Helicity, Cohomology, and Configuration Spaces

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Helicity is a cohomology class

Today's Goal

We demonstrate that helicity is a cup product in cohomology.
Therefore, it is a diffeomorphism invariant.

Let Diff($\mathbf{R}^{2k+1}$) be the diffeomorphism group of $\mathbf{R}^{2k+1}$.
Let SDiff be the diffeos. homotopic to the identity.

Question

Are there other SDiff *invariants of vector fields?*

- Obstructions to other invariants
- What we gain by this cohomological view of helicity.

Diffeomorphism invariance

Question

Under which diffeomorphisms $\Omega \to \Omega'$ is helicity an invariant?

Theorem (Invariance of helicity theorem; Arnol'd, Cantarella-DeTurck-Gluck-Tetyel)

1. *Helicity is invariant under any volume-preserving element of* $\mathrm{SDiff}(\Omega)$.
2. *If* Ω *is simply connected or the vector field is fluxless, helicity is invariant under any volume-preserving diffeo. in* $\mathrm{Diff}(\Omega)$.

Question

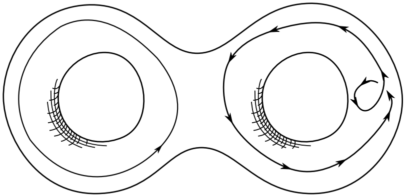
Do other helicity-preserving diffeos exist?

Cohomology

Suppose we want to answer: for a curl-free field V, does $V = \nabla f$?

Cohomology.

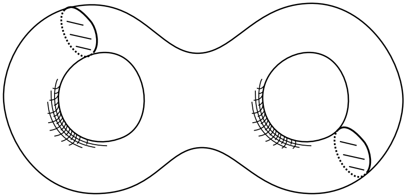
answer depends on the periods of the field: integrals around certain curves in the domain.



Cohomology is a way of keeping track of periods for differential forms.

Relative Cohomology.

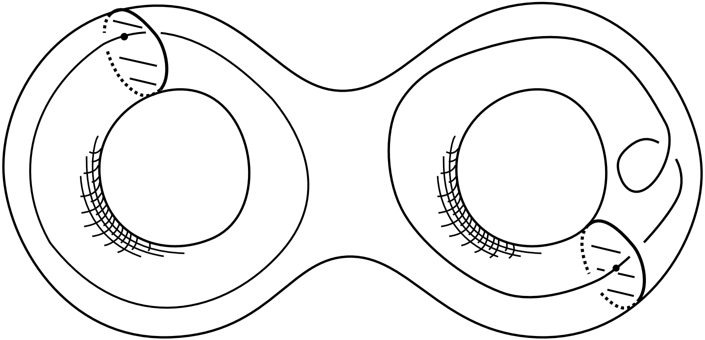
answer depends on the fluxes of the field: integrals over certain surfaces in the domain.



Relative cohomology is a way of keeping track of fluxes for differential forms. It is “relative” to the boundary of the domain.

Poincaré Duality and the Cup Product

It turns out that curves and spanning surfaces are dual to one another, and so periods and fluxes are dual to one another as well.



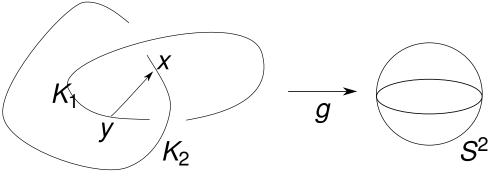
Idea

The cup product is an operation which pairs periods and fluxes and corresponds to the pairing of curves and spanning surfaces. In differential forms, it is the wedge product.

Definition of Linking

Definition

The linking number of two curves a and b is the degree of the map $g\colon S^1 \times S^1 \to S^2$ given by $(x, y) \mapsto x - y/|x - y|$.



Equivalently, it is the (de Rham) cohomology class of the pullback of the volume form on S^2 over $S^1 \times S^1$.

$$\mathrm{Lk}(K_1, K_2) = \int_{S^1 \times S^1} g^* \, \mathrm{dVol}_{S^2} \, .$$

The point is that $S^1 \times S^1$ has cohomology in dimension 2.

