Adaptive Numerical Simulation
Of Idealized Cyclones

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Outline

1 Motivation
2 Adaptation Technique
3 Adaptation Criteria
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Motivation

- devastating geophysical hazard: e.g. Katrina (2005) killed 1836 people, cost US$ 1.6 billion

- accurate and early numerical prediction crucial

- numerical prediction $\rightarrow$ physical model $\rightarrow$ discretization

- challenge: multiscale phenomenon
Adaptation Technique

- locally adapt spatial discretization

- FE discretization
  \[ \rightarrow \text{h-, p- and r-adaptivity} \]

- adapted mesh causes artificial asymmetry

- influence on storm’s track and its intensity?
**Numerical Tests**

- storm track $\rightarrow$ weighted barycenter of vorticity ($\nabla \times \vec{v}$)
- intensity $\rightarrow$ kinetic energy of vortex
- experiment carried out with two different models: CG Navier-Stokes, DG Shallow-Water model
### Models

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<td><strong>Heuveline, EMCL, KIT, <a href="http://www.hiflow3.org">www.hiflow3.org</a></strong></td>
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Numerical Tests – Results

Kinetic Energy of Vortex

- no further effect of frontier in resolution on energy budget
Numerical Tests – Results

Storm track

![Vorticity](image1)

![Vorticity](image2)
Numerical Tests – Results

- storm tracks are affected
- deflection is very small
Adaptation Criteria

a posteriori error estimators
- local contributions to total error
  \[ E \approx \| u - u_h \| \text{ in global norm} \]

goal-oriented approach
- local contributions to total a post. error \( E \) in goal functional \( J(u) \)
  \[ E \approx |J(u) - J(u_h)| \]

physical criteria
- physical features of contemplated problem

\[
\text{efficiency} = \frac{\text{error}}{\text{costs}}
\]
Benchmark-Problem

Cyclone-Cyclone-Interaction

- analysis of tropical-cyclone-scale vortices in the western North Pacific region, Lander et al. (1993)
- Fujiwhara-effect (1921)
- interaction highly sensitive to initial distance of vortices, Ritchie et al. (1993)
Benchmark-Problem

- numerical simulation sensitive to
  - initial distance
  - viscosity
  - mesh width

existence of a *binary system* will make the predictability of the storm tracks difficult, Brand et al. (1970)
Model Scenario

- Navier-Stokes equations
- quads
- Taylor-Hood-elements (Braess, 2007)
- cGP(1) (Schieweck, 2010)

Adaptivity

- h-adaptivity
- fixed mesh-fraction strategy (Becker & Rannacher, 2003)

adaptation indicator $\psi_K$

a posteriori? goal-oriented? physical?
A Posteriori Error Estimators

\[ \| u - u_h \| \leq c \sum_{K \in T_h} \eta_K(u_h) \]

- residual error estimator (Navier-Stokes equation and Poisson equation)
- adaptation indicator \( \psi_K = \text{error indicator } \eta_K(u_h) \)
Goal-Oriented Adaptivity

- interest in a goal functional $J(u)$

$$|J(u) - J(u_h)| \leq c \sum_{K \in T_h} \eta_K$$

Dual Weighted Residual Error Estimator (Becker & Rannacher, 2003)

- primal problem: seek $u \in V$ with

$$A(u, \varphi) = F(\varphi) \quad \forall \varphi \in V$$

- adjoint dual problem: seek $z \in V$ with

$$\left. \frac{dA(u, z)}{du} \right|_{\varphi} = \left. \frac{dJ(u)}{du} \right|_{\varphi} \quad \forall \varphi \in V$$
Goal-Oriented Adaptness

