,1}v_{1,2,1,2} - v_{1,1,1,2}v_{1,2,2,1} + v_{1,1,1,1}v_{1,2,2,2}$$

$$h_{5,1,3} = -v_{1,2,1,2}v_{1,2,2,1} + v_{1,2,1,1}v_{1,2,2,2}$$

$$h_{5,2,1} = v_{1,1,2,2}v_{2,1,1,1} - v_{1,1,2,1}v_{2,1,1,2} - v_{1,1,1,2}v_{2,1,2,1} + v_{1,1,1,1}v_{2,1,2,2}$$

$$\begin{aligned} h_{5,2,2} = {} & v_{1,2,2,2}v_{2,1,1,1} - v_{1,2,2,1}v_{2,1,1,2} - v_{1,2,1,2}v_{2,1,2,1} + v_{1,2,1,1}v_{2,1,2,2} \\ & + v_{1,1,2,2}v_{2,2,1,1} - v_{1,1,2,1}v_{2,2,1,2} - v_{1,1,1,2}v_{2,2,2,1} + v_{1,1,1,1}v_{2,2,2,2} \end{aligned}$$

$$h_{5,2,3} = v_{1,2,2,2}v_{2,2,1,1} - v_{1,2,2,1}v_{2,2,1,2} - v_{1,2,1,2}v_{2,2,2,1} + v_{1,2,1,1}v_{2,2,2,2}$$

$$h_{5,3,1} = -v_{2,1,1,2}v_{2,1,2,1} + v_{2,1,1,1}v_{2,1,2,2}$$

Classical invariants for $n = 4$, $D = (2, 2, 2, 2)$

$$
\begin{aligned}
h_{5,3,2} &= v_{2,1,2,2}v_{2,2,1,1} - v_{2,1,2,1}v_{2,2,1,2} - v_{2,1,1,2}v_{2,2,2,1} + v_{2,1,1,1}v_{2,2,2,2} \\
h_{5,3,3} &= -v_{2,2,1,2}v_{2,2,2,1} + v_{2,2,1,1}v_{2,2,2,2} \\
h_6 &= \det(h_{6,i,j})_{1\leq i\leq 3;1\leq j\leq 3} \\
h_{6,1,1} &= -v_{1,1,1,2}v_{1,2,1,1} + v_{1,1,1,1}v_{1,2,1,2} \\
h_{6,1,2} &= -v_{1,1,2,2}v_{1,2,1,1} + v_{1,1,2,1}v_{1,2,1,2} - v_{1,1,1,2}v_{1,2,2,1} + v_{1,1,1,1}v_{1,2,2,2} \\
h_{6,1,3} &= -v_{1,1,2,2}v_{1,2,2,1} + v_{1,1,2,1}v_{1,2,2,2} \\
h_{6,2,1} &= v_{1,2,1,2}v_{2,1,1,1} - v_{1,2,1,1}v_{2,1,1,2} - v_{1,1,1,2}v_{2,2,1,1} + v_{1,1,1,1}v_{2,2,1,2} \\
h_{6,2,2} &= v_{1,2,2,2}v_{2,1,1,1} - v_{1,2,2,1}v_{2,1,1,2} + v_{1,2,1,2}v_{2,1,2,1} - v_{1,2,1,1}v_{2,1,2,2} \\
&\quad - v_{1,1,2,2}v_{2,2,1,1} + v_{1,1,2,1}v_{2,2,1,2} - v_{1,1,1,2}v_{2,2,2,1} + v_{1,1,1,1}v_{2,2,2,2} \\
h_{6,2,3} &= v_{1,2,2,2}v_{2,1,2,1} - v_{1,2,2,1}v_{2,1,2,2} - v_{1,1,2,2}v_{2,2,2,1} + v_{1,1,2,1}v_{2,2,2,2} \\
h_{6,3,1} &= -v_{2,1,1,2}v_{2,2,1,1} + v_{2,1,1,1}v_{2,2,1,2} \\
h_{6,3,2} &= -v_{2,1,2,2}v_{2,2,1,1} + v_{2,1,2,1}v_{2,2,1,2} - v_{2,1,1,2}v_{2,2,2,1} + v_{2,1,1,1}v_{2,2,2,2} \\
h_{6,3,3} &= -v_{2,1,2,2}v_{2,2,2,1} + v_{2,1,2,1}v_{2,2,2,2}
\end{aligned}
$$

Classical invariants for $n = 4$, $D = (2,2,2,2)$

$$h_7 = \det(h_{7,i,j})_{1\le i\le 3;1\le j\le 3}$$

$$h_{7,1,1} = -v_{1,1,2,1}v_{1,2,1,1} + v_{1,1,1,1}v_{1,2,2,1}$$

$$h_{7,1,2} = -v_{1,1,2,2}v_{1,2,1,1} - v_{1,1,2,1}v_{1,2,1,2} + v_{1,1,1,2}v_{1,2,2,1} + v_{1,1,1,1}v_{1,2,2,2}$$

$$h_{7,1,3} = -v_{1,1,2,2}v_{1,2,1,2} + v_{1,1,1,2}v_{1,2,2,2}$$

$$h_{7,2,1} = v_{1,2,2,1}v_{2,1,1,1} - v_{1,2,1,1}v_{2,1,2,1} - v_{1,1,2,1}v_{2,2,1,1} + v_{1,1,1,1}v_{2,2,2,1}$$