Helicity and the Gauss Linking Integral

In coordinates, Gauss (1833) showed

$$\mathrm{Lk}(K_1, K_2) = \frac{1}{\mathrm{Vol}(S^2)} \int_{S^1 \times S^1} x'(\theta) \times y'(\phi) \cdot \frac{x(\theta) - y(\phi)}{|x(\theta) - y(\phi)|^3} d\theta\, d\phi.$$

Definition

For a divergence-free vector field V tangent to boundary of a subdomain Ω, the **helicity** of V is defined by

$$\mathrm{H}(V) = \frac{1}{\mathrm{Vol}(S^2)} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x - y}{|x - y|^3}\, \mathrm{dVol}_x\, \mathrm{dVol}_y$$

Our goal: recognize helicity in an algebraic topology framework.

However, $\Omega \times \Omega$ may not have any cohomology in dimension 2.

Outline of rest of talk

- Differential forms are more natural than vector fields for computing helicity.
- All results extend from $\mathbf{R}^3$ to $\mathbf{R}^{2k+1}$ – take helicity of $(k+1)$-form there.

1. Configuration spaces
2. We give 3 equivalent definitions of helicity (via cohomology, Bott-Taubes, potentials).
3. We show helicity of forms is invariant under SDiff.
4. Moreover, we find formula for change of helicity under arbitrary diffeo.
5. In particular, $\exists$ diffeos $\notin$ SDiff on double solid torus that preserve helicity
6. Applications
7. Examine the obstacles to generalizing helicity and producing SDiff invariants

Vector fields and forms: the basic setup

Suppose we have a domain $\Omega \subset \mathbf{R}^3$ with smooth boundary.

- Vector fields in $\mathbf{R}^3 \iff$ 2-forms in $\mathbf{R}^3$ by

$$\alpha(W_1, W_2) = \mathrm{Vol}(V, W_1, W_2).$$

- Divergence-free vector fields tangent to the boundary of $\Omega \iff$ closed 2-forms which vanish when pulled back to $\partial\Omega$, *closed Dirichlet 2-form*.
- α closed Dirichlet on Eucl. subdomain $\Omega \Rightarrow \alpha$ exact

Question

α represents a relative cohomology class.
But Ω might not have any cohomology.

So how can we recognize helicity in terms of cohomology?

Answer: Configuration spaces

Configuration Spaces

Issue

The integrand in the definition of helicity is not really defined on the diagonal of $\Omega \times \Omega$. To define helicity as integral of a form, we must handle the diagonal singularity.

$$\mathrm{H}(V) = \frac{1}{\mathrm{Vol}(S^2)} \int_{\Omega \times \Omega} V(x) \times V(y) \cdot \frac{x-y}{|x-y|^3} \,\mathrm{dVol}_x \,\mathrm{dVol}_y$$

Definition

Consider taking n points on a space X. The set of distinct n-tuples forms the **configuration space** $C_n(X)$.

If $\Omega \subset \mathbf{R}^3$ then the configuration space

$$C_2(\Omega) = \Omega \times \Omega - \{\text{diagonal}\}.$$

$C_2(\Omega)$ is no longer compact.

Configuration spaces, compactified

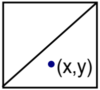
Fulton-MacPherson Compactification

Compactify to a manifold with boundary (and corners) $C_2[\Omega]$.

n.b., The space $C_2[\Omega]$ has a new cohomology class which is nontrivial even if Ω is contractible.

Examples.

1. $C_2[0,1]$



2. $C_3[S^1]$ is 2 solid tori

Note: The topology of $C_n[X]$ will inherit the topology of X, plus extra structure.

Example 3

Suppose that Ω is a closed disk D^{2k+1}.
Then $\Omega \times \Omega = D^{4k+2}$. But,

$$\begin{aligned} \Omega \times \Omega - \Delta &\cong D^{2k+1} \times (D^{2k+1} - \{pt\}) \\ &\cong D^{2k+1} \times S^{2k} \times (0,1] \\ &\simeq D^{2k+2} \times S^{2k} \end{aligned}$$

In $\mathbf{R}^3$, $\Omega \times \Omega - \Delta \simeq D^4 \times S^2$.

By Poincare duality (in real coefficients),

$$H^{2k+2}(C_2[\Omega], \partial C_2[\Omega]) \cong H^{2k}(S^{2k}) \cong \mathbf{R}.$$

Lemma for constructing helicity

On Ω^3, start with field V, divergence-free, tangent to the boundary. Make the corresponding 2-form α, which is closed, Dirichlet.

