Electrodynamic Fields and Flow Fields Combined: Magnetohydrodynamics, Group Theory and Topology

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Basic Quantities

1. The magnetic field \( \vec{B} \) 3 functions.
2. The velocity field \( \vec{V} \) 3 functions.
3. The density field \( \rho \) 1 function.

Total of 7 functions.
Derived quantities

**Current:**

\[ \vec{J} = \frac{\nabla \times \vec{B}}{4\pi} \]

**Pressure:**

\[ p(\rho) \]

For Barotropic flows.
Basic Equations

\[
\begin{align*}
\frac{\partial \vec{B}}{\partial t} &= \vec{\nabla} \times (\vec{v} \times \vec{B}) \\
\vec{\nabla} \cdot \vec{B} &= 0 \\
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) &= 0 \\
\rho \frac{d\vec{v}}{dt} &= \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p(\rho) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi}
\end{align*}
\]

Total of 7 equations – one equation is just an initial condition requirement.
The total magnetic field flux into a closed volume is null – what comes in must come out.
The magnetic field lines are co-moving that is they move with the fluid.

Hence the topology of the field lines is conserved.
Physical Content of the Equation

Ohms law:
\[ \vec{J} = \sigma \left( \vec{E} + \vec{v} \times \vec{B} \right) \]

Take the limit:
\[ \lim_{\sigma} \frac{\vec{J}}{\sigma} = 0 \]  (Current density small & Conductivity very high)

\[ \vec{E} = -\vec{v} \times \vec{B} \]
Physical Content of the Equation

Faraday’s law:

$$\vec{E} = -\vec{v} \times \vec{B}$$

$$\frac{\partial \vec{B}}{\partial t} = -\nabla \times \vec{E}$$

$$\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})$$
Content of the equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \]

The continuity equation – mass is conserved.
Content of the equations

\[
\rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} \right) = -\nabla p(\rho) + \frac{(\nabla \times \vec{B}) \times \vec{B}}{4\pi}
\]

The Euler equation: Newton’s second law for continuous matter moving under the influence of pressure and magnetic forces.
Questions

1. Can those equations be derived from a variational principle?
2. Can the numbers of functions describing the problem be reduced?
Bibliography


Sakurai’s Idea


Let us represent the magnetic field lines as intersection of two co-moving surfaces $\chi$ and $\eta$.

This is essentially introducing a coordinate system connected to the magnetic field lines. The third coordinate parameterizes the distance along the magnetic field lines and was denoted “Magnetic Metage” $\mu$ by Yahalom & Lynden-Bell (2008).
Suppose:

\[ \mathbf{B} = \mathbf{\nabla} \chi \times \mathbf{\nabla} \eta \]

The magnetic field lines are an intersection of two surfaces (Euler potentials)—this is true only for some field topologies.

\[ \frac{d\chi}{dt} = 0, \quad \frac{d\eta}{dt} = 0 \]

The surfaces are co-moving
The two equations:

\[
\frac{d\chi}{dt} = 0, \quad \frac{d\eta}{dt} = 0
\]

Replace the four equations:

\[
\mathbf{\nabla} \cdot \mathbf{B} = 0
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = \mathbf{\nabla} \times (\mathbf{v} \times \mathbf{B})
\]
The Variational Principle

\[ A \equiv \int \mathcal{L}d^3xdt \]

\[ \mathcal{L} \equiv \mathcal{L}_1 + \mathcal{L}_2 \]

\[ \mathcal{L}_1 \equiv \rho \left( \frac{1}{2} \bar{v}^2 - \varepsilon(\rho) \right) + \frac{\bar{B}^2}{8\pi} \]

\[ \mathcal{L}_2 \equiv \nu \left[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \bar{v}) \right] - \rho \alpha \frac{d\chi}{dt} - \rho \beta \frac{d\eta}{dt} - \frac{\bar{B}}{4\pi} \cdot (\nabla \chi \times \nabla \eta) \]
\[ A \equiv \int \mathcal{L} d^3 x dt \]

