Anderson localization, topology, and interaction

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in collaboration with

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Outline

1 Introduction
   - Weak (anti)localization
   - Quantum Hall effect, quantum spin-Hall effect
   - Symmetry classification
   - Topological insulators

2 Localization in chiral systems
   - Absence of weak localization
   - Non-perturbative localization by vortices
   - Chiral topological insulators

3 Localization in symplectic systems
   - Interaction effects
   - QSHE phase diagram
   - Effect of vortices

4 Summary and outlook
Introduction
Scaling theory of localization

Abrahams, Anderson, Licciardello, Ramakrishnan ’79

Dimensionless conductance [in units $e^2/h$]:

- Metallic sample (Ohm’s law): $g \sim L^{d-2}$
- Insulating sample (tunneling): $g \sim e^{-L/\xi}$

Universal scaling function:

$$\frac{d \ln g}{d \ln L} = \beta(g) = \begin{cases} d - 2, & g \gg 1, \quad \text{(metal)}, \\ \ln g, & g \ll 1, \quad \text{(insulator)}. \end{cases}$$

![Graph showing the scaling function for different dimensions: 3D, 2D, and 1D.](image)
Quantum interference correction

- Metal with time-reversal symmetry $\implies$ Cooperon loop

\[
\Delta \sigma = \pm \frac{2e^2}{h} \int \frac{(d^d q)}{q^2} = \pm \frac{e^2}{\pi h} \left\{ \frac{1}{\pi l} - \frac{1}{\pi L}, \begin{array}{c} \text{3D,} \\ \ln(L/l), \end{array} \begin{array}{c} \text{2D,} \\ L - l, \end{array} \begin{array}{c} \text{1D} \end{array} \right. 
\]

Correction diverging in 2D and 1D
Spin preserved (broken) $\implies$ weak (anti)localization

- No time-reversal symmetry $\implies$ two diffuson loops

Very weak localization in 2D:

\[
\Delta \sigma = -\frac{e^4}{2\pi h^2 \sigma} \ln(L/l)
\]
Weak localization correction in 2D

Gor’kov, Larkin, Khmelnitskii ’79; Hikami, Larkin, Nagaoka ’80

\[
\frac{d \ln g}{d \ln L} = \begin{cases} 
-(\pi g)^{-1}, & \text{orthogonal (TR preserved, spin preserved)}, \\
-(2\pi g^2)^{-1}, & \text{unitary (TR broken)}, \\
+(\pi g)^{-1}, & \text{symplectic (TR preserved, spin broken)}
\end{cases}
\]

\[\Sigma_{\text{Sp}} \approx 1.4\]
Nonlinear sigma model

- Effective theory for interacting diffusons and Cooperons

\[ S[Q] = \frac{\pi \nu}{8} \int d^2 r \ \text{tr} \ [D(\nabla Q)^2 + 2i \omega \wedge Q] \]

- Matrix field

\[
Q \in \begin{cases} 
U(2N)/U(N) \times U(N), & \text{unitary (A)}, \\
Sp(2N)/Sp(N) \times Sp(N), & \text{orthogonal (AI)}, \\
O(2N)/O(N) \times O(N), & \text{symplectic (All)}. 
\end{cases}
\]

- Replica limit \( N \to 0 \) is assumed

- Renormalization of \( D \implies \text{weak (anti)localization} \)
Quantum Hall effect
von Klitzing, Dorda, Pepper ‘80

Electron states

Quantized resistance

Landau levels

- Anderson localization
- Topology
- Interaction
Quantum Hall effect
Pruisken '83; Khmelnitskii '83

Low-energy theory: unitary class sigma model with $\theta$-term

$$S[Q] = \int d^2 r \ tr \left[ \frac{\sigma_{xx}}{8} (\nabla Q)^2 + \frac{\sigma_{xy}}{4} (Q \nabla_x Q \nabla_y Q) \right]$$

$Q: S^2 \mapsto \mathcal{M} = U(2N)/U(N) \times U(N) \quad \pi_2(\mathcal{M}) = \mathbb{Z}$

Top. invariant: $\int d^2 r \ tr \ Q \nabla_x Q \nabla_y Q = 8\pi i n \implies e^{-S}$ is periodic in $\sigma_{xy}$!

