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The leapfrog is dead. Long live the leapfrog!

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Time-stepping methods

weather and climate models

computational fluid dynamics

increasing accuracy

Method	Order	Formula	Storage factor	Efficiency factor	Amplitude error	Phase error	Maximum $\omega\Delta t$
Forward (Adams–Bashforth)	1	$\phi^{n+1} = \phi^n + hF(\phi^n)$	2	0	$1 + \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	0
Backward (Adams–Moulton)	1	$\phi^{n+1} = \phi^n + hF(\phi^{n+1})$	Implicit	∞	$1 - \frac{p^2}{2}$	$1 - \frac{p^2}{3}$	∞
Matsuno	1	$\phi^* = \phi^n + hF(\phi^n)$ $\phi^{n+1} = \phi^n + hF(\phi^*)$	2	.5	$1 - \frac{p^2}{2}$	$1 + \frac{2}{3}p^2$	1
Asselin-filtered leapfrog	1	$\phi^{n+1} = \overline{\phi^{n-1}} + 2hF(\phi^n)$ $\overline{\phi^n} = \phi^n + \gamma(\overline{\phi^{n-1}} - 2\phi^n + \phi^{n+1})$	3	<1	$1 - \frac{\gamma}{2(1-\gamma)}p^2$	$1 + \frac{1+2\gamma}{6(1-\gamma)}p^2$	<1
Leapfrog	2	$\phi^{n+1} = \phi^{n-1} + 2hF(\phi^n)$	2	1	1	$1 + \frac{p^2}{6}$	1
Runge–Kutta (Williamson/Huen)	2	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1$ $q_2 = hF(\phi_1) - q_1, \quad \phi^{n+1} = \phi_1 + q_2/2$	2	0	$1 + \frac{p^4}{8}$	$1 + \frac{p^2}{6}$	0
Adams–Bashforth	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	0	$1 + \frac{p^4}{4}$	$1 + \frac{5}{12}p^2$	0
Adams–Moulton (Trapizoidal)	2	$\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^{n+1}) + F(\phi^n)]$	Implicit	∞	1	$1 - \frac{p^2}{12}$	∞
Leapfrog, then Adams–Bashforth (Magazenkov)	2	$\phi^n = \phi^{n-2} + 2hF(\phi^{n-1})$ $\phi^{n+1} = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$	3	.67	$1 - \frac{p^4}{4}$	$1 + \frac{p^2}{6}$	.67
Leapfrog-Trapizoidal (Kurihara)	2	$\phi^* = \phi^{n-1} + 2hF(\phi^n)$ $\phi^{n+1} = \phi^n + \frac{h}{2}[F(\phi^n) + F(\phi^*)]$	3	.71	$1 - \frac{p^4}{4}$	$1 - \frac{p^2}{12}$	1.41
Young's method A	2	$\phi_1 = \phi^n + hF(\phi^n)/2$ $\phi_2 = \phi_1 + hF(\phi_1)/2$ $\phi^{n+1} = \phi^n + hF(\phi_2)$	3	0	$1 + \frac{p^6}{128}$	$1 + \frac{p^2}{24}$	0
Runge–Kutta (Williamson)	3	$q_1 = hF(\phi^n), \quad \phi_1 = \phi^n + q_1/3,$ $q_2 = hF(\phi_1) - 5q_1/9, \quad \phi_2 = \phi^1 + 15q_2/16$ $q_3 = hF(\phi_2) - 153q_2/128, \quad \phi^{n+1} = \phi_2 + 8q_3/15$	2	.58	$1 - \frac{p^4}{24}$	$1 + \frac{p^4}{30}$	1.73
ABM predictor-corrector	3	$\phi^* = \phi^n + \frac{h}{2}[3F(\phi^n) - F(\phi^{n-1})]$ $\phi^{n+1} = \phi^* + \frac{5h}{12}[F(\phi^*) - 2F(\phi^n) + F(\phi^{n-1})]$	4	.60	$1 - \frac{19}{144}p^4$	$1 + \frac{1243}{8640}p^4$	1.20
Adams–Moulton	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[5F(\phi^{n+1}) + 8F(\phi^n) - F(\phi^{n-1})]$	Implicit	0	$1 + \frac{p^4}{24}$	$1 - \frac{11}{720}p^4$	0
Adams–Bashforth	3	$\phi^{n+1} = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n+1}) + 5F(\phi^{n-2})]$	4	.72	$1 - \frac{3}{8}p^4$	$1 + \frac{289}{720}p^4$	0.72
Runge–Kutta (Classical)	4	$k_0 = hF(\phi^n)$ $k_1 = hF(\phi^n + k_0/2)$ $k_2 = hF(\phi^n + k_1/2)$ $k_3 = hF(\phi^n + k_2)$ $\phi^{n+1} = \phi^n + \frac{1}{6}(k_0 + 2k_1 + 2k_2 + k_3)$	3*	.70	$1 - \frac{p^6}{144}$	$1 - \frac{p^4}{120}$	2.82
ABM predictor-corrector	4	$\phi^* = \phi^n + \frac{h}{12}[23F(\phi^n) - 16F(\phi^{n-1}) + 5F(\phi^{n-2})]$ $\phi^{n+1} = \phi^* + \frac{3h}{8}[F(\phi^*) - 3F(\phi^n) + 3F(\phi^{n-1}) - F(\phi^{n-2})]$	5	.59	$1 - \frac{265}{1536}p^6$	$1 - \frac{329}{2880}p^4$	1.18
Adams–Moulton	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[9F(\phi^{n+1}) + 19F(\phi^n) - 5F(\phi^{n-1}) + F(\phi^{n-2})]$	Implicit	0	$1 + \frac{p^6}{48}$	$1 + \frac{19}{720}p^4$	0
Adams–Bashforth	4	$\phi^{n+1} = \phi^n + \frac{h}{24}[55F(\phi^n) - 59F(\phi^{n-1}) + 37F(\phi^{n-2}) - 9F(\phi^{n-3})]$	5	.43	$1 - \frac{13}{24}p^6$	$1 - \frac{251}{720}p^4$	.43

(Durran 1991)

Impact of time-stepping schemes

- “In the weather and climate prediction community, when thinking in terms of model predictability, there is a tendency to associate model error with the physical parameterizations. In this paper, it is shown that time truncation error can be a substantial part of the total forecast error.” (Teixeira et al. 2007)

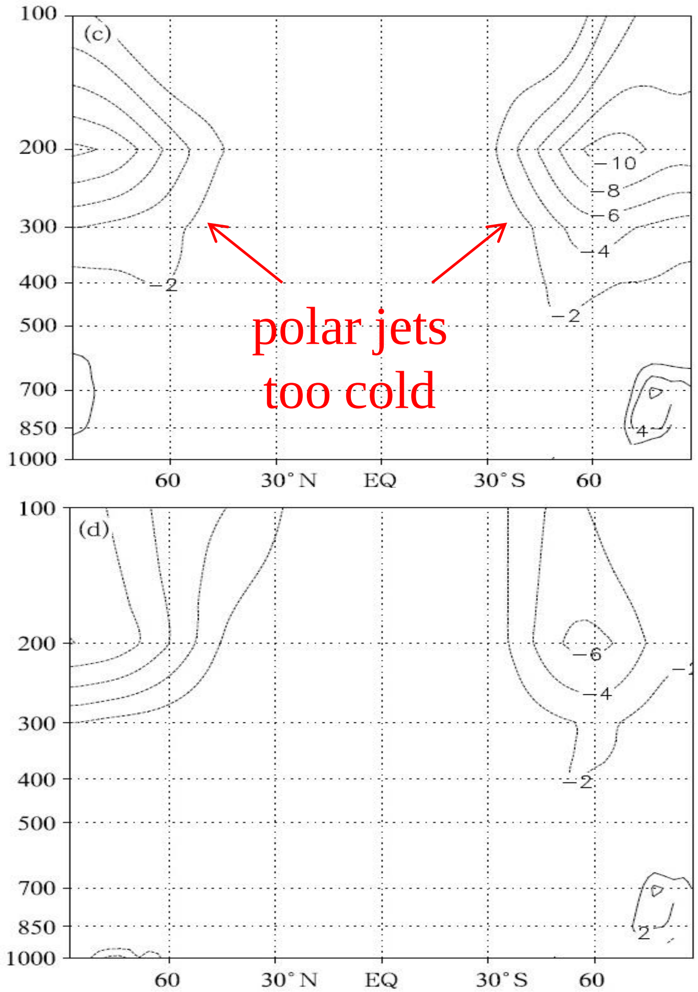
- “Climate simulations are sensitive not only to physical parameterizations of subgrid-scale processes but also to the numerical methodology employed.” (Pfeffer et al. 1992)