Is the Action

\[ \mathcal{L} \]

Is the Lagrangian density
The first part of the Lagrangian density contains the Kinetic Energy term and two potential energy terms, one is due to the internal energy (pressure) and the other is due to magnetic fields.
The second part of the Lagrangian density contains four terms which enforce by using Lagrange multipliers:

1. The equation of continuity.
2. The existence of two comoving surfaces.
3. The magnetic field lines being at the intersection of those two surfaces.
The Variational Equations

Variation with respect to $\nu$, $\alpha$ and $\beta$ yields the continuity equations and two equations that express the fact that surfaces $\chi$ and $\eta$ are co-moving:

\[
\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0
\]

\[
\rho \frac{d\chi}{dt} = 0
\]

\[
\rho \frac{d\eta}{dt} = 0.
\]
The Variational Equations

Variation with respect to magnetic field results in the equation:

\[ \vec{\mathcal{B}} = \hat{\mathcal{B}} \equiv \vec{\nabla}_\chi \times \vec{\nabla}_\eta \]

This equation is the Sakurai representation which expresses the fact that the magnetic field lines are the intersection of the two co-moving surfaces.
The Variational Equations

Variation with respect to velocity vector results in the equation:

\[ \delta \vec{v} = \frac{\partial \mathcal{L}}{\partial \vec{v}} + \frac{\partial \mathcal{L}}{\partial \vec{\chi}} + \frac{\partial \mathcal{L}}{\partial \vec{\eta}} \]

This equation is a “Clebsch” like representation of the velocity.
The Variational Equations

Variation with respect to the density results in the Bernoulli type equation:

\[ \frac{dv}{dt} = \frac{1}{2}v^2 - w \]

In which \( w \) is specific enthalpy.
The Variational Equations

Finally variation with respect to chi and eta results in the equations:

\[
\frac{d\alpha}{dt} = \frac{\nabla \eta \cdot \vec{J}}{\rho}, \quad \frac{d\beta}{dt} = -\frac{\nabla \chi \cdot \vec{J}}{\rho}
\]
Relations to the Physical Equations

The two equations:

\[
\frac{d\chi}{dt} = 0, \quad \frac{d\eta}{dt} = 0
\]

Replace the four equations:

\[
\nabla \cdot \vec{B} = 0
\]

\[
\frac{\partial \vec{B}}{\partial t} = \nabla \times (\vec{v} \times \vec{B})
\]
Relations to the Physical Equations

Mass conservation is also assured:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0$$
Relations to the Physical Equations

The Euler equations are obtained by:

\[
\frac{d\vec{v}}{dt} = \frac{d\nabla \nu}{dt} + \frac{d\alpha}{dt} \nabla \chi + \alpha \frac{d\nabla \chi}{dt} + \frac{d\beta}{dt} \nabla \eta + \beta \frac{d\nabla \eta}{dt}
\]
After substituting the relevant equations we obtain the Euler equations

\[
\frac{d \vec{v}}{dt} = -\vec{\nabla} v_k \left( \frac{\partial \nu}{\partial x_k} + \alpha \frac{\partial \chi}{\partial x_k} + \beta \frac{\partial \eta}{\partial x_k} \right) + \vec{\nabla} \left( \frac{1}{2} \vec{v}^2 - w \right) \\
+ \frac{1}{\rho} \left( \left( \vec{\nabla} \eta \cdot \vec{J} \right) \vec{\nabla} \chi - \left( \vec{\nabla} \chi \cdot \vec{J} \right) \vec{\nabla} \eta \right) \\
= -\vec{\nabla} v_k v_k + \vec{\nabla} \left( \frac{1}{2} \vec{v}^2 - w \right) + \frac{1}{\rho} \vec{J} \times \left( \vec{\nabla} \chi \times \vec{\nabla} \eta \right) \\
= -\vec{\nabla} p \frac{1}{\rho} + \frac{1}{\rho} \vec{J} \times \vec{B}
\]
Intermediate Account

1. We have succeeded in deriving a variational principle for magnetohydrodynamics of the topology assumed.

2. The variational equation contains the 7 physical variables + alpha, beta, nu, chi, eta 5 functions = 12 functions. This is too much!

