# Variational derivation of energy-conserving schemes for geophysical fluid equations

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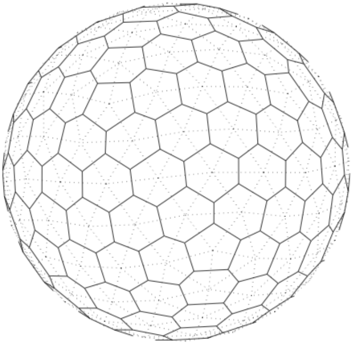
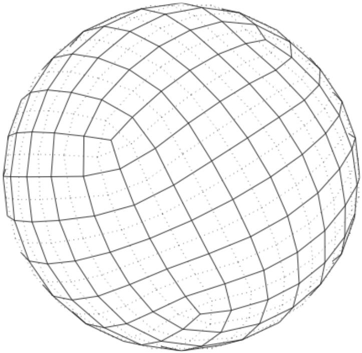
# Energy-conserving schemes

- Arakawa (1966) : non-divergent, + enstrophy
- Sadourny et al. (1968) : non-divergent, icosahedron, + enstrophy
- Sadourny (1972) : rotating shallow-water, icosahedron/cubed sphere, collocated A-grid
- Sadourny (1975) : RSW, Cartesian staggered C-grid
- Simmons & Burridge (1981) : hydrostatic, hybrid vertical coordinate
- Janjic (1977, 1984), Rancic (1988, 2009) : RSW, staggered E-grid, cubed sphere / octagonal
- Arakawa & Lamb (1982) : RSW, C-grid, + enstrophy
- Hollingsworth et al. (1983) : RSW, Cartesian staggered C-grid
- Taylor et al. (2010) : RSW / hydrostatic, cubed-sphere, high-order "collocated" finite elements
- Ringler et al. (2010) : RSW, unstructured C- grid with orthogonal dual
- Thuburn & Cotter (2012) : RSW, unstructured C- grid with non-orthogonal dual
- Gassmann (2012) : $z-$based coordinate, non-hydrostatic, unstructured C-grid with orthogonal dual

# Energy-conserving C-grid schemes



- orthogonal Delaunay/Voronay meshes
- too many velocity DOFs ⇒ numerical modes



- good balance of mass/velocity DOFs
- no quasi-uniform orthogonal primal/dual pair

# Vector field on a non-orthogonal C-grid (Thuburn & Cotter, 2012)

**Describe a vector field by a finite set of scalars ?**

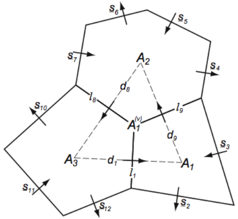
1. cell-averaged components
2. pointwise components
3. *integrals along curves*
4. *integrals across surfaces*

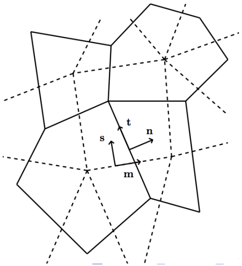
**Not orthogonal ⇒ distinguish between :**

- circulations along dual mesh edges $\Gamma_e$
  ⇒ discrete vorticity budget
- fluxes across primal mesh faces
  ⇒ discrete mass budget

**Discrete vorticity budget**

prognose circulations $v_e = \int_{\Gamma_e} \underline{u} \cdot d\underline{l}$
⇒ diagnose mass fluxes $u_e = \int_{\Sigma_e} \underline{u} \cdot \underline{n} ds$



# Conservation of energy

**Linear wave equation**

$$\partial_t p + \mathrm{div}\ \underline{u} = 0,\ \partial_t \underline{u} + \underline{\nabla} p = 0$$

define column-vectors $\mathbf{p} = (p_i)$ and $\mathbf{v} = (v_e)$

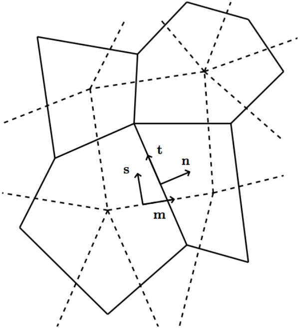
$$p_i = \int_{V_i} p \mathrm{d}A, \qquad v_e = \int_{\Gamma_e} \underline{u} \cdot \mathrm{d}\underline{l}$$