- “Many published conclusions on parameter sensitivity, calibrated values and associated uncertainty may be questionable due to numerical artifacts introduced by unreliable time stepping schemes.” (Kavetski & Clark 2010)

Impact of time stepping in weather and climate prediction

annual-mean zonal-mean temperature error (C) in CAM relative to ERA40

pressure (hPa)



pressure (hPa)

The default scheme (leapfrog + RA filter)

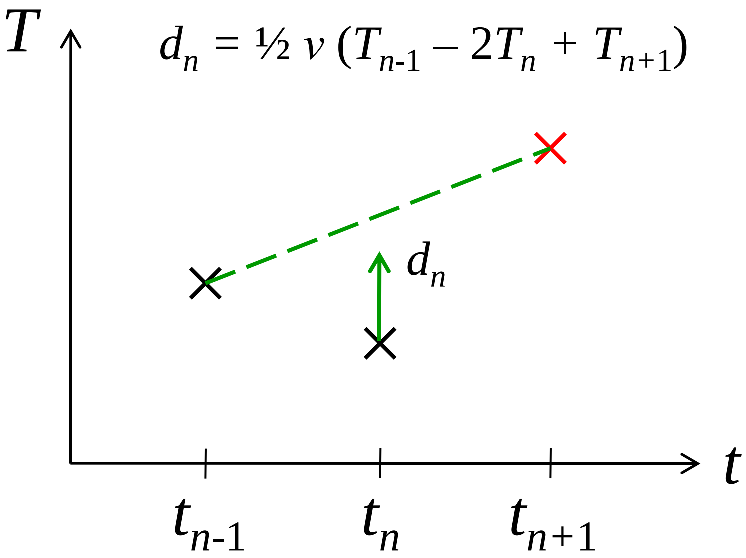
A more accurate scheme (AB2)

(Zhao & Zhong 2009)

Leapfrog with Robert–Asselin filter

LF+RA

(Robert 1966, Asselin 1972)



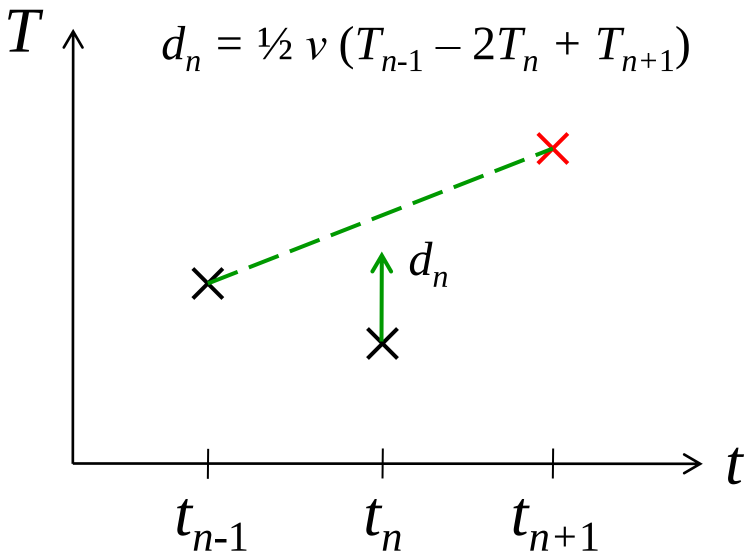
- use leapfrog to calculate T_{n+1}
- RA filter nudges T_n
- reduces curvature but does not conserve mean
- amplitude accuracy is 1st order

Leapfrog with Robert–Asselin filter

- Widely used in current numerical models
  - atmosphere: ECHAM, MAECHAM, MM5, CAM, MESO-NH, HIRLAM, KMCM, LIMA, SPEEDY, IGCM, PUMA, COSMO, FSU-GSM, FSU-NRSM, NCEP-GFS, NCEP-RSM, NSEAM, NOGAPS, RAMS, CCSR/NIES-AGCM
  - ocean: OPA, ORCA, NEMO, HadOM3, DieCAST, TIMCOM, GFDL-MOM, POM, MICOM, HYCOM, POSEIDON, NCOM, ICON, OFES, SOM
  - coupled: HiGEM (oce), COAMPS (atm), PlaSim (atm), ECHO (atm), MIROC (atm), FOAM (oce), NCAR-CCSM (atm), BCM (oce), NCEP-CFS (atm/oce), QESM (oce), CHIME (oce), FORTE (atm)
  - others: GTM, ADCIRC, QUAGMIRE, MORALS, SAM, ARPS, CASL, CReSS, JTGCM, ECOMSED, UKMO-LEM, MPI-REMO
- Asselin (1972) has received over 450 citations
- Has many problems
  - "*The Robert–Asselin filter has proved immensely popular, and has been widely used for over 20 years. However, it is not the last word…*" (Lynch 1991)
  - "*Replacement of the Asselin time filter… can be a feasible way to improve the ability of climate models*" (Zhao & Zhong 2009)
  - "*The Robert–Asselin filter can produce slewing frequency as well as the well-known damping and phase errors*" (Thrastarson & Cho 2011)

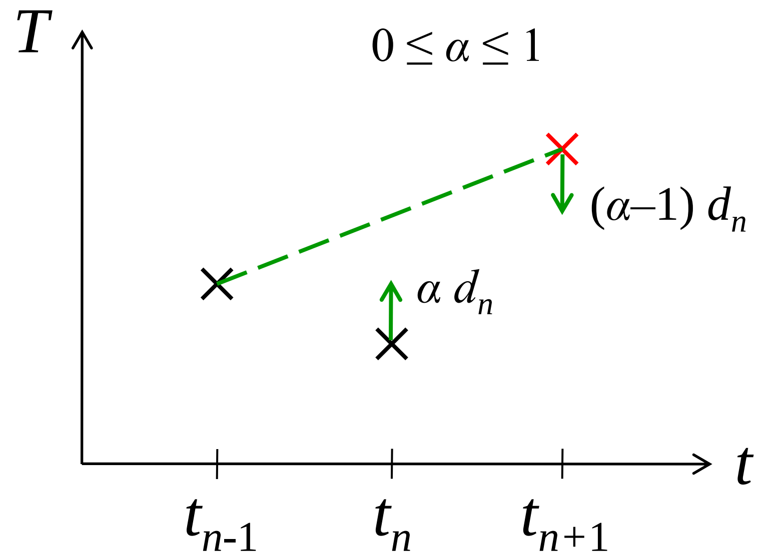
A proposed improvement

LF+RA
(Robert 1966, Asselin 1972)



- use leapfrog to calculate T_{n+1}
- RA filter nudges T_n
- reduces curvature but does not conserve mean
- amplitude accuracy is 1st order

LF+RAW
(Williams 2009, 2011)



- use leapfrog to calculate T_{n+1}
- RAW filter nudges T_n and T_{n+1}
- reduces curvature and conserves mean (for α=½)
- amplitude accuracy is 3rd order