3. Solution rearrange terms in the Lagrangian!
The Simplified Variational Principle

\[ \mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}_v + \mathcal{L}_{\vec{B}} + \mathcal{L}_{\text{boundary}} \]

\[ \hat{\mathcal{L}} = -\rho \left[ \frac{\partial \nu}{\partial t} + \alpha \frac{\partial \chi}{\partial t} + \beta \frac{\partial \eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2} (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta)^2 \right] \]

\[ -\frac{1}{8\pi} (\nabla \chi \times \nabla \eta)^2 \]

\[ \mathcal{L}_v = \frac{1}{2} \rho (\tilde{v} - \hat{v})^2 \]

\[ \mathcal{L}_{\vec{B}} = \frac{1}{8\pi} (\vec{B} - \hat{B})^2 \]

\[ \mathcal{L}_{\text{boundary}} = \frac{\partial (\nu \rho)}{\partial t} + \nabla \cdot (\nu \rho \vec{v}) \]
The terms:

\[ \mathcal{L}_{\vec{v}} \equiv \frac{1}{2} \rho (\vec{v} - \hat{v})^2 \]
\[ \mathcal{L}_{\vec{B}} \equiv \frac{1}{8\pi} (\vec{B} - \hat{B})^2 \]

Result in the equations:

\[ \vec{v} = \hat{v} \equiv \vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta \]

\[ \vec{B} = \hat{B} \equiv \vec{\nabla} \chi \times \vec{\nabla} \eta \]
The magnetic and velocity fields can be calculated using alpha, beta, nu, chi, eta after the relevant equations for those quantities are solved.
Hence we are left with:

\[
\mathcal{L} = \hat{\mathcal{L}} + \mathcal{L}_\pi + \mathcal{L}_B + \mathcal{L}_{\text{boundary}}
\]

\[
\hat{\mathcal{L}} \equiv -\rho \left[ \frac{\partial \nu}{\partial t} + \alpha \frac{\partial \chi}{\partial t} + \beta \frac{\partial \eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2} \left( \nabla \nu + \alpha \nabla \chi + \beta \nabla \eta \right)^2 \right]
\]

\[
- \frac{1}{8\pi} (\nabla \chi \times \nabla \eta)^2
\]

\[
\mathcal{L}_\pi \equiv \frac{1}{2} \rho (\vec{v} - \hat{\mathbf{v}})^2
\]

\[
\mathcal{L}_B \equiv \frac{1}{8\pi} \left( \vec{B} \right)^2
\]

\[
\mathcal{L}_{\text{boundary}} \equiv \frac{\partial (\nu \rho)}{\partial t} + \nabla \cdot (\nu \rho \vec{v})
\]
Or with:

\[
\hat{\mathcal{L}} \equiv -\rho \left[ \frac{\partial \nu}{\partial t} + \alpha \frac{\partial \chi}{\partial t} + \beta \frac{\partial \eta}{\partial t} + \varepsilon(\rho) + \frac{1}{2}(\vec{\nabla} \nu + \alpha \vec{\nabla} \chi + \beta \vec{\nabla} \eta)^2 \right] \\
- \frac{1}{8\pi} (\vec{\nabla} \chi \times \vec{\nabla} \eta)^2
\]
The final Lagrangian contains only 6 functions and yields 6 equations:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) &= 0 \\
\rho \frac{d\chi}{dt} &= 0 \\
\rho \frac{d\eta}{dt} &= 0 \\
\frac{d\nu}{dt} &= \frac{1}{2} \vec{v}^2 - w \\
\frac{d\alpha}{dt} &= \frac{\nabla \eta \cdot \vec{J}}{\rho}, \\
\frac{d\beta}{dt} &= -\frac{\nabla \chi \cdot \vec{J}}{\rho}
\end{align*}
\]
Analogy to Electromagnetic Theory

