A New Approach to Implement Sigma Coordinate in a Numerical Model

Yiyuan Li¹, Bin Wang¹, Donghai Wang²

¹. Institute of Atmospheric Physics, Chinese Academy of Sciences
². Chinese Academy of Meteorological Sciences

Isaac Newton Institute for Mathematical Sciences
2012.08.24
Outline

1. Motivation
2. A New Approach
3. The Idealised Experiments
4. Conclusion
1. Motivation – the Importance of Vertical Coordinate

- The numerical models become one of the most important tool for forecast and research
  - weather, climate, ocean wave, ...

- The vertical coordinate is a basic element of a numerical model
  - To tackle the terrain

- The choice of vertical coordinate can affect the performance of a model. (Steppeler et al., 2003; Ji et al., 2005; Staniforth and Wood, 2008)
1. Motivation – the Importance of Terrain

◆ Multiscale terrain affects the atmosphere
  – Motion, cloud, and precipitation

(Mountain waves and downslope winds, 2004)
1. Motivation – the Necessity of Terrain in a Model

- We need the terrain in all scales in a numerical model

Blue arrow line: wind; white: cloud; green: terrain

http://www.meted.ucar.edu/mesoprim/models/print.htm
1. Motivation – Terrain in a Model

- Dynamical core ← vertical coordinate for the simple boundary condition

- Terrain-following coordinate
  (σ-coordinate)
  - Adv: continuous, no internal boundary
  - Disadv: non-horizontal coordinate surface, non-orthogonal

- Step-mountain coordinate
  (η-coordinate)
  - Adv: quasi-horizontal coordinate surface
  - Disadv: internal boundary, discontinuous

(Mountain waves and downslope winds, 2004)

1. Motivation – the PGF Errors in the \( \sigma \)-coordinate

- The pressure gradient force errors (PGF errors) are increased by one order of magnitude.
- The PGF errors are significant in high resolution models.

<table>
<thead>
<tr>
<th>Relative errors:</th>
<th>40%</th>
<th>1%</th>
<th>1%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Computational value:</td>
<td>0.7 = 10.1 – 9.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Analytical value:</td>
<td>0.5 = 10 – 9.5</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
- \frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_z = - \left[ \frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_\sigma + \frac{1}{\rho} \frac{\partial p}{\partial \sigma} \left( \frac{\partial \sigma}{\partial x} \right)_z \right]
\]

- Same order of magnitude
- Opposite in sign

\(\text{(Smagorinsky et al., 1967; Sundquist, 1975, 1976; Janjić, 1977; Mesinger, 1982; Haney, 1991; Steppeler et al., 2003; Yu and Xu, 2004; Ji et al., 2005; Hu and Wang, 2007)}\)
1. Motivation – The Existing Methods to Overcome the PGF Errors

To alleviate the PGF errors to an acceptable level, upon the two-term computational form (Zeng, 1963; Corby et al., 1972; Gary, 1973; Qian and Zhong, 1986; Blumberg and Mellor, 1987; Yu, 1989; Sikirić et al., 2009; Berntsen, 2011)

- Adjusting parameters before calculating the PGF, including the pressure, temperature, geopotential height, and terrain slope, eg: subtraction of a standard atmosphere, Linear Programming.

- Computational schemes for calculating the PGF, eg: Corby scheme, Jacobian method.

\[
-\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_z = - \left[ \frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_\sigma + \frac{1}{\rho} \frac{\partial p}{\partial \sigma} \left( \frac{\partial \sigma}{\partial x} \right)_z \right]
\]

Keep the computational form of PGF being one in the \( \sigma \)–coordinate?

✔ Design a new approach to implement the \( \sigma \)-coordinate.
Outline

1. Motivation
2. A New Approach
3. The Idealised Experiments
4. Conclusion
2. A new method – Design

Categorising the existing methods to obtain the scalar equations in a $\sigma$–coordinate

$\downarrow$

Analysing the different reasons of inducing the PGF errors

Based on the scalar gradient formula in a curvilinear coordinate

$\downarrow$

Using the covariant scalar equations of the $\sigma$–coordinate
2. A new method – Analysis of the PGF Errors

Two types of the method to obtain the scalar equations of $\sigma$-coordinate

- Scalar to scalar (method one): chain rules, the relationship of spatial partial derivatives from $z$-coordinate to $\sigma$-coordinate
- Vector to scalar (method two): the scalar gradient in a curvilinear coordinate

Scalar equations of $z$-coordinate  Vector equations

Scalar equations of the $\sigma$-coordinate

Chain rules  Basis vectors
Both methods obtain the **two-term** computational form of PGF due to different reasons.