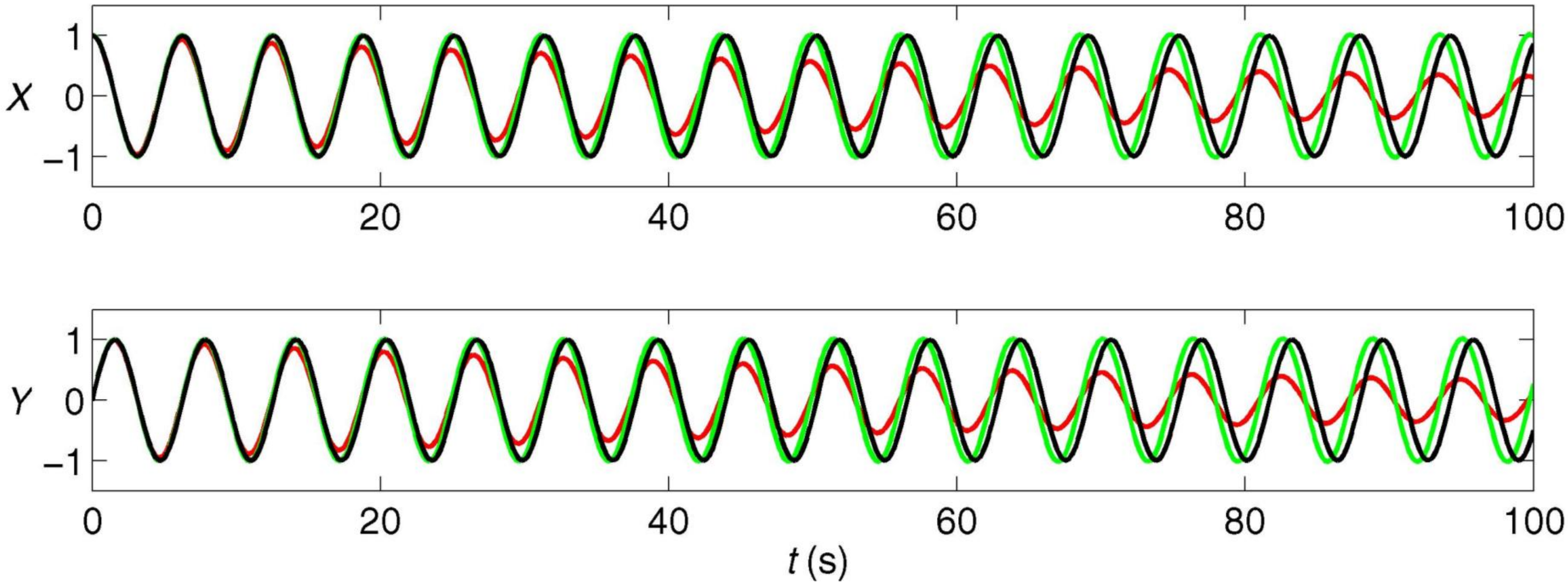
Simple test integration

$$\frac{dX}{dt} = -\omega Y$$
$$\frac{dY}{dt} = +\omega X$$

$\omega = 1 \text{ rad s}^{-1}$

exact
LF+RA
LF+RAW$_{\alpha=1/2}$

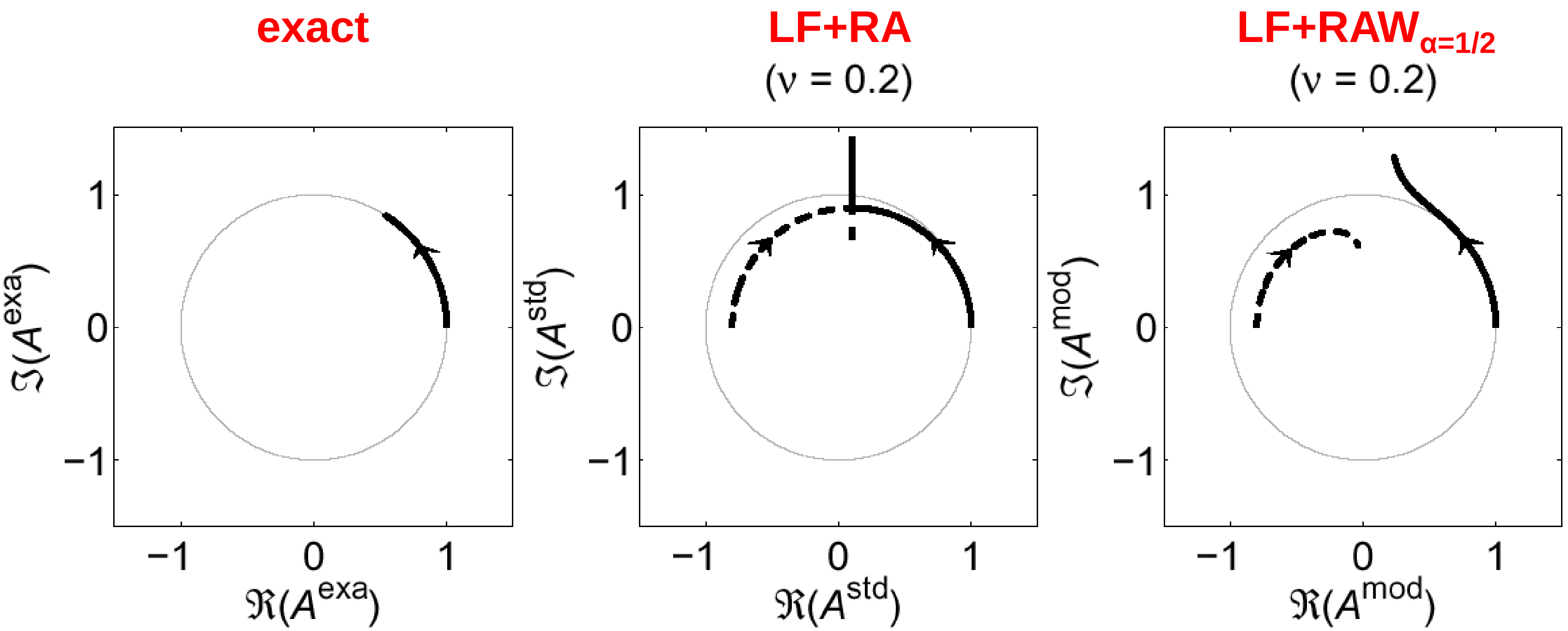
$\left.\right\}$ $\Delta t = 0.2$ s, $\nu = 0.2$



(Williams 2009)