Dual Weighted Residual Error Estimator (Becker & Rannacher, 2003)

\[ \eta_K = \rho_K \omega_K \]

\[ \rho_K := \| R_K(u_h) \|_K + h_K^{-1/2} \| R_{\Gamma_K}(u_h) \|_{\Gamma_K} \]

\[ \omega_K := \| z - I_h z \|_K + h_K^{1/2} \| z - I_h z \|_{\Gamma_K}, \]

\[ \forall K \in T_h; \ u_h \in V_h; \ z \in V \]
Goal-Oriented Adaptivity

- $z$ is unknown $\rightarrow$ use patch-wise interpolation $I_h^+$ of $z_h$ into FE space of higher-order instead: $Z = I_h^+ z_h \approx z$

\[ Z = I_h^+ z_h \]

$\rightarrow \eta_K = \eta_K(u_h, z_h)$
Goal-Oriented Adaptivity

- $J$ is correlated with **storm position** after 96 h
- goal functional
  \[
  J(u) = \int_B (\nabla \times u(x, 96\text{ h})) \, dx
  \]
  \[
  B = \{x \in \Omega : \|x - x_{\text{ref}}\|_{L^2} \leq r_{\text{ref}}\}
  \]
- adaptation indicator $\psi_K = \text{error indicator } \eta_K(u_h, z_h)$
Physical Criteria

- **idea:** relate mesh resolution with solution, e.g. low pressure $\rightarrow$ high resolution

- adaptation indicator $\psi_K \neq$ physical quantity $\Phi$
  
  $$\psi_K = h_K^\beta \Phi(K), \quad \forall K \in T_h$$

- $\Phi = (\nabla \times \vec{v})^2$, $\beta = 1$ $\beta = 4$ $\beta = 16$

- **vorticity-indicator:** $\Phi = (\nabla \times \vec{v})^2$, $\beta = 16$

- **energy-indicator:** $\Phi = u^2 + v^2$, $\beta = 2$
Numerical Results

![Graph showing error in storm position vs. DOFs for different indicators: A posteriori Poisson, A posteriori Navier-Stokes, Vorticity, Energy, Dual-Weighted Residual, Uniform.](image)

**Indicator**
- A posteriori Poisson
- A posteriori Navier-Stokes
- Vorticity
- Energy
- Dual-Weighted Residual
- Uniform

**Axes:**
- **Y-axis:** Error in storm position [km]
- **X-axis:** DOFs
Costs

Residual Error And Physical Indicators

- evaluation of $\psi_K(u_h)$ at arbitrary timesteps
- one floating-point number per cell additional storage required
Costs

Dual Weighted Residual Error Indicator

\[ \psi_K = \max(\psi_{K,n}) \]

\[ \psi_{K,n} \]

- \( \psi_K(u_h, z_h) \)
- Solution of (linear) dual problem backward in time
- Complete primal solution has to be stored
Efficiency

$$\text{efficiency} = \frac{\text{error}}{\text{costs}}$$

Error

- all indicators lead to an efficient mesh
- best results for goal-oriented approach
- good choice of $\beta$ for physical indicators is crucial

Costs

- additional effort to find optimal $\beta$
- higher computational costs and additional storage required for goal-oriented approach
Conclusion

h-adaptivity ✔
- no influence on intensity
- minor influence on storm track

r-adaptivity
→ Poster

Goal-oriented adaptation of time-discretization
→ Poster
Conclusion

Adaptation Criteria

- for this scenario all indicators lead to a more efficient mesh
- difference in computational costs and storage requirements

Outlook

- generic choice of $\beta$ possible?
- more realistic scenario (3D)

source: Diploma-Thesis Andrea Richter, IMK, KIT
Publications


Baumann, Heuveline: Evaluation of different strategies for goal oriented adaptivity in CFD - Part I: The stationary case, Preprint Series EMCL, No. 2010-06, 2010; submitted in GEM.


**In preparation**

Scheck, Jones, Heuveline: Shear-enhanced Barotropic Perturbation Growth in Hurricane-like vortices.