$$\partial_t \mathbf{p} = -D \cdot H \cdot \mathbf{v}, \qquad \partial_t \mathbf{v} = -G \cdot A^{-1} \cdot \mathbf{p}$$

with $V_i$ primal mesh cells, $A$ diagonal matrix

$$A_{ii} = \int_{V_i} \mathrm{d}A$$

$G, D$ incidence matrices : ${}^tG + D = 0$




**$H$ symmetric positive definite $\Rightarrow$ conservation of energy**

$$\begin{aligned} E(\mathbf{p},\mathbf{v}) &= \frac{1}{2}\mathbf{p}^* \cdot A \cdot \mathbf{p} + \frac{1}{2}\mathbf{v}^* \cdot H \cdot \mathbf{v} \\ \partial_t E &= \partial_t \mathbf{p}^* \cdot A \cdot \mathbf{p} + \partial_t \mathbf{v}^* \cdot H \cdot \mathbf{v} \\ &= -\mathbf{v}^* \cdot H \cdot D^* \cdot A \cdot \mathbf{p} - \mathbf{p}^* \cdot A^{-1} \cdot G^* \cdot H \cdot \mathbf{v} = 0 \end{aligned}$$

# A 2D non-diagonal discrete Hodge star

**$H$ symmetric positive definite $\Rightarrow$ conservation of energy**

$$\begin{aligned} E(\mathbf{p},\mathbf{v}) &= \frac{1}{2}\mathbf{p}^* \cdot A^{-1} \cdot \mathbf{p} + \frac{1}{2}\mathbf{v}^* \cdot H \cdot \mathbf{v} \\ \partial_t E &= \partial_t \mathbf{p}^* \cdot A^{-1} \cdot \mathbf{p} + \partial_t \mathbf{v}^* \cdot H \cdot \mathbf{v} \\ &= -\mathbf{v}^* \cdot H \cdot D^* \cdot A^{-1} \cdot \mathbf{p} - \mathbf{p}^* \cdot A^{-1} \cdot G^* \cdot H \cdot \mathbf{v} = 0 \end{aligned}$$

**Strategy (inspired by Perot, Vidovic & Wesseling)**

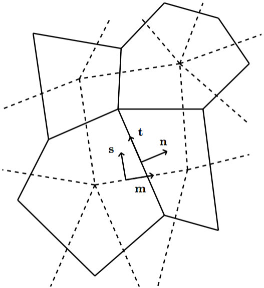
- Estimate the kinetic energy $K(\mathbf{v}) \simeq \frac{1}{2}\int \underline{u} \cdot \underline{u} d\Omega$
- Define $\mathbf{u} = \partial K / \partial \mathbf{v} = H \cdot \mathbf{v}$
- Find the primal mesh such that $u_e \simeq \int_{\Sigma_e} \underline{u} \cdot \underline{n} ds$ !

Here $\simeq$ means that $=$ holds when $\underline{u}$ is uniform (consistency).

# Corner-wise velocity reconstruction

Consider two consecutive edges $AB$ and $BC$ of a dual cell, counter-clockwise :

$$v_{AB} = \int_{AB} \underline{u} \cdot \mathrm{d}\underline{l} \simeq \underline{u} \cdot \underline{AB}$$
$$v_{BC} = \int_{BC} \underline{u} \cdot \mathrm{d}\underline{l} \simeq \underline{u} \cdot \underline{BC}$$




Define

$$\underline{z} = \text{unit normal vector}$$
$$ABC = \underline{z} \cdot (\underline{AB} \times \underline{BC})$$
$$\underline{u}_{ABC}(v_{AB}, v_{BC}) = \frac{1}{ABC} \underline{z} \times (v_{AB} \underline{BC} - v_{BC} \underline{AB})$$

Then

$$\underline{u}_{ABC}(v_{AB}, v_{BC}) \simeq \underline{u}.$$

# Discrete Hodge star operator

## Triangular dual cell

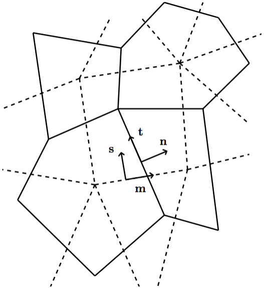
$$\text{area}(ABC) = \frac{1}{2} ABC$$

$$K_{ABC} = \frac{ABC}{6}\left(\underline{u}^2_{ABC} + \underline{u}^2_{BCA} + \underline{u}^2_{CAB}\right)$$