Two-parameter scaling:

Quantum-Hall transition at half-integer $\sigma_{xy}$ and $\sigma_{xx} = \sigma^*_U \approx 0.6$
Quantum spin-Hall effect
Kane, Mele ’05, Sheng et al ’05 Bernevig, Zhang ’06

No magnetic field but strong spin-orbit interaction

Electrons with opposite spins feel opposite effective magnetic field

Electric current leads to spin accumulation at the edges

⇒ spin-Hall effect

Extreme spin-orbit coupling opens a band gap

⇒ quantum spin-Hall effect
Quantum (spin-)Hall effect with disorder

Impurities do not destroy the edge (spin) current:

QHE

due to chirality of carriers
any disorder

$\mathbb{Z}$ top. term in 2D

QSHE

due to time-inversion symmetry
no magnetic impurities

$\mathbb{Z}_2$ top. term in 1D
Quantum spin-Hall effect: experiment
Molenkamp group '07

HgTe/CdTe quantum well

I — $d = 5.5\text{nm}$: normal insulator
II, III, IV — $d = 7.3\text{nm}$: inverted band gap — topological insulator
Symmetry Classification

Wigner ’51, Dyson ’62, Altland and Zirnbauer ’97, Schnyder et al ’08

- Always present: diffuson

- Time-reversal symmetry (T):

  \[ H = U H^T U^{-1}, \quad T^2 = U U^* = \pm 1 \]
  Cooperon

- Chiral symmetry (C):

  \[ H = -U H U^{-1} \]
  RR diffuson

- Particle-hole symmetry (CT):

  \[ H = -U H^T U^{-1}, \quad CT^2 = U U^* = \pm 1 \]
  RR Cooperon
### Symmetry Classification

Wigner ’51, Dyson ’62, Altland and Zirnbauer ’97, Schnyder et al ’08

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## Classification: Bott periodicity

Kitaev ’08, Schnyder et al ’08

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Classification of topological insulators

Kitaev '08, Schnyder et al '08

\[ \mathbb{Z} \text{ topology} \]

Let \( \pi_d = \mathbb{Z} \) (as in QHE at \( d = 2 \))

\[ \Rightarrow \text{topological term may appear in } d \text{ dimensions} \]

\[ \Rightarrow (d - 1)\text{-dimensional surface states delocalized} \]

\[ \Rightarrow d\text{-dimensional } \mathbb{Z} \text{ topological insulator} \]

\[ \mathbb{Z}_2 \text{ topology} \]

Let \( \pi_d = \mathbb{Z}_2 \) (as on the \( d = 2 \) surface of 3D BiSb)

\[ \Rightarrow \text{topological term with } \theta = \pi \]

\[ \Rightarrow \text{absence of localization (as in graphene with potential disorder)} \]

\[ \Rightarrow (d + 1)\text{-dimensional } \mathbb{Z}_2 \text{ topological insulator} \]
## Classification of topological insulators

Kitaev '08, Schnyder et al '08

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Localization in chiral systems
From metal to insulator

Gradually remove sites from graphene

**Metal-insulator transition expected!**
From metal to insulator

Gradually remove sites from graphene

Metal-insulator transition expected!
From metal to insulator

Gradually remove sites from graphene

Metal-insulator transition expected!
From metal to insulator

Gradually remove sites from graphene

Metal-insulator transition expected!
Chiral unitary (AIII) network model
Both critical (Gade) and localized phase observed
Similar results for dimerized lattice model \[\text{[Motrunich et al.'02]}\]
Localization from metallic perspective

Gade and Wegner ’91, Gade ’93

- 2D nonlinear sigma model for a chiral system

\[ S[Q] = \int d^2x \left\{ \frac{\sigma}{8\pi} \text{tr} \left[ \nabla Q^{-1} \nabla Q \right] - \frac{c}{8\pi} \left[ \text{tr} Q^{-1} \nabla Q \right]^2 \right\} \]

- Matrix field

\[ Q \in \left\{ \begin{array}{c}
U(N), \quad \text{unitary (AIII),} \\
U(N)/Sp(N), \quad \text{orthogonal (BDI),} \\
U(N)/O(N), \quad \text{symplectic (CII).}\end{array} \right. \]

- Replica limit \( N \to 0 \) is assumed

- \( \sigma \) – conductivity per square
Localization from metallic perspective

Gade and Wegner ’91, Gade ’93

- Rewrite $Q = e^{i\phi}U$ (det $U = 1$)

$$S[U, \phi] = \int d^2x \left\{ \frac{\sigma}{8\pi} \text{tr} \left[ \nabla U^{-1} \nabla U \right] + N \left( \frac{\sigma + Nc}{8\pi} \right) (\nabla \phi)^2 \right\}$$

- Decoupled Gaussian theory in $\phi$:

$$\frac{d}{d\ln L} (\sigma + Nc) = 0$$

- Replica limit

$$\frac{d\sigma}{d\ln L} = -N \lim_{N \to 0} \frac{dc}{d\ln L} = 0$$

Absence of localization to all orders in perturbation theory!
Status quo

Apparent controversy

- Strong disorder induces localization in a chiral system (intuition + numerics)
- No traces of localization in the perturbation theory in the metallic limit (Gade and Wegner)
- No room for topological protection against localization

How to resolve?