Baumann, Heuveline, Scheck, Jones: Goal-Oriented Adaptivity for Idealised Tropical Cyclones.
References


2D Navier Stokes equations incompressible, Newtonian fluid

\[
\begin{align*}
\frac{\partial \vec{u}}{\partial t} - \nu \Delta \vec{u} + (\vec{u} \cdot \nabla) \vec{u} + \frac{1}{\rho} \nabla p &= 0 & \text{in } \Omega \times (0, T] \\
\nabla \cdot \vec{u} &= 0 & \text{in } \Omega \times (0, T]
\end{align*}
\]

Discretization in Space

- continuous Galerkin method
- triangulation of \( \Omega \rightarrow \) quads
- Taylor-Hood elements \( \rightarrow \) \( Q_2 - Q_1 \)

Discretization in Time

- implicit, continuous Galerkin-Petrov-Method \( cGP(1) \)
StormFlash (Behrens, NumGeo, KlimaCampus Hamburg)

Shallow-Water equations $\approx$ barotropic for $h \to \infty$

\[
\begin{align*}
\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} gh^2 \\ hv \end{pmatrix} + \partial_y \begin{pmatrix} hv \\ huv \\ hv^2 + \frac{1}{2} gh^2 \end{pmatrix} &= \begin{pmatrix} 0 \\ -gh \partial_x b \\ -gh \partial_y b \end{pmatrix} \\
\text{in } \Omega \times (0, T]
\end{align*}
\]

Discretization in Space

- discontinuous Galerkin method
- triangulation of $\Omega \rightarrow$ triangles
- linear elements $\rightarrow P_1$

Discretization in Time

- explicit, two-step Runge-Kutta method of order 2
Numerical Tests – Results

- storm tracks are affected
- deflection is very small
Overview Adaptation Indicators

A Posteriori Error Estimators

$$\|u - u_h\| \leq c \sum_{K \in T_h} \eta_K(u_h)$$

- residual error estimator for abstract example:
  seek $u \in V$, $A(u, \varphi) = F(\varphi)$, $\forall \varphi \in V$

- area-specific residual $R_K = A(u_h, \cdot) - F(\cdot)$
- edge-specific jump terms $R_e = [\partial_n u_h]_e$
Overview Adaptation Indicators

A Posteriori Error Estimators

Residual in Poisson-equation

\[ \eta_K := [h_K^2 \| (f + \Delta u_h)|_K \|_{L^2(\Omega)} + \frac{1}{2} \sum_{e \in \partial K} h_e \| [\partial_n u_h]|_e \|_{L^2(\Omega)}^2]^{\frac{1}{2}} \]

Residual in Navier-Stokes-equation

\[ \eta_K := [h_K^2 \|(f + \nu \Delta \bar{u}_h - (\bar{u}_h \cdot \nabla) \bar{u}_h - \frac{1}{\rho} \nabla p_h)|_K \|_{L^2(\Omega)} + \frac{1}{2} \sum_{e \in \partial K} h_e \| [\partial_n u_h]|_e \|_{L^2(\Omega)}^2]^{\frac{1}{2}} \]
Overview Adaptation Indicators

Goal-Oriented Adaptivity

**Dual-Weighted Residual Error Indicator**

\[ \eta_K = \rho_K(u_h)\omega_K(z_h) \]
\[ \rho_K := \|R_K(u_h)\|_K + h_K^{-1/2}\|R_{\Gamma_K}(u_h)\|_{\Gamma_K} \]
\[ \omega_K := \|l_h^+z_h - z_h\|_K + h_K^{1/2}\|l_h^+z_h - z_h\|_{\Gamma_K} \]

**goal-functional**: error in storm position after 96 h
Overview Adaptation Indicators

Physical Criteria

**Energy-Indicator**

$$\eta_K = \int_K (u^2 + v^2) \, dK$$

**Vorticity-Indicator**

$$\eta_K = h_K^{14} \int_K ((\nabla \times \vec{v})^2) \, dK$$