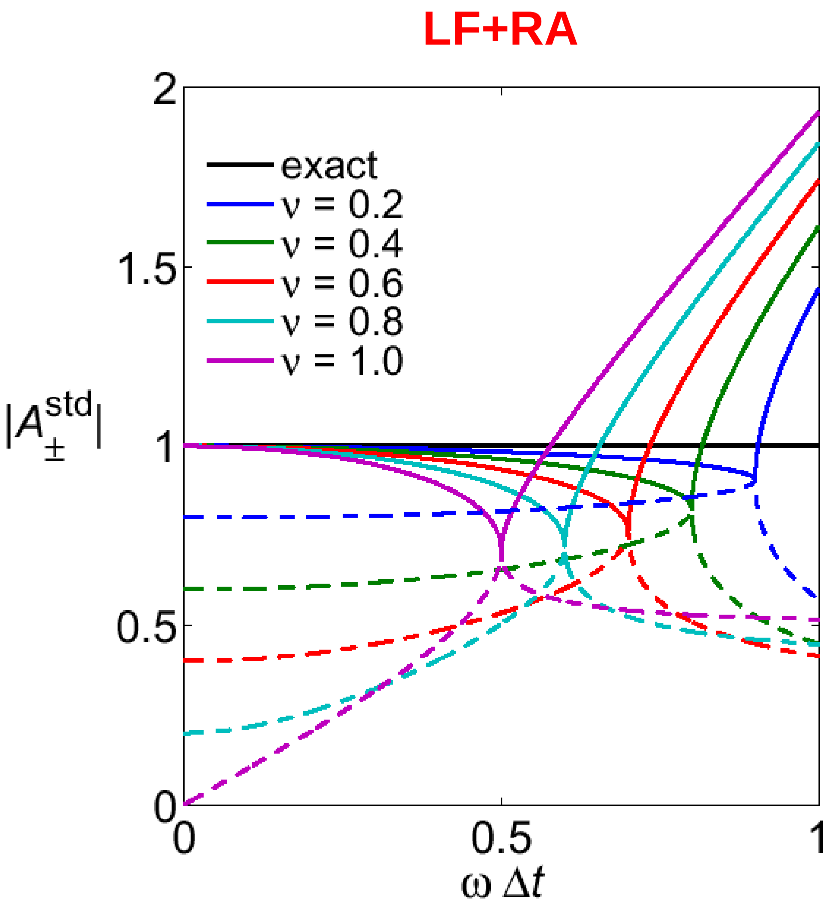
Analysis: numerical amplification

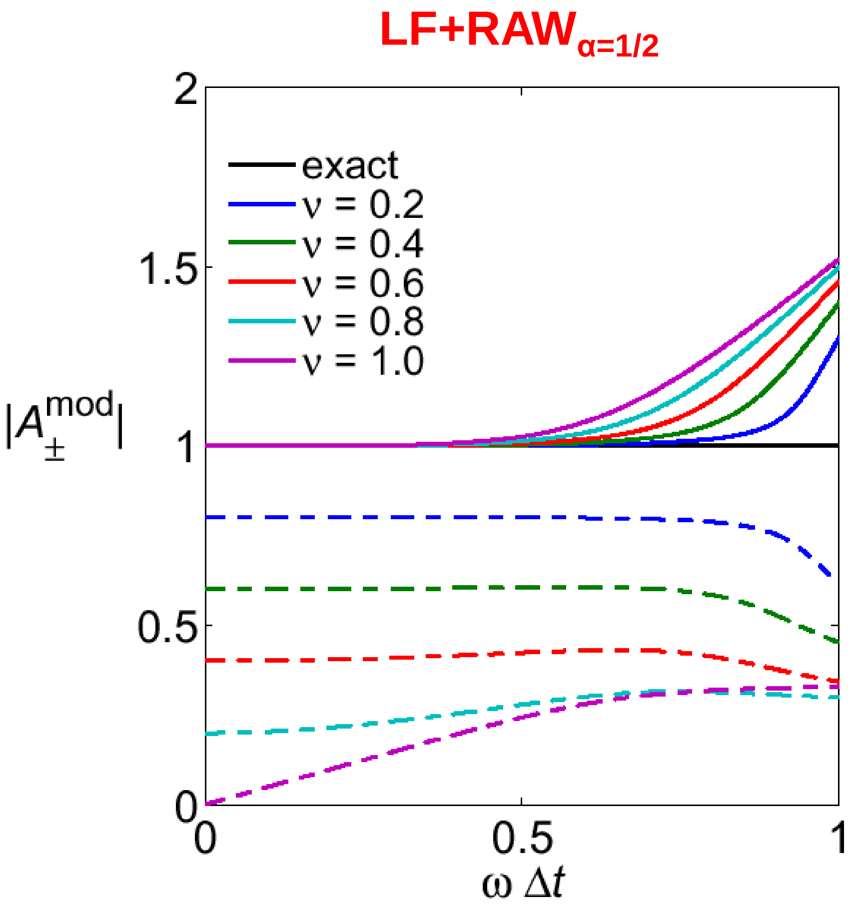
Let $\dot{F} = i\omega F$ and $A = F(t+\Delta t)\,/\,F(t)$ and trace A as $\omega\Delta t = 0 \to 1$:



(Williams 2009)

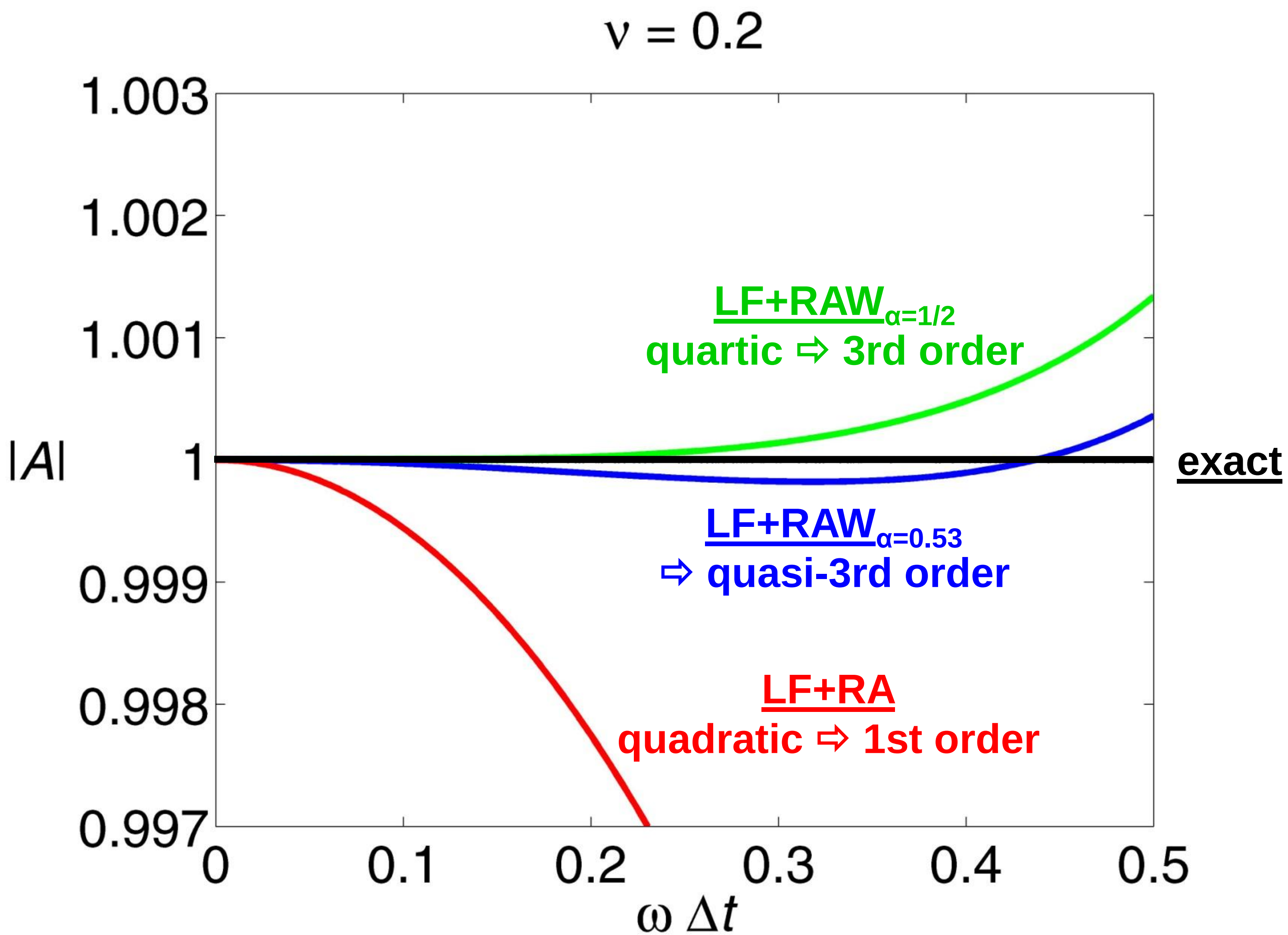
Analysis: numerical amplification





(Williams 2009)