Take into account non-perturbative effects
Loophole to escape Gade and Wegner argument:
\[ \det Q = e^{i\phi} \in U(1) \simeq S^1 \]
\[ \Rightarrow \text{vortex excitations allowed!} \]

Recalls Berezinskii-Kosterlitz-Thouless transition!
BKT Transition
Berezinskii ’70, Kosterlitz and Thouless ’73

- Continuum limit of xy-model: \[ S[\phi] = \frac{J}{2} \int d^2 x (\nabla \phi)^2 \]
- Vortex excitations with core energy \( S_{\text{core}} \)
- Large \( J \) (low temperature):
  \[ \implies \text{vortices strongly bound in tiny dipoles} \]
  \[ \implies \text{ordered phase (quasi long-range order)} \]
- Small \( J \) (high temperature):
  \[ \implies \text{vortex plasma, disordered phase} \]
- Renormalization group (fugacity \( y = L^2 e^{-S_{\text{core}}} \))
  \[ \frac{dJ}{d \ln L} = -y^2 J^2 \]
  \[ \frac{dy}{d \ln L} = (2 - \pi J) y \]
BKT Transition
Berezinskii '70, Kosterlitz and Thouless '73

\[ y_2 - p J \]

RG-flow in the vicinity of the critical “end” point.
Renormalization group

- Expand the fast field $\tilde{Q}$ near 1
  \[ \implies \text{One-loop perturbative RG:} \]
  \[
  \frac{d\sigma}{d \ln L} = 0, \quad \frac{dc}{d \ln L} = 1
  \]

  Exact in AIII class [Guruswamy et al'00]

- Include one vortex-antivortex dipole in $\tilde{Q}$
  (lowest order in fugacity $y = e^{-S_{\text{core}}}$)
  \[
  \frac{d\sigma}{d \ln L} = -\sigma y^2, \quad \frac{dc}{d \ln L} = 1 - (\sigma + 2c)y^2, \\
  \frac{dy}{d \ln L} = \left(2 - \frac{\sigma + c}{4}\right)y
  \]
Flow diagram

In terms of stiffness parameter \( K = (\sigma + c)/4 \)

\[
\frac{d\sigma}{d \ln L} = -\sigma y^2, \quad \frac{dK}{d \ln L} = \frac{1}{4} - 2Ky^2, \quad \frac{dy}{d \ln L} = (2 - K)y
\]

Fixed points:
- metal (Gade)
- critical
- insulator

No minimal metallic conductivity
Vortices vs. topology

- Chiral symplectic class CII admits $\mathbb{Z}_2$ $\theta$-term
  - Vortices attract instantons
  - Vortex-instanton fusion changes $S_{\text{core}} \mapsto S_{\text{core}} + i\pi$
  - Internal $\mathbb{Z}_2$ degree of freedom in each vortex

- Chiral unitary class AIII admits Wess-Zumino term
  - Vortices break global gauge symmetry
  - Internal “Goldstone” degree of freedom in each vortex
  - Random $\text{Im} S_{\text{core}}$

Presence of topological terms in sigma-model action prevents the theory from vortices!
Localization in symplectic systems
Interaction in disordered system

Altshuler, Aronov ’79; Finkelstein ’83

- Coulomb interaction $\Rightarrow$ new soft mode: plasmon

- Corrections to diffusion:

- Logarithmic in 2D: $\Delta g = -\frac{2e^2}{\pi hN} \log(L_T/l)$

- To be added to weak (anti)localization

- $N =$ number of independent flavours (valleys, spin etc.)

- Finkelstein sigma model
Scaling with interaction

Coulomb interaction “kills” metallic phase in symplectic class

\[ \beta(g) \]

**no interaction**

\[ g^* \approx 1.4 \]

**supermetal**

**with interaction**

\[ \beta(g) \]

0

\[ g^* \approx 1.4 \]

**insulator**
QSHE: Coulomb interaction

- Edge modes are protected w.r.t. Coulomb interaction
  - Distinction between normal and QSH insulator is robust
- Coulomb interaction "kills" supermetal phase

Interaction restores direct quantum spin-Hall transition
via a novel critical state
$\mathbb{Z}_2$ vortices in symplectic system

Fu, Kane, arXiv:1208.3442

Fu&Kane: **Localization in symplectic class is due to $\mathbb{Z}_2$ vortices!!!**

no vortices $\implies$ no localization

Combine everything

**No vortices + Coulomb interaction = critical state for QSHE transition**
Summary

Results
1. Renormalization of sigma model due to $\mathbb{Z}$ vortices
2. Metal-insulator transition in chiral systems
3. Topological prevention from vortex-induced localization
4. Interaction-induced critical state in QSHE
5. $\mathbb{Z}_2$ vortices + Coulomb interaction = critical state for QSHE transition

Outlook
1. Numerical study of localization transition in chiral systems
2. Higher-dimensional effects (vortex loops)
3. $\mathbb{Z}_2$ vortices in DIII

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