Potentials (vector + scalar):

\[ E = -\nabla \phi - \frac{\partial A}{\partial t} \]
\[ B = \nabla \times A. \]

\[ \mathcal{L} = -\frac{1}{16\pi} F_{\alpha\beta} F^{\alpha\beta} - \frac{1}{c} J_\alpha A^\alpha \]

\[ F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \]
Intermediate Account

1. We have managed to reduce the magnetohydrodynamic equations from 7 equations containing 7 quantities to 6 equations with 6 variables.

2. Furthermore we have managed to represent magnetohydrodynamics as a Variational Problem in terms of those 6 variables.

3. Can one reduce the number of functions further? The answer is yes.
Consider the two equations:

\[
\frac{d\chi}{dt} = \frac{\partial \chi}{\partial t} + \vec{v} \cdot \nabla \chi = \frac{\partial \chi}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \chi = 0,
\]

\[
\frac{d\eta}{dt} = \frac{\partial \eta}{\partial t} + \vec{v} \cdot \nabla \eta = \frac{\partial \eta}{\partial t} + (\nabla \nu + \alpha \nabla \chi + \beta \nabla \eta) \cdot \nabla \eta = 0,
\]

Those can be viewed as a set of two equations with two variables for \( \alpha \) and \( \beta \).
The solution is:

\[
\alpha[\chi, \eta, \nu] = \frac{(\vec{\nabla}\eta)^2 \left( \frac{\partial \chi}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\chi \right) - (\vec{\nabla}\eta \cdot \vec{\nabla}\chi) \left( \frac{\partial \eta}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\eta \right)}{(\vec{\nabla}\eta \cdot \vec{\nabla}\chi)^2 - (\vec{\nabla}\eta)^2 (\vec{\nabla}\chi)^2}
\]

\[
\beta[\chi, \eta, \nu] = \frac{(\vec{\nabla}\chi)^2 \left( \frac{\partial \eta}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\eta \right) - (\vec{\nabla}\eta \cdot \vec{\nabla}\chi) \left( \frac{\partial \chi}{\partial t} + \vec{\nabla}\nu \cdot \vec{\nabla}\chi \right)}{(\vec{\nabla}\eta \cdot \vec{\nabla}\chi)^2 - (\vec{\nabla}\eta)^2 (\vec{\nabla}\chi)^2}
\]

Those expressions can now be inserted into the equation of the velocity:

\[
\vec{v} = \vec{\nabla}\nu + \alpha[\chi, \eta, \nu] \vec{\nabla}\chi + \beta[\chi, \eta, \nu] \vec{\nabla}\eta
\]
And we obtain after some manipulations:

\[
\bar{v} = \vec{\nabla} \nu + \frac{1}{\bar{B}^2} \left[ \frac{\partial \eta}{\partial t} \vec{\nabla} \chi - \frac{\partial \chi}{\partial t} \vec{\nabla} \eta + \vec{\nabla} \nu \times \bar{B} \right] \times \bar{B}
\]

\[
= \frac{1}{\bar{B}^2} \left[ \left( \frac{\partial \eta}{\partial t} \vec{\nabla} \chi - \frac{\partial \chi}{\partial t} \vec{\nabla} \eta \right) \times \bar{B} + \bar{B} (\vec{\nabla} \nu \cdot \bar{B}) \right]
\]
Thus the velocity field is partitioned naturally into a tangential and normal components with respect to the magnetic field:

\[ \vec{v} = \vec{v}_\perp + \vec{v}_\parallel \]

\[ \vec{v}_\perp = \frac{1}{B^2} \left( \frac{\partial \eta}{\partial t} \vec{\nabla} \chi - \frac{\partial \chi}{\partial t} \vec{\nabla} \eta \right) \times \vec{B}, \quad \vec{v}_\parallel = \frac{\vec{B}}{B^2} (\vec{\nabla} \nu \cdot \vec{B}) \]