Analysis: numerical amplification



(Williams 2009)

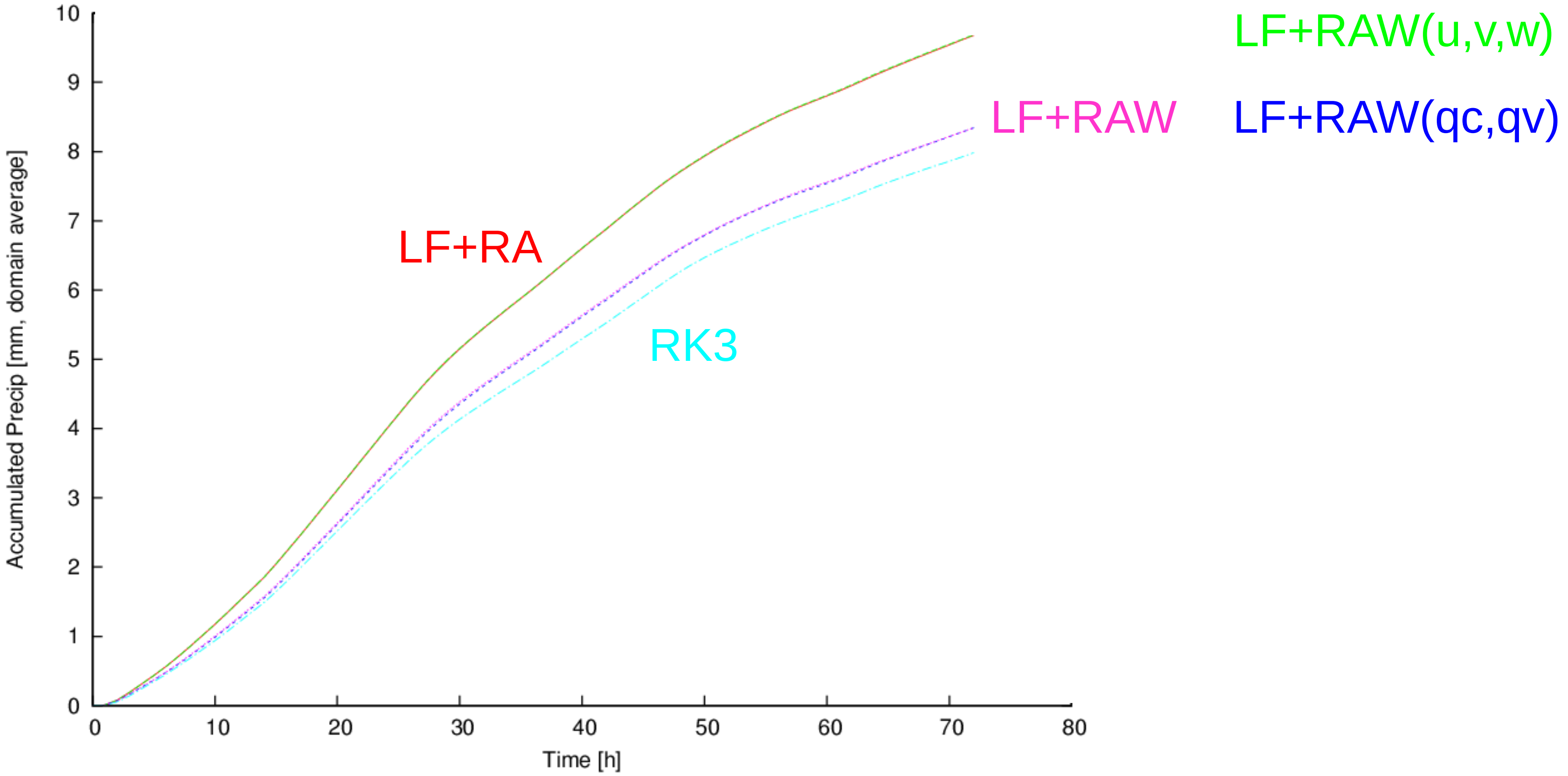
Implementation in existing code

```
! Compute tendency at this time step
tendency = […]

! Leapfrog step
x_next = x_last + tendency*2*delta_t

! Compute filter displacement
d = nu*(x_last - 2*x_this + x_next)/2

! Apply filter
x_this = x_this + d*alpha
x_next = x_next + d*(alpha-1)
```

(Williams 2011)

Implementation in COSMO-7

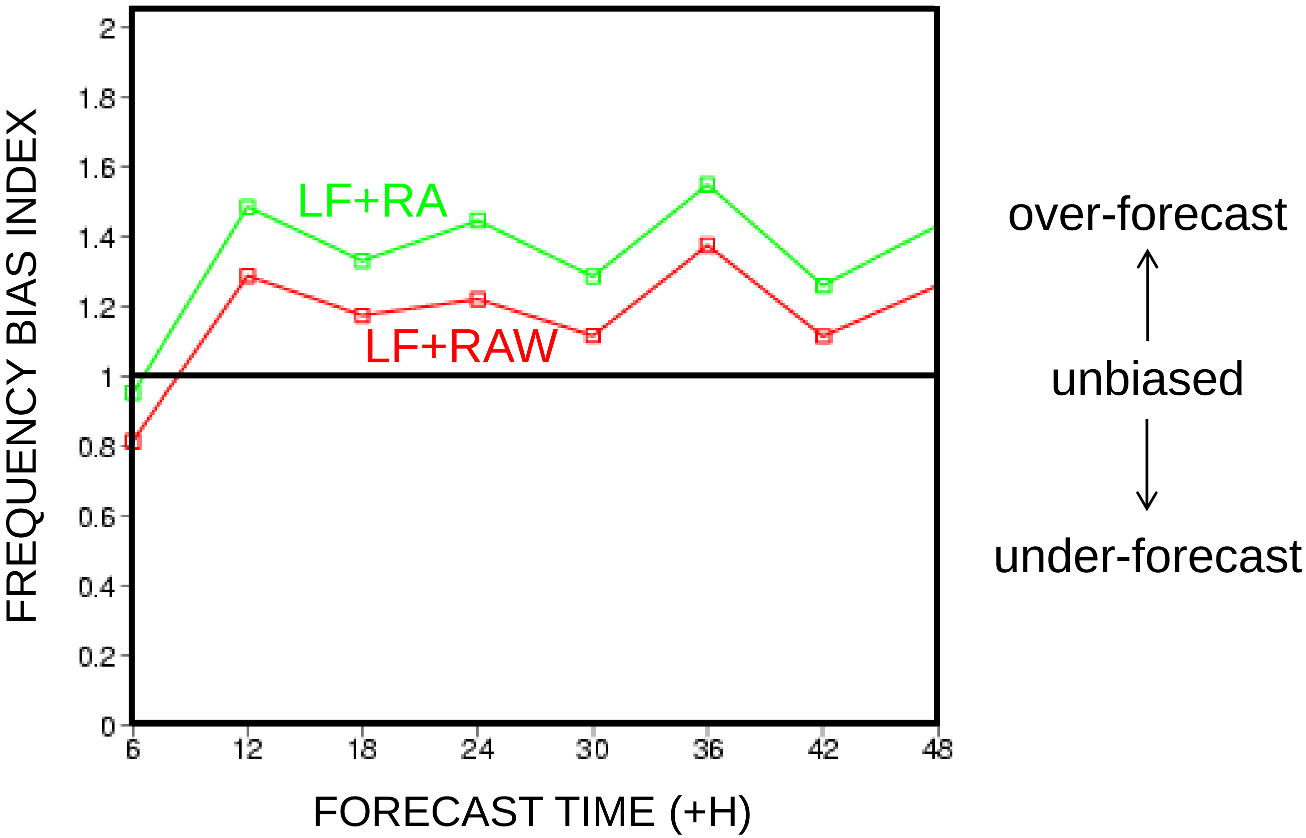
- Test case: 72h forecast over Europe starting at 00Z on 9 Feb 2009
- LF+RAW is almost as accurate as the (more expensive) RK3 integration