**Method one**

\[
\left( \frac{\partial \phi}{\partial x} \right)_z = \left( \frac{\partial \phi}{\partial x} \right)_\sigma + \frac{\partial \phi}{\partial \sigma} \cdot \left( \frac{\partial \sigma}{\partial x} \right)_z
\]

**Method two**

\[
\nabla \phi = g_1^j \frac{\partial \phi}{\partial q^j} e_1 + g_2^j \frac{\partial \phi}{\partial q^j} e_2 + g_3^j \frac{\partial \phi}{\partial q^j} e_3
\]

\[
g^j = \begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

\[
-\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_z = -\frac{1}{\rho} \left( \frac{\partial p}{\partial x} \right)_\sigma - \frac{1}{\rho} \frac{\partial p}{\partial \sigma} \cdot \left( \frac{\partial \sigma}{\partial x} \right)_z
\]

\(j\) represents the summation form 1 to 3, \(q^j\) are the coordinates of \(\sigma\)–coordinate.
How to keep the **one-term** computational form of PGF as those in z-coordinate?

- Using “**Method two**” to obtain the scalar equations of $\sigma$-coordinate
- Using the “**contravariant basis vectors $e^j$**” to expand the vector equations of atmosphere

$$
\nabla \phi = \frac{\partial \phi}{\partial q^j} e^j = g^{ij} \frac{\partial \phi}{\partial q^j} e_i
$$

$$
\frac{\partial \phi}{\partial q^1} e^1 + \frac{\partial \phi}{\partial q^2} e^2 + \frac{\partial \phi}{\partial q^3} e^3 = g^{1j} \frac{\partial \phi}{\partial q^j} e_1 + g^{2j} \frac{\partial \phi}{\partial q^j} e_2 + g^{3j} \frac{\partial \phi}{\partial q^j} e_3
$$

e^j and $e_i$ represent the contravariant and covariant basis vectors of a curvilinear coordinate
2. A new method – Comparison with the Classic Method

- Using the contravariant basis vectors $e^j$ of $\sigma$-coordinate to expand the vector equation of the atmosphere
  - The computational form of PGF are one-term
  - Similar with the scalar equations of $z$- and classic $\sigma$-coordinate

\[
\frac{\partial v_1}{\partial t} + u^1 \frac{\partial v_1}{\partial x} + v^2 \frac{\partial v_1}{\partial y} + v^3 \frac{\partial v_1}{\partial \sigma} - \frac{1}{2} v^m v^n \frac{\partial g_{nm}}{\partial x} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - 2\sqrt{g}\left(\Omega^2 v^3 - \Omega^3 v^2\right) + g_1
\]

\[
\frac{\partial v_2}{\partial t} + u^1 \frac{\partial v_2}{\partial x} + v^2 \frac{\partial v_2}{\partial y} + v^3 \frac{\partial v_2}{\partial \sigma} - \frac{1}{2} v^m v^n \frac{\partial g_{nm}}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} + 2\sqrt{g}\left(\Omega^1 v^3 - \Omega^3 v^1\right) + g_2
\]

The new method

The classic method

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial \sigma} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial p}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial x} - 2(\Omega_2 \omega - \Omega_3 v)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial \sigma} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{1}{\rho} \frac{\partial p}{\partial \sigma} \cdot \frac{\partial \sigma}{\partial y} + 2(\Omega_1 \omega - \Omega_3 u)
\]
2. A new method – A Unified Framework

◆ The unified framework of \( z \)-coordinate and the two methods of using the \( \sigma \)-coordinate

- Difference in form ← setting corresponding parameters
- Difference in value ← defining the value of scalars

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial q_1} + v \frac{\partial u}{\partial q_2} + w \frac{\partial u}{\partial q_3} + \lambda_{11} = -\frac{1}{\rho} \delta_{1i} \frac{\partial p}{\partial q_i} - 2 \lambda_2 \left( \Omega_2 w - \Omega_3 v \right) + g_1
\]

<table>
<thead>
<tr>
<th>( \lambda_{11} )</th>
<th>( \delta_{1i} )</th>
<th>( \lambda_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(1, 0, 0)</td>
<td>1</td>
</tr>
<tr>
<td>The classic sigma method</td>
<td>( \frac{H - h}{H - h} )</td>
<td>(H-h)/H</td>
</tr>
<tr>
<td>The new sigma method</td>
<td>( \frac{H - h}{H - h} )</td>
<td>(H-h)/H</td>
</tr>
</tbody>
</table>

\( \alpha, \beta, \gamma \) = \( \begin{pmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \)^{-1}

\( a, b, \) and \( c \) represent the scalars in the \( z \)-coordinate.

H is the top of a model, and \( h \) is the terrain height.
It’s **not** the $\sigma$-coordinate but the way it is implemented that causes the well-known PGF errors.