## Quadragular dual cell

$$\text{area}(ABCD) = \frac{1}{4}\left(ABC + BCD + CDA + DAB\right)$$

$$K_{ABCD} = \frac{1}{8}\left(ABC\, \underline{u}^2_{ABC} + BCD\, \underline{u}^2_{BCD} + CDA\, \underline{u}^2_{CDA} + DAB\, \underline{u}^2_{DAB}\right)$$




Algebra shows that

$$K(\mathbf{v}) = \sum_{dual\ cells} K_{cell} \quad \Rightarrow \quad \frac{\partial K}{\partial v_e} \simeq \int_{\Sigma_e} \underline{u} \cdot \underline{n} ds$$

where the primal mesh vertices are barycenters of dual mesh cells.

# Connection to the discrete Poisson bracket approach

e.g. Gassmann (2012)

## Discrete

$$E(\mathbf{p},\mathbf{v}) = \frac{1}{2}\mathbf{p}^* \cdot A^{-1} \cdot \mathbf{p} + \frac{1}{2}\mathbf{v}^* \cdot H \cdot \mathbf{v}$$

$$\partial_t \mathbf{p} = -D \cdot \frac{\partial E}{\partial \mathbf{v}}, \qquad \partial_t \mathbf{v} = -G \cdot \frac{\partial E}{\partial \mathbf{p}}$$

$$\begin{aligned}
\partial_t F(\mathbf{p},\mathbf{v}) &= \frac{\partial F}{\partial \mathbf{p}} \cdot \frac{\partial \mathbf{p}}{\partial t} + \frac{\partial F}{\partial \mathbf{v}} \cdot \frac{\partial \mathbf{v}}{\partial t} \\
&= -\frac{\partial F}{\partial \mathbf{p}} \cdot D \cdot \frac{\partial E}{\partial \mathbf{v}} - \frac{\partial F}{\partial \mathbf{v}} \cdot G \cdot \frac{\partial E}{\partial \mathbf{p}} \\
&= -\left(\frac{\partial F}{\partial \mathbf{v}} \cdot G \cdot \frac{\partial E}{\partial \mathbf{p}} - \frac{\partial E}{\partial \mathbf{v}} \cdot G \cdot \frac{\partial F}{\partial \mathbf{p}}\right) \\
\partial_t F(\mathbf{p},\mathbf{v}) &= \{F, E\}
\end{aligned}$$

## Continuous

$$\begin{aligned}
E[p,\underline{u}] &= \frac{1}{2}\int \left(p^2 + \underline{u}\cdot\underline{u}\right) d\Omega \\
\delta E &= \int (p\delta p + \underline{u}\cdot\delta\underline{u})\, d\Omega
\end{aligned}$$

$$\partial_t p + \nabla \cdot \frac{\delta E}{\delta \underline{u}} = 0, \quad \partial_t \underline{u} + \nabla \frac{\delta E}{\delta p} = 0,$$

$$\begin{aligned}
\partial_t F[p,\underline{u}] &= \int \left(\frac{\delta F}{\delta p} \cdot \frac{\partial p}{\partial t} + \frac{\delta F}{\delta \underline{u}} \cdot \frac{\partial \underline{u}}{\partial t}\right) d\Omega \\
&= -\int \left(\frac{\delta F}{\delta \underline{u}} \cdot \nabla \frac{\delta E}{\delta p} - \frac{\delta E}{\delta \underline{u}} \cdot \nabla \frac{\delta F}{\delta p}\right) d\Omega \\
\partial_t F[p,\underline{u}] &= \{F, E\}
\end{aligned}$$

Caveat : here, discrete Poisson brackets only have antisymmetry : $\{F,G\} + \{G,F\} = 0$,