Oliver Fuhrer, MeteoSwiss

Implementation in COSMO-ME

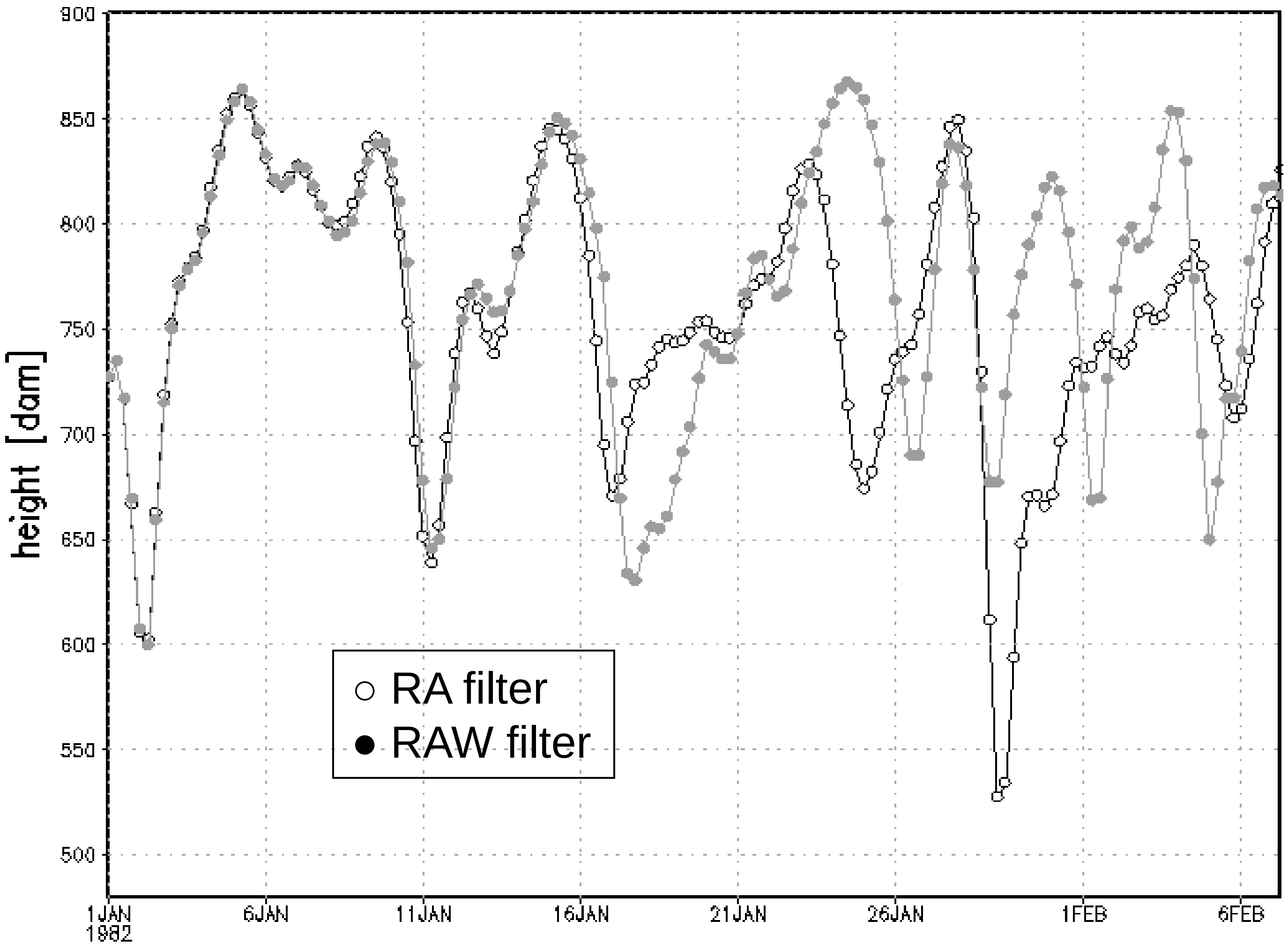
- Statistical analysis: two 48h forecasts per day from 16 Dec 2008 to 18 Jan 2009
- The modified filter significantly improves precipitation forecasts (2-10 mm / 6h)



Lucio Torrisi, CNMCA, Italy

Implementation in SPEEDY

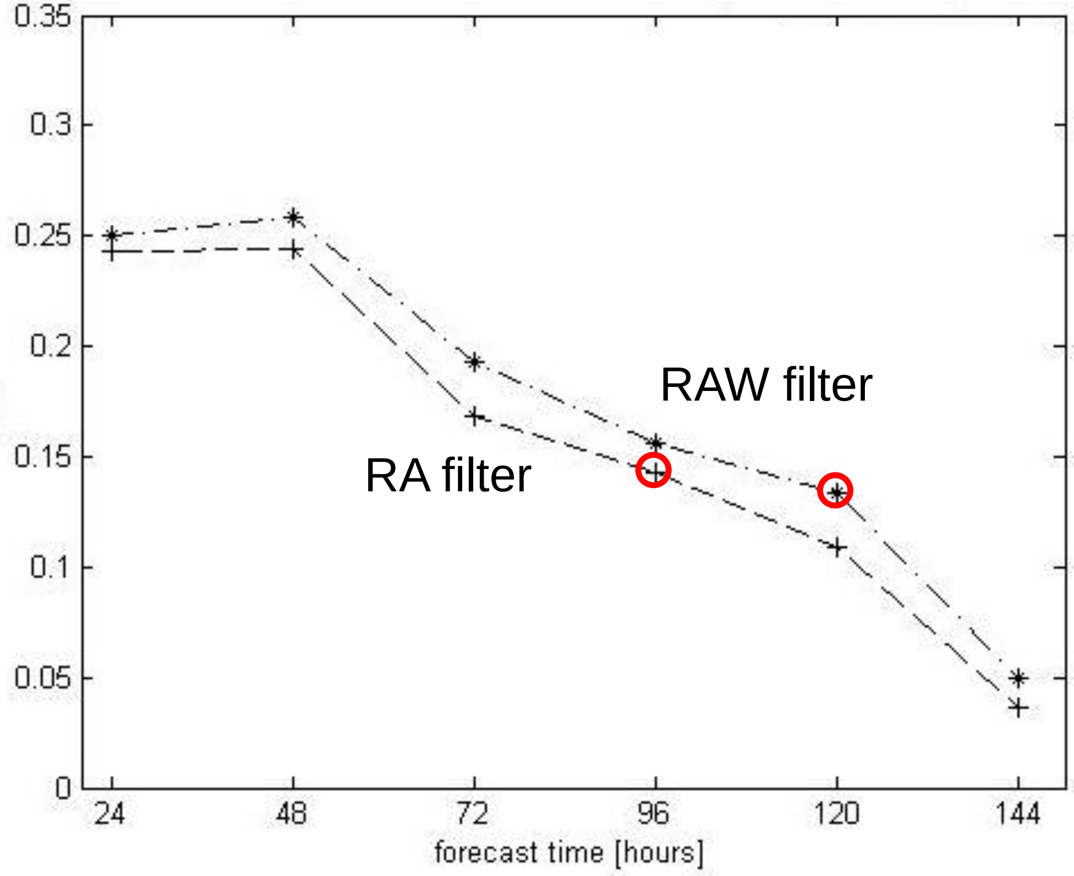
500 hPa geopotential height in Maryland



(Amezcua, Kalnay & Williams 2011)