By construction the two equations below are now satisfied automatically:

\[ \frac{d\chi}{dt} = 0, \quad \frac{d\eta}{dt} = 0 \]
Inserting those expressions yields a four function Lagrangian:

\[ \mathcal{L}[\chi, \eta, \nu, \rho] \equiv -\rho \left[ \frac{\partial \nu}{\partial t} + \alpha[\chi, \eta, \nu] \frac{\partial \chi}{\partial t} + \beta[\chi, \eta, \nu] \frac{\partial \eta}{\partial t} + \varepsilon(\rho) \right] + \frac{1}{2} (\nabla \nu + \alpha[\chi, \eta, \nu] \nabla \chi + \beta[\chi, \eta, \nu] \nabla \eta)^2 \]

\[ - \frac{1}{8\pi} (\nabla \chi \times \nabla \eta)^2. \]
Which can also be written as:

\[ \mathcal{L}[\chi, \eta, \nu, \rho] = \rho \left[ \frac{1}{2} \vec{v}^2 - \frac{d\nu}{dt} - \varepsilon(\rho) \right] - \frac{1}{8\pi} \vec{B}^2 \]

Or more explicitly as:

\[
\mathcal{L}[\chi, \eta, \nu, \rho] = \frac{1}{2} \frac{\rho}{(\nabla \chi \times \nabla \eta)^2} \left[ \nabla \eta \frac{\partial \chi}{\partial t} - \nabla \chi \frac{\partial \eta}{\partial t} + (\nabla \chi \times \nabla \eta) \times \nabla \nu \right]^2 \\
- \rho \left[ \frac{\partial \nu}{\partial t} + \frac{1}{2} (\nabla \nu)^2 + \varepsilon(\rho) \right] - \frac{(\nabla \chi \times \nabla \eta)^2}{8\pi}.
\]
We are left with the four variational equations:

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0, \]

\[ \frac{d\nu}{dt} = \frac{1}{2} \vec{v}^2 - w, \]

\[ \frac{d\alpha[\chi, \eta, \nu]}{dt} = \frac{\nabla \eta \cdot \vec{J}}{\rho}, \]

\[ \frac{d\beta[\chi, \eta, \nu]}{dt} = -\frac{\nabla \chi \cdot \vec{J}}{\rho}. \]
Conclusion

We have proved that Barotropic magnetohydrodynamics for the topologies assumed is a four function field theory which is defined by a Lagrangian.

From this Lagrangian the equations and boundary conditions of the theory can be derived.
Group Theory
\[ \mathcal{L}[\chi, \eta, \nu, \rho] = \frac{1}{2} \frac{\rho}{(\nabla \chi \times \nabla \eta)^2} [\nabla \eta \frac{\partial \chi}{\partial t} - \nabla \chi \frac{\partial \eta}{\partial t} + (\nabla \chi \times \nabla \eta) \times \nabla \nu]^2 \\
- \rho \left[ \frac{\partial \nu}{\partial t} + \frac{1}{2} (\nabla \nu)^2 + \varepsilon(\rho) \right] - \frac{(\nabla \chi \times \nabla \eta)^2}{8\pi}. \]

Notice that \( \chi \) and \( \eta \) are not defined uniquely!
Notice the infinite symmetry group:

\[ \hat{\eta} = \hat{\eta}(\chi, \eta), \quad \hat{\chi} = \hat{\chi}(\chi, \eta) \]

Such that:

\[ \left| \frac{\partial (\hat{\eta}, \hat{\chi})}{\partial (\eta, \chi)} \right| = 1 \]
Noether’s Theorem

Amalie Emmy Noether (23 March 1882 – 14 April 1935) was an influential German (Jewish) mathematician known for her groundbreaking contributions to abstract algebra and theoretical physics. Described by Hilbert, Einstein and others as the most important woman in the history of mathematics, she revolutionized the theories of rings, fields, and algebras.
In physics, Noether's theorem explains the fundamental connection between symmetry and conservation laws.