Using the **covariant scalar equations** of the $\sigma$-coordinate, there is only one term in the computational form of PGF, which **bypass the PGF errors completely**.

Via setting corresponding parameters and defining the value of scalar, a **unified framework** is proposed for implementing the new method in a numerical model.

Further investigations are advised as the curvilinear terms and gravity terms are in the momentum equations of the new method.
Outline

1. Motivation
2. A New Approach
3. The Idealised Experiments
4. Conclusion
3. The Idealised Experiments – Design

◆ Basic parameters
- Two-dimension
- Bell-shaped mountain
- Idealised pressure fields
- Leapfrog scheme in the horizontal discretization and forward scheme in the vertical

<table>
<thead>
<tr>
<th>The classic sigma method (Cs)</th>
<th>The new sigma method (Ns)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Analytical PGF (PGFa)

\[ p = p_0 \cdot e^{-\frac{z-h}{H-h}} \]

Relative errors (RE) = \( \frac{\text{Absolute errors (AE)}}{\text{PGF}} = \frac{\text{PGF-PGFa}}{\text{PGFa}} \)
3. The Idealised Experiments – Design

◆ A basic comparison of PGF errors in both methods

- $dx = 5.0 \, \text{km}$, $dz = 3.7 \, \text{km}$
- None horizontal PGF (NonePGF), $p = p(z)$
- With horizontal PGF (WithPGF), $p = p(x, z)$

◆ The sensitivity experiments

- Increasing the terrain slope (ITS) (30° ~ 60°, 121 in total)
- Increasing the horizontal resolution (IHR) (6.25 m ~ 5000 m, 800 in total)
3. The Idealised Experiments – Results of NonePGF

The absolute errors of PGF in the new method are reduced by orders of magnitude than those in the Classic method.

**The Absolute Errors of PGF in Cs**
Max = 0.6 × 10^{-2}

**The Absolute Errors of PGF in Ns**
Max = 0.6 × 10^{-8}

cold color: the negative value, warm color: the positive value;
the intervals of contours: left 0.0008, right 2 × 10^{-9}
The pattern of PGF in the new method are more consistent with the PGFa than those in classic method.

<table>
<thead>
<tr>
<th></th>
<th>Classic sigma</th>
<th>New sigma</th>
</tr>
</thead>
<tbody>
<tr>
<td>The analytical PGF (PGFa)</td>
<td>Max = 0.0028</td>
<td>Max = 0.0375</td>
</tr>
<tr>
<td>Max = 0.0045</td>
<td>Max = 0.0288</td>
<td></td>
</tr>
</tbody>
</table>

cold color: the negative value, warm color: the positive value;
the intervals of contours: left 0.0004, right 0.004
3. The Idealised Experiments – Results of WithPGF

- The relative errors of PGF in the new method are one order of magnitude less than those in the classic method.

The Relative Errors of PGF in Cs

Max = 5.77  RMSE=3.3

The Relative Errors of PGF in Ns

Max = 0.42  RMSE=0.17

cold color: the negative value, warm color: the positive value; the intervals of contours: 0.001
3. The Idealised Experiments – Results of WithPGF

◆ The pattern of absolute errors of PGF in the new method can be more easily predicted
  – Keeps the same pattern as those in the classic method
  – Coinsides with the pattern of slopes of $\sigma$ levels

Warm lines: lower levels; cold lines: higher levels
The relative errors of PGF in the new method are outperformance in all the 121 experiments
  – One orders of magnitude less than those in the classic method
  – Stays almost constant despite the increase in terrain slope

3. The Idealised Experiments – Results of ITS

The three dot-line are drawn by all the 121 experiments.
3. The Idealised Experiments – Results of IHR

- The PGF in the new method are outperformance in all the 800 experiments
  - Relative errors of PGF are at least one order of magnitude less than those in classic method
  - Both the absolute and relative errors of PGF decrease more significantly than those in classic method according to the increasing horizontal resolution

The three dot-line are drawn by all the 800 experiments.
Outline

1. Motivation
2. A New Approach
3. The Idealised Experiments
4. Conclusion
Proposing a new approach to implement the $\sigma$-coordinate

- Covariant scalar equation
- Bypass the PGF errors

4. Conclusion

Analysis of PGF errors

Using the covariant scalar equations of the $\sigma$-coordinate

Obtaining the one-term computational form of PGF in the $\sigma$-coordinate
All experiments:
  - Relative errors of PGF in the new method have been reduced consistently by at least one order of magnitude than those in the classic method

Sensitivity experiments of terrain slope and horizontal resolution:
  - The PGF errors in the new method decrease more significantly than those in the classic one according to both the increased terrain slope and horizontal resolution

Thank you!

Mountains