NOT the Jacobi property : $\{F,\{G,H\}\} + \{H,\{F,G\}\} + \{G,\{H,F\}\} = 0.$

# Rotating shallow-water Poisson bracket

## Hamiltonian ($f$-plane)

- $\underline{u}$ : horizontal velocity ; $\underline{R}(x,y) = \Omega \underline{e_z} \times \underline{x}$ : entrainment velocity
- $\underline{v} = \underline{u} + \underline{R}$: absolute velocity
- curl $\mathbf{R} = 2\Omega = f$, potential vorticity $q = \text{curl } \underline{v}/h$

$$H[h,\underline{v}] = \frac{1}{2}\left\langle h\left(\underline{v} - \underline{R}(x,y)\right)^2 + gh^2 \right\rangle \qquad \langle \cdot \rangle = \int \mathrm{d}x\mathrm{d}y$$

$$\Rightarrow \quad \frac{\delta H}{\delta \underline{v}} = h\underline{u}, \qquad \frac{\delta H}{\delta h} = gh + \frac{\underline{u}\cdot\underline{u}}{2}$$

## Vector-invariant form and Poisson bracket

$$\frac{\partial h}{\partial t} + \nabla\cdot\frac{\delta H}{\delta \underline{v}} = 0, \qquad \frac{\partial \underline{v}}{\partial t} + \frac{\text{curl } \underline{v}}{h} \times \frac{\delta H}{\delta \underline{v}} + \nabla \frac{\delta H}{\delta h} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}t} F[h,\underline{v}] = \left\langle \frac{\delta H}{\delta \underline{v}} \cdot \nabla \frac{\delta F}{\delta h} - \frac{\delta F}{\delta \underline{v}} \cdot \nabla \frac{\delta H}{\delta h} + \frac{\text{curl } \underline{v}}{h} \cdot \left( \frac{\delta F}{\delta \underline{v}} \times \frac{\delta H}{\delta \underline{v}} \right) \right\rangle = \{F,H\}$$

# Rotating shallow-water Poisson bracket

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## Recipe for an energy-conserving scheme

- discretize in vector-invariant form
- approximate total energy as a function of DOFs
- define mass flux and Bernoulli function from derivatives of total energy
- ensure that div and grad are *compatible*
- antisymmetrize the Coriolis bracket

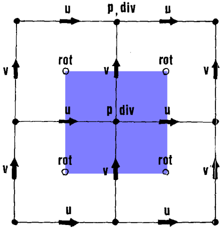
*Was this recipe followed in the past for the RSW ?*

# Sadourny's energy-conserving scheme

## Sadourny, 1975

$$H = \frac{1}{2}\left\langle gh^2 + h\overline{u^2}^x + h\overline{v^2}^y \right\rangle$$

$$\begin{aligned}
\partial_t h + \delta_x U + \delta_y V &= 0 & U = u\overline{h}^x = \frac{\partial H}{\partial u}, \\
\partial_t u - \overline{q\overline{V}^x}^y + \delta_x B &= 0 & V = v\overline{h}^y = \frac{\partial H}{\partial v} \\
\partial_t v + \overline{q\overline{U}^y}^x + \delta_y B &= 0 & B = h + \frac{\overline{u^2}^x + \overline{v^2}^x}{2} = \frac{\partial H}{\partial h}
\end{aligned}$$




$$\frac{dF}{dt} = -\left\langle \frac{\partial F}{\partial u}\delta_x\frac{\partial H}{\partial h} - \frac{\partial H}{\partial u}\delta_x\frac{\partial F}{\partial h} + \frac{\partial F}{\partial v}\delta_y\frac{\partial H}{\partial h} - \frac{\partial H}{\partial v}\delta_y\frac{\partial F}{\partial h} + \overline{\frac{\partial F}{\partial v}}^x q\,\overline{\frac{\partial H}{\partial u}}^y - \overline{\frac{\partial F}{\partial u}}^y q\,\overline{\frac{\partial H}{\partial v}}^x \right\rangle$$

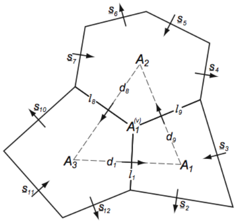
Hollingsworth et al., 1983 : Modified kinetic energy to control a numerical instability

$$H = \frac{1}{2}\left\langle gh^2 + \frac{h}{3}\left(\overline{u^2}^x + \overline{v^2}^y\right) + \frac{2h}{3}\left(\overline{u^2}^{xyy} + \overline{v^2}^{yxx}\right)\right\rangle$$

$$\Rightarrow U = \left(\frac{1}{3}\overline{h}^x + \frac{2}{3}\overline{h}^{xyy}\right)u = \frac{\partial H}{\partial u}, \qquad V = \left(\frac{1}{3}\overline{h}^y + \frac{2}{3}\overline{h}^{yxx}\right)v = \frac{\partial H}{\partial v}$$