Implementation in SPEEDY

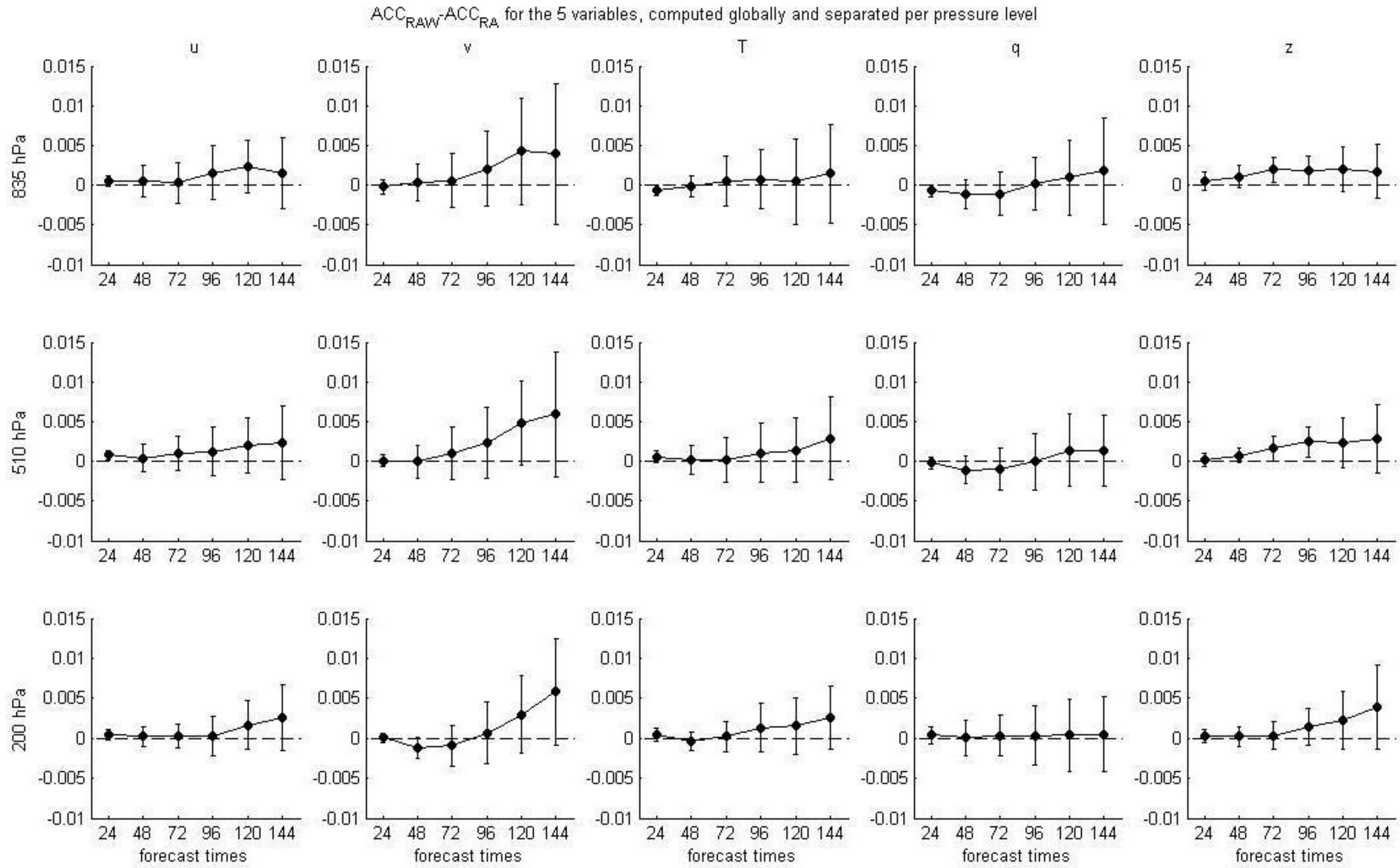
ACC for surface pressure in the tropics (25°S-25°N)



5-day forecasts made using the RAW filter have approximately the same skill as 4-day forecasts made using the RA filter

(Amezcua, Kalnay & Williams 2011)

Implementation in SPEEDY



(Amezcua, Kalnay & Williams 2011)

Composite-tendency leapfrog with (1, -2, 1) filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \begin{cases} x_{n+1} = \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] \\ \overline{\overline{x}}_n = \overline{x}_n + \dfrac{\nu\alpha}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \\ \overline{x}_{n+1} = x_{n+1} - \dfrac{\nu(1-\alpha)}{2}\left[\overline{\overline{x}}_{n-1} - 2\overline{x}_n + x_{n+1}\right] \end{cases}$$

leapfrog (first equation)

filter (second and third equations)

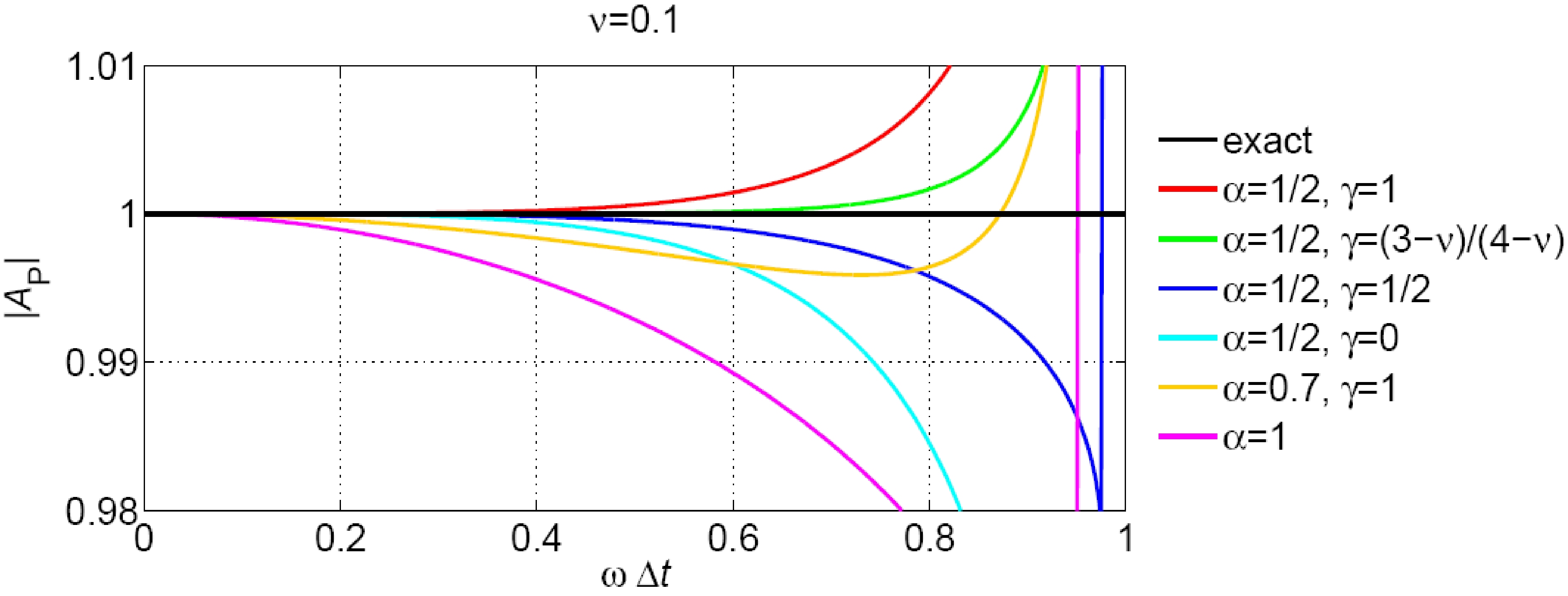
$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \begin{cases} |A_{\mathrm{P}}| - 1 = \dfrac{\nu(1-2\alpha)}{2(2-\nu)}(\omega\Delta t)^2 + \mathcal{O}[(\omega\Delta t)^4] \\ \alpha = \dfrac{1}{2} \quad |A_{\mathrm{P}}| - 1 = \dfrac{\nu(4\gamma - 3 + \nu - \nu\gamma)}{4(2-\nu)^2}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] \\ \gamma = \dfrac{3-\nu}{4-\nu} \quad |A_{\mathrm{P}}| - 1 = \dfrac{\nu}{4(4-\nu)(2-\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] \end{cases}$$