Every Continuous Symmetry group generates conserved currents.
\[ \hat{\eta} = \eta + \delta \eta(\chi, \eta), \quad \hat{\chi} = \chi + \delta \chi(\chi, \eta), \]

To first order:

\[ \partial_\eta \delta \eta + \partial_\chi \delta \chi = 0. \]

\[ \delta \eta = \partial_\chi \delta f, \quad \delta \chi = -\partial_\eta \delta f, \]
Assuming that the relevant equations of motion and boundary conditions hold, we have:

\[ \delta_H L = -\frac{d}{dt} \int d^3 x \rho \alpha \delta \chi, \quad \delta_H L = -\frac{d}{dt} \int d^3 x \rho \beta \delta \eta \]

\[ \delta_H L + \delta_H L = -\frac{d}{dt} \int d^3 x \rho (\alpha \delta \chi + \beta \delta \eta), \]
\[ \frac{d}{dt} \int d^3x \rho (\alpha \partial_\eta \delta f - \beta \partial_\chi \delta f) = 0. \]

\[ \delta G = \int d\chi d\eta d\mu (\alpha \partial_\eta \delta f - \beta \partial_\chi \delta f) = \int d\chi d\mu \alpha \delta f \bigg|_{\eta_1}^{\eta_2} - \int d\eta d\mu \beta \delta f \bigg|_{\chi_1}^{\chi_2} \]

\[ + \int d\chi d\eta d\mu \delta f (\partial_\chi \beta - \partial_\eta \alpha). \]
Since \( \delta f \) is arbitrary, we arrive at the following conserved Noether current:

\[
J = \int d\mu (\partial_\chi \beta - \partial_\eta \alpha).
\]
Intermediate Account

- The MHD Lagrangian has a Diffeomorphism Symmetry Group.
- The Noether current connected with the symmetry group was derived.
- A physical interpretation of the conserved current is still lacking.
Topology
Aharonov – Bohm Effect

\[ \vec{B} = \vec{\nabla} \times \vec{A} = 0 \Rightarrow \vec{A} = \vec{\nabla} \vec{S} \]

\( \vec{S} \) is not single valued.

\[ \Phi = \int \vec{B} \cdot d\vec{S} = \oint \vec{A} \cdot d\vec{l} = [\vec{S}] \]
Aharonov-Bohm 1959 Example

\[ \vec{A} = A_\theta \hat{\theta} = \frac{\Phi}{2\pi r} \hat{\theta} = \vec{\nabla} \vec{S} \implies \vec{S} = \frac{\Phi}{2\pi} \theta + \vec{S}_0 \]
Main features of the Aharanov - Bohm effect

1. A domain that is not simply connected, but can be made simply connected by introducing a cut. Mathematically speaking the domain has a non-trivial fundamental Homotopy group.

2. The electron (or its wave function) do not feel the magnetic field – non locality.

3. The potential vector field is a gradient of a non-single valued function.

4. Gauge freedom is not gone but only limited to single-valued gauges.
Bohm’s Causal Interpretation of Quantum Mechanics: Quantum - Classical Correspondence

According to Bohm the phase of a wave function should be interpreted as a potential of the velocity field:

\[ \vec{V} = \frac{1}{m} \nabla S \]

However, this correspondence can go the other way around!! If the velocity field has a potential part it can be interpreted as a phase of a wave function.
Topological Constants


A MHD flow has two topological constants of motion:

1. Magnetic Helicity

\[ \mathcal{H}_M \equiv \int B \cdot A \, d^3x \]

\[ B = \nabla \times A \]

Which is known to measure the degree of knottiness of lines of the magnetic field B.
Topological Constants

2. Cross Helicity

\[ \mathcal{H}_c \equiv \int B \cdot v \, d^3x \]

Characterizing the degree of cross-knottiness of the magnetic field and velocity lines.
Magnetic Aharonov-Bohm Effect