# TRiSK scheme : Ringler et al. (2010)

| | | primal | | dual |
|---|---|---|---|---|
| $i$ | $\Leftrightarrow$ | cell $V_i$ | $\Leftrightarrow$ | vertex |
| $v$ | $\Leftrightarrow$ | vertex $\mathbf{x}_v$ | $\Leftrightarrow$ | cell |
| $e$ | $\Leftrightarrow$ | face $l_e = \lvert E_e \rvert$ | $\Leftrightarrow$ | edge $d_e = \lvert \Gamma_e \rvert$ |




$$M_i = \int_{V_i} h \mathrm{d}A, \quad u_e = \int_{\Gamma_e} \mathbf{u} \cdot \mathbf{dl}$$

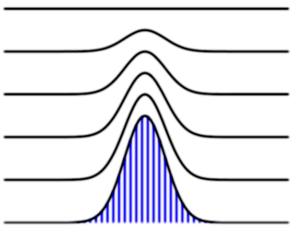
$$\partial_t M_i + \sum_e n_{ei} U_e = 0 \qquad \partial_t u_e - \eta_e U_e^{\perp} + \sum_i n_{ei} B_i = 0, \qquad n_{ei} = -1, 0, 1$$

$$H = \frac{1}{2} \sum_i A_i g \left( \frac{M_i}{A_i} \right)^2 + \frac{1}{2} \sum_e l_e d_e h_e \left( \frac{u_e}{d_e} \right)^2, \qquad h_e = \sum_i w_{ei} \frac{M_i}{A_i}, \qquad \sum_i w_{ei} = 1$$

$$U_e = h_e l_e \frac{u_e}{d_e} = \frac{\partial H}{\partial u_e}, \qquad B_i = g \frac{M_i}{A_i} + \frac{1}{A_i} \sum_e w_{ei} l_e d_e \left( \frac{u_e}{d_e} \right)^2 = \frac{\partial H}{\partial M_i}$$

# Hamiltonian in a general vertical coordinate $\eta$

- $\mathbf{R}(x,y,z)$ : entrainment velocity;
- curl $\mathbf{R} = 2\mathbf{\Omega}$ ; need not be along $z$ (complete Coriolis force, see Dellar & Salmon, 2005)
- $\underline{u}$ : horizontal velocity ; $\underline{v} = \underline{u} + \underline{R}$
- pseudo-density $\mu = \rho \partial z / \partial \eta$
- mass-weighted entropy $S = \mu s$
- grad, div taken along $\eta = cst$
- mass flux $W$ through interfaces $\eta = cst$



$$H[\mu, \underline{v}, S, z] = \left\langle \mu \left( \frac{(\underline{v} - \underline{R})^2}{2} + U\left( \frac{1}{\mu}\frac{\partial z}{\partial \eta}, \frac{S}{\mu} \right) + \Phi(z) \right) \right\rangle,$$

$$\frac{\delta H}{\delta \mu} = \frac{\underline{u}\cdot\underline{u}}{2} + \Phi + U + pv - Ts, \qquad \frac{\delta H}{\delta \underline{v}} = \mu \underline{u}, \qquad \frac{\delta H}{\delta S} = T$$

$$\mu \frac{\partial}{\partial \eta}\left( \frac{\delta H}{\delta \mu} \right) + \Theta \frac{\partial}{\partial \eta}\left( \frac{\delta H}{\delta \Theta} \right) - \frac{\partial \underline{v}}{\partial \eta} \cdot \frac{\delta H}{\delta \underline{v}} - \frac{\partial z}{\partial \eta} \frac{\delta H}{\delta z} = 0$$