$$h_{7,2,2} = v_{1,2,2,2}v_{2,1,1,1} + v_{1,2,2,1}v_{2,1,1,2} - v_{1,2,1,2}v_{2,1,2,1} - v_{1,2,1,1}v_{2,1,2,2} - v_{1,1,2,2}v_{2,2,1,1} - v_{1,1,2,1}v_{2,2,1,2} + v_{1,1,1,2}v_{2,2,2,1} + v_{1,1,1,1}v_{2,2,2,2},$$

$$h_{7,2,3} = v_{1,2,2,2}v_{2,1,1,2} - v_{1,2,1,2}v_{2,1,2,2} - v_{1,1,2,2}v_{2,2,1,2} + v_{1,1,1,2}v_{2,2,2,2}$$

$$h_{7,3,1} = -v_{2,1,2,1}v_{2,2,1,1} + v_{2,1,1,1}v_{2,2,2,1}$$

$$h_{7,3,2} = -v_{2,1,2,2}v_{2,2,1,1} - v_{2,1,2,1}v_{2,2,1,2} + v_{2,1,1,2}v_{2,2,2,1} + v_{2,1,1,1}v_{2,2,2,2}$$

$$h_{7,3,3} = -v_{2,1,2,2}v_{2,2,1,2} + v_{2,1,1,2}v_{2,2,2,2}$$

Classical invariants for $n = 4$, $D = (2,2,2,2)$

Relations:

$$\begin{aligned} h_2 + h_3 + h_4 &= 0 \\ h_1 h_2 - h_6 + h_7 &= 0 \\ h_1 h_3 - h_7 + h_5 &= 0 \\ h_1 h_4 - h_5 + h_6 &= 0 \end{aligned}$$

Classical invariants for new classes for $n = 4$, $D = (2,2,2,2)$

h_1	h_2	h_3	h_4	h_5	h_6	h_7	
0	0	0	0	0	0	0	$C_0, C_1, \{C_2, \ldots, C_7\}, \{C_{11}, \ldots, C_{14}\}, \{C_{15}, \ldots, C_{18}\}, C_{19}, \{C_{20}, \ldots, C_{25}\}, \{C_{34}, \ldots, C_{37}\}, \{C_{50}, \ldots, C_{53}\}$
$\neq 0$	0	0	0	0	0	0	$C_{26}, \{C_{38}, C_{39}, C_{40}\}, \{C_{54}, \ldots, C_{59}\}, C_{60}, \{C_{68}, \ldots, C_{71}\}$
0	$\neq 0$	$\neq 0$	0	0	0	0	$C_{47}, C_{61}, C_{73}, C_{74}$
0	$\neq 0$	0	$\neq 0$	0	0	0	$C_{48}, C_{62}, C_{75}, C_{76}$
0	0	$\neq 0$	$\neq 0$	0	0	0	$C_{49}, C_{63}, C_{77}, C_{78}$
0	0	0	0	$\neq 0$	$\neq 0$	$\neq 0$	C_{72}
0	$\neq 0$	$\neq 0$	$\neq 0$	0	0	0	C_{82}
$\neq 0$	$\neq 0$	$\neq 0$	0	0	0	$\neq 0$	$C_8, C_{27}, C_{41}, C_{42} C'_{47}, C'_{61}$
$\neq 0$	$\neq 0$	0	0	$\neq 0$	$\neq 0$	0	$C_9, C_{28}, C_{43}, C_{44}, C'_{48}, C'_{62}$
$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	0	0	$C_{10}, C_{29}, C_{45}, C_{46} C'_{49}, C'_{63}$
0	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	C_{64}, C_{79}
0	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{65}, C_{80}
0	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{66}, C_{81}

Classical invariants for new classes for $n = 4$, $D = (2,2,2,2)$

h_1	h_2	h_3	h_4	h_5	h_6	h_7	
$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	0	C'_{64}
$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	0	$\neq 0$	C'_{65}
$\neq 0$	0	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	C'_{66}
$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	C_{30}, C''_{64}
$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{31}, C''_{65}
$\neq 0$	0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{32}, C''_{66}
0	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{67}
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	C'_{67}
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	C''_{67}
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	0	$\neq 0$	$\neq 0$	C'''_{67}
$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	$\neq 0$	C_{33}, C''''_{67}

Summary

Old method

- conceptual simplicity, computational difficulties
- uses the classical theory of geometric invariants
- all known results are obtained in this way

New method

- conceptual and computational simplicity
- one-to-one correspondence between entanglement classes and classes of linear maps
- a classification via analysis of algebraic properties of maps
- all found results agree with known results
- a large number of new results
- insights into geometry and topology of spaces of entangled states

Outlook

Present and future work:

- full classification for $n = 3$; $D = \{d_1, d_2, d_3\}$
- full classification for $n = 4, 5, \ldots$; $D = \{2, \ldots, 2\}$

Applications:

- continuous measures of entanglement (entanglement entropy)
- discrete measures of entanglement (topological charges)
- black hole entropy, holography, AdS/CFT
- extension to superspaces