Using Sakurai representation one can show that:

\[ A = \chi \nabla \eta + \nabla \zeta \]

\[ B \cdot A = (\nabla \chi \times \nabla \eta) \cdot \nabla \zeta \]

Hence:

\[ B \cdot A = \frac{\partial \zeta}{\partial \mu} (\nabla \chi \times \nabla \eta) \cdot \nabla \mu = \frac{\partial \zeta}{\partial \mu} \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)} \]

\( \mu \) is a variable changing along magnetic field lines denoted by Yahalom & Lynden-Bell (2008) as the magnetic metage.
Magnetic Aharonov-Bohm Effect

The magnetic helicity takes the form:

\[ \mathcal{H}_M = \int \frac{\partial \zeta}{\partial \mu} \, d\mu \, d\chi \, d\eta \]

Integrating along a single magnetic field line:

\[ \oint_{\chi,\eta} \frac{\partial \zeta}{\partial \mu} \, d\mu = [\zeta]_{\chi,\eta} \]

Hence Zeta must be discontinuous (non single valued) in the case that the magnetic helicity is not null.
Main features of the Magnetic Aharonov-Bohm effect

1. A domain that is not simply connected, since the internal magnetic flux is knotted inside the external magnetic flux line.
2. The external magnetic field line does not touch the internal flux yet the $\zeta$ function is not single valued due to that line – non locality.
3. The potential vector field has a gradient of a non-single valued function part.
4. Gauge freedom is not gone but only limited to single-valued gauges.
Magnetic Helicity

Thus we obtain:

\[ H_M = \int [\zeta]_{\chi,\eta} \, d\chi \, d\eta = \int [\zeta] \, d\Phi \]

This result can be rephrased as:

\[ [\zeta] = \frac{dH_M}{d\Phi} \]

Hence the discontinuity of Zeta is the magnetic helicity per unit flux.
Cross Helicity

In a completely analogue way:

\[ v \cdot B = \frac{\partial v}{\partial \mu} (\nabla \chi \times \nabla \eta) \cdot \nabla \mu = \frac{\partial v}{\partial \mu} \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)} \]

Hence the cross helicity can be written as:

\[ \mathcal{H}_C = \int \frac{\partial v}{\partial \mu} d\mu \, d\chi \, d\eta \]
Cross Aharonov-Bohm Effect

Integrating along a single magnetic field line:

\[ \int_{x,\eta} \frac{\partial \nu}{\partial \mu} d\mu = [\nu]_{x,\eta} \]

Hence \( \nu \) must be discontinuous (non single valued) in the case that the cross helicity is not null. Thus:

\[ \mathcal{H}_C = \int [\nu]_{x,\eta} d\chi d\eta = \int [\nu] d\Phi \]
Main features of the Cross Aharanov-Bohm effect

1. A domain that is not simply connected, since the internal magnetic flux is knotted inside the external vortex line.
2. The external vortex line does not touch the internal flux yet the nu function is not single valued due to that line – non locality.
3. The velocity field has a gradient of a non-single valued function part, this part is interpreted as a phase according to Bohm’s causal interpretation correspondence.
Cross Helicity

The result can be rephrased as:

\[ [\nu] = \frac{d\mathcal{H}}{d\Phi} c \]

Hence the discontinuity of \( \nu \) is the cross helicity per unit flux. Moreover:

\[ \frac{d\nu}{dt} = \frac{1}{2} \nu^2 - w \]
Cross Helicity

Since the right hand side is single valued so must be the left hand side, hence:

$$\frac{d[v]}{dt} = 0$$

Thus not only the cross helicity is conserved but also the cross helicity per unit flux is conserved as well:

$$[v] = \frac{dHc}{d\Phi}$$

This is a new conservation law of MHD!
Conclusion

It is shown that there are two inherent Aharonov - Bohm effects in MHD, in each case a magnetic flux induces a "phase" on quantities that do not come under the influence of the magnetic field directly. Those quantities include the velocity fields and "external" magnetic field. Those phases quantify two well known Topological conservation laws of the magnetic and cross helicities. One of those phases is useful for introducing a very efficient variational principle for MHD which is given in terms of only four independent functions for non-stationary flows of appropriate topology. This is less than the seven variables which appear in the standard equations of MHD.
An Analytic Example
Analogy with Moffat 1969:

\[ A = \nabla \psi \times \nabla \phi + \alpha \psi \nabla \phi = \frac{1}{R} \partial_R \psi \hat{z} - \frac{1}{R} \partial_z \psi \hat{R} + \frac{\alpha \psi}{R} \hat{\phi}, \quad \nabla \phi = \frac{\hat{\phi}}{R} \]

\[ R, \phi, z \text{ are the standard cylindrical coordinates} \]

\[ B = \frac{\alpha}{R} \partial_R \psi \hat{z} - \frac{\alpha}{R} \partial_z \psi \hat{R} - \frac{D^2 \psi}{R} \hat{\phi} \]

\[ D^2 = \partial^2_z + R \partial_R \left( \frac{1}{R} \partial_R \right) \]
\[ \nabla \psi \cdot A = \nabla \psi \cdot B = 0 \]

Let us define the variable \( r \):

\[ r = \sqrt{z^2 + (R - 1)^2} \]

\( \psi = \psi(r) \). In this case surfaces of constant \( \psi \) are nested tori.
A numerically integrated field line assuming that $\Psi = r + r^3$, $\alpha = 1$ and starting from the point $R = 0.6$, $\phi = 0$, $z = 0$. The plot shows twenty rotations.
\[ \chi = \frac{1}{2\pi} \int \mathbf{B} \cdot d\mathbf{S} = \frac{1}{2\pi} \oint \mathbf{A} \cdot dl = \frac{1}{2\pi} \int_0^{2\pi} A_\phi R \, d\phi = A_\phi R = \alpha \Psi \]

\[ \eta = \phi + C(z, R) \]

\[ C = \frac{1}{\alpha} \left[ \frac{r \Psi''}{\Psi'} I(r, \eta^*) + II(r, \eta^*) \right] \]

\[ I(r, \eta^*) \equiv \int \frac{d\eta^*}{1 + r \cos \eta^*} = \frac{2}{\sqrt{1 - r^2}} \left[ \arctan \left( \sqrt{\frac{1 - r}{1 + r}} \tan \left( \frac{\eta^*}{2} \right) \right) + \left\{ \begin{array}{l} 0, \quad 0 \leq \eta^* < \pi \\ \pi, \quad \pi \leq \eta^* < 2\pi. \end{array} \right. \right] \]

\[ II(r, \eta^*) \equiv \int \frac{d\eta^*}{(1 + r \cos \eta^*)^2} = \frac{I(r, \eta^*)}{1 - r^2} - \frac{r \sin \eta^*}{(1 - r^2)(1 + r \cos \eta^*)} \]
\[
\zeta(r, \eta^*) = r \Psi' I(r, \eta^*) - \alpha \Psi C = r I(r, \eta^*) \left( \psi' - \frac{\Psi \Psi''}{\psi'} \right) - \Psi I I(r, \eta^*).
\]

\[
[\zeta(r, \eta^*)] = \frac{2\pi}{\sqrt{1 - r^2}} \left( r \left( \psi' - \frac{\Psi \Psi''}{\psi'} \right) - \frac{\Psi}{1 - r^2} \right).
\]

\[
\mathcal{H}_M = \int [\zeta] d\Phi = \int_0^a [\zeta] 2\pi \alpha \psi' dr
\]

\[
= (2\pi)^2 \alpha \int_0^a \frac{dr}{\sqrt{1 - r^2}} \left( r((\psi')^2 - \Psi \Psi'') - \frac{\Psi \Psi'}{1 - r^2} \right).
\]

Same can be obtained by a direct calculation using the traditional formula.