# Vector-invariant form

$$\frac{\delta H}{\delta \mu} = \frac{\underline{u}\cdot\underline{u}}{2} + \Phi + U + pv - Ts, \qquad \frac{\delta H}{\delta \underline{v}} = \mu \underline{u}, \qquad \frac{\delta H}{\delta S} = T$$

$$\begin{aligned}
\frac{\partial \mu}{\partial t} + \frac{\partial W}{\partial \eta} + \mathrm{div}\, \frac{\delta H}{\delta \underline{v}} &= 0 \\
\frac{\partial S}{\partial t} + \frac{\partial}{\partial \eta}(sW) + \mathrm{div}\left( s \frac{\delta H}{\delta \underline{v}} \right) &= 0 \\
\frac{\partial \underline{v}}{\partial t} + \frac{W}{\mu}\frac{\partial \underline{v}}{\partial \eta} + \frac{\mathrm{curl}\ \underline{v}}{\mu} \times \frac{\delta H}{\delta \underline{v}} + \nabla \frac{\delta H}{\delta \mu} + T\nabla \frac{\delta H}{\delta S} &= 0 \\
\frac{\delta H}{\delta z} &= 0
\end{aligned}$$

# Energy-conserving schemes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}F[\mu,\mathbf{v},S,z] &= \left\langle \frac{\delta H}{\delta \mathbf{v}}\cdot\nabla\frac{\delta F}{\delta\mu} - \frac{\delta F}{\delta \mathbf{v}}\cdot\nabla\frac{\delta H}{\delta\mu} + \frac{\mathrm{curl}\,\mathbf{v}}{\mu}\cdot\left(\frac{\delta F}{\delta \mathbf{v}}\times\frac{\delta H}{\delta \mathbf{v}}\right)\right\rangle \\ &+ \left\langle \theta\left(\frac{\delta H}{\delta \mathbf{v}}\cdot\nabla\frac{\delta F}{\delta\Theta} - \frac{\delta F}{\delta \mathbf{v}}\cdot\nabla\frac{\delta H}{\delta\Theta}\right)\right\rangle + \left\langle \frac{\delta F}{\delta z}\cdot\frac{\partial z}{\partial t}\right\rangle \\ &+ \left\langle W\left(\frac{\partial}{\partial\eta}\left(\frac{\delta F}{\delta\mu}\right) + \theta\frac{\partial}{\partial\eta}\left(\frac{\delta F}{\delta\Theta}\right) - \frac{1}{\mu}\frac{\delta F}{\delta \mathbf{v}}\cdot\frac{\partial \mathbf{v}}{\partial\eta}\right)\right\rangle\end{aligned}$$

## Finite-Difference

- Define $H(\mu,\mathbf{v},S,\mathbf{z})$ and $\mathbf{U}=\partial H/\partial\mathbf{v},\ldots$
- Poisson bracket $\Rightarrow$ Coriolis and compatible div/grad
- Find $W$ according to type of vertical coordinate
- Solve $\partial H/\partial\mathbf{z}=0$ for $\mathbf{z}$
- Hydrostasy defines $\theta$ at layer interfaces (Simmons & Burridge, 1981)

## Galerkin

- Define spaces $\mathscr{M},\mathscr{V}.\mathscr{S},\mathscr{Z}$ and $H(\mu,\mathbf{v},S,\mathbf{z})$
- Define dual spaces $\mathscr{B},\mathscr{U}.\mathscr{T},\mathscr{W}$ for $\delta H/\delta\mu,\ldots,$
- Poisson bracket $\Rightarrow$ weak form
- Find $W\in\mathscr{W}$ according to type of vertical coordinate
- Enforce $\delta H/\delta\mathbf{z}=0$ weakly, solve for $z$
- $\mathscr{W}=\mathscr{Z}\Rightarrow$ Hydrostasy

# Summary

## Rotating shallow-water equations

- Energy-conserving shallow-water scheme on a non-orthogonal C-grid
- Interpreted as a discrete Poisson bracket
- Sadourny's scheme and TRiSK are discrete Poisson brackets !

## 3D geophysical equations

- Variational principles for HPE in a general vertical coordinate (generalizes Holm & Long, Brice & Roulstone, Bokhove)
- Prognose absolute velocity / angular momentum
- Several possible paths (FD/FE) to energy conservation
- Applications : deep-atmosphere QHE, non-hydrostatic, variable gravity, arbitrary equation of state, moist dynamics, ocean

Conserving energy is "easy", work is needed for :

- good *linear* properties : numerical modes, dispersion, geostrophic balance
- higher-order