1st order

3rd order

5th order

Composite-tendency leapfrog with (1, -2, 1) filter



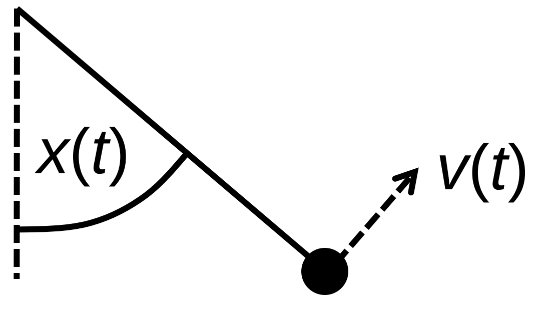
(Williams 2013)

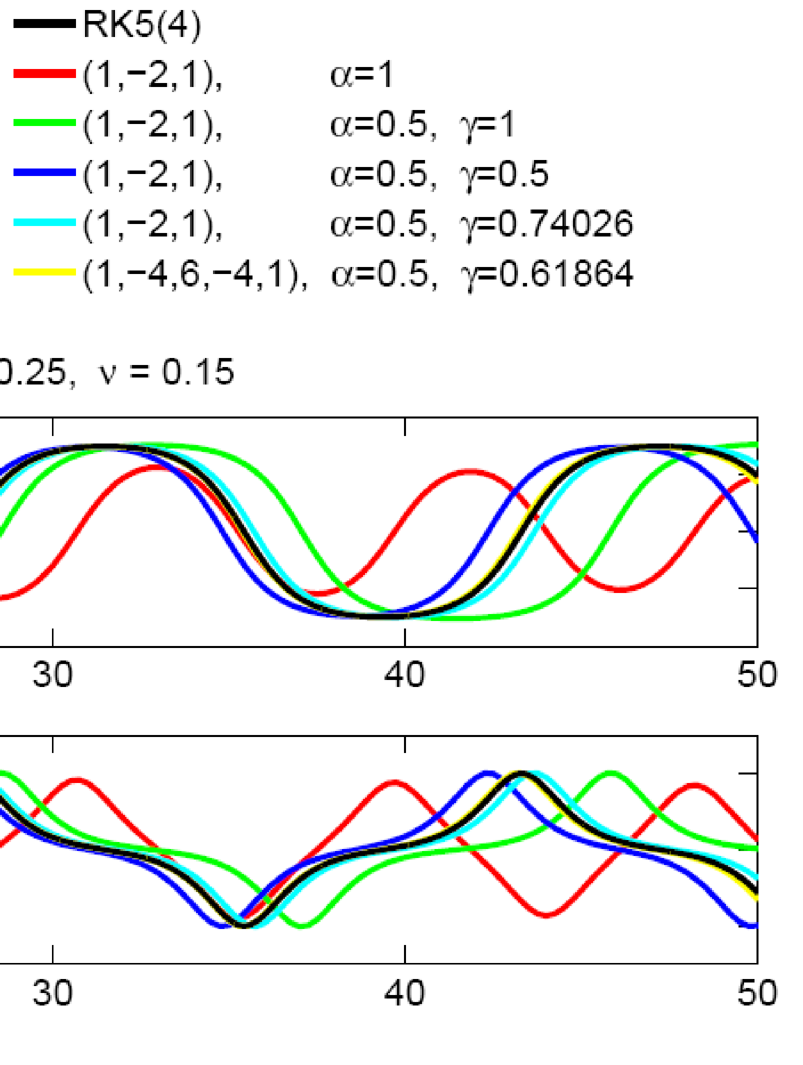
Composite-tendency leapfrog with (1, -4, 6, -4, 1) filter

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x) \quad \begin{cases} x_{n+1} = \overline{\overline{x}}_{n-1} + 2\Delta t\left[\gamma f(\overline{x}_n) + (1-\gamma) f(x_n)\right] & \text{leapfrog} \\ \overline{\overline{x}}_n = \overline{x}_n + \nu\alpha\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] & \text{filter} \\ \overline{x}_{n+1} = x_{n+1} - \nu(1-\alpha)\left[\overline{\overline{x}}_{n-3} - 4\overline{\overline{x}}_{n-2} + 6\overline{\overline{x}}_{n-1} - 4\overline{x}_n + x_{n+1}\right] & \end{cases}$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{i}\omega x \quad \begin{cases} |A_\mathrm{P}| - 1 = -\dfrac{\nu(1-2\alpha)}{2(1-\nu-2\alpha\nu)}(\omega\Delta t)^4 + \mathcal{O}[(\omega\Delta t)^6] & \text{3rd order} \\ \alpha = \dfrac{1}{2} \qquad |A_\mathrm{P}| - 1 = \dfrac{\nu(5-8\gamma-9\nu+14\nu\gamma)}{8(1-2\nu)^2}(\omega\Delta t)^6 + \mathcal{O}[(\omega\Delta t)^8] & \text{5th order} \\ \gamma = \dfrac{5-9\nu}{2(4-7\nu)} \qquad |A_\mathrm{P}| - 1 = -\dfrac{5\nu(4-13\nu+11\nu^2)}{32(1-2\nu)^2(4-7\nu)}(\omega\Delta t)^8 + \mathcal{O}[(\omega\Delta t)^{10}] & \text{7th order} \end{cases}$$

Nonlinear simple pendulum





Summary table

filter	α	γ	no. computational modes	no. function evaluations per time step	order	amplitude order of accuracy	phase order of accuracy	maximum $\omega\Delta t$ for stability ($\nu = 0.1$)	maximum $\omega\Delta t$ for accuracy ($\nu = 0.1$)
$(1, -2, 1)$	1	–	1	1	1	1	2	0.951	0.030
	1/2	1	1	1	2	(3)	2	0	(0.448)
	1/2	$(3-\nu)/(4-\nu)$	2	2	2	(5)	2	0	(0.712)
	1/2	1/2	2	2	2	3	2	0.975	0.475
	1/2	0	2	1	2	3	2	0.832	0.333
	0	1	1	1	1	(1)	2	0	(0.030)
$(1, -4, 6, -4, 1)$	1	–	3	1	2	(3)	2	0	(0.228)
	1/2	1	3	1	2	5	2	0.690	0.424
	1/2	$(5-9\nu)/[2(4-7\nu)]$	4	2	2	7	2	0.616	0.586
	1/2	0	4	1	2	(5)	2	0	(0.394)
	0	1	3	1	2	3	2	0.777	0.240

Summary

- Time stepping is an important contributor to model error
- The Robert–Asselin filter is widely used but is dissipative and reduces accuracy
- The RAW filter has approximately the same stability but much greater accuracy
- Implementation in an existing code is trivial and there is no extra computational cost
- 5th-order and even 7th-order amplitude accuracy may be achieved, by using a composite tendency and/or a more discriminating filter

Further information

Williams PD. Achieving seventh-order amplitude accuracy in leapfrog integrations. *Monthly Weather Review*, in preparation.

Williams PD (2011) The RAW filter: An improvement to the Robert–Asselin filter in semi-implicit integrations. *Monthly Weather Review* **139**(6), 1996-2007.

Williams PD (2009) A proposed modification to the Robert–Asselin time filter. *Monthly Weather Review* **137**(8), 2538-2546.

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