Consider the projection map

$$\pi_x : C_2[\Omega] \to \Omega$$

Lemma

1. *α pulls back to a pair of closed forms α_x, α_y on $C_2[\Omega]$.*

$$\alpha_x := \pi_x^* \alpha, \qquad \alpha_y := \pi_y^* \alpha$$

2. *$\alpha_x \wedge \alpha_y$ is closed on $C_2[\Omega]$.*
3. *$\alpha_x \wedge \alpha_y$ vanishes on $\partial C_2[\Omega]$.*
4. *Hence $[\alpha_x \wedge \alpha_y]$ represents in $H^4(C_2[\Omega], \partial C_2[\Omega])$.*

Proof: Only effort is to prove (3) – on the "diagonal" face diffeomorphic to $\Omega \times S^2$; a calculation in coordinates.

Cup Product Definition of Helicity

We can now define helicity for $(k+1)$-forms on Ω^{2k+1}:

Definition

The *Gauss map* $g\colon\ C_2(\Omega) \to S^{2k}$ is given by $(x,y) \mapsto \dfrac{x-y}{|x-y|}$.

Lemma

g is a smooth map defined on all of $C_2[\Omega]$, including the boundary. If $\mathrm{dVol}_{S^{2k}}$ is the unit volume form on S^{2k}, $g^*\,\mathrm{dVol}$ is a closed $2k$-form on $C_2[\Omega]$ and represents in $H^{2k}(C_2[\Omega])$.

Definition (Helicity as a cup product)

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^*\,\mathrm{dVol}_{S^{2k}} = [\alpha_x \wedge \alpha_y] \cup [g^*\,\mathrm{dVol}_{S^{2k}}](C_2[\Omega]).$$

Therefore, helicity is SDiff invariant – the cup product structure is preserved under SDiff action.

Helicity only works for $(4i+3)$-dimensions

All of our results are for odd-dimensional subdomains of Euclidean space: take α, a closed Dirichlet $(k+1)$-form on $\Omega^{2k+1} \subset \mathbf{R}^{2k+1}$

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \alpha_x \wedge \alpha_y \wedge g^* \, \mathrm{dVol}_{S^{2k}}$$

Proposition

For k even, helicity is a trivial invariant, i.e., $\mathrm{H}(\alpha) \equiv 0$.

Proof: Interchanging x, y preserves helicity but reverses the orientation of the configuration space.

Integrand changes by factor of $(-1)^k$, so overall change $(-1)^{k+1}$. So k even $\Rightarrow$ helicity trivial.

Bott-Taubes integration

Theorem (Bott-Taubes)

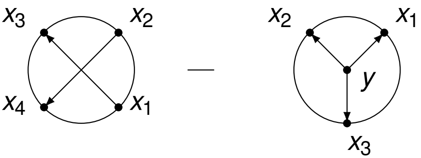
Let $\mathscr{K}$ be the space of embeddings of knots into $\mathbf{R}^3$. Certain knot invariants (Vassiliev invts of type $n/2$) may be computed from the following diagram, where g pairs the configuration points via Gauss maps.

$$\begin{array}{ccc} \alpha = g^*\omega & \xleftarrow{g^*} & \omega = \mathrm{dVol} \\ C_n[S^1] \times \mathscr{K} & \xrightarrow{g} & (S^2)^{n/2} \\ \downarrow I & & \\ \mathscr{K} & & \end{array}$$

Here I represents integration against the first factor:

$$I : \alpha \mapsto \int_{C_n[S^1]} \alpha$$

Computing the Vassiliev type 2 invariant



X integral uses $C_4[S^1]$ with $g = (g_{13}, g_{24})$

$$dI_X = d\int_{C_4[S^1]} g_{13}^*\omega \, g_{24}^*\omega \; = -\int_{\partial C_4[S^1]} g_{13}^*\omega \, g_{24}^*\omega \; \neq 0$$

But we cancel these terms via appropriate terms from the

Y integral, uses $C_{3;1}[S^1; \mathbf{R}^3]$ with $g = (g_{y1}, g_{y2}, g_{y3})$.

$$d(I_X - cI_Y) = 0 \qquad \text{(constant } c\text{)}$$

Thus, $v_2(K) := I_X(K) - cI_Y(K)$ is the Vassiliev type 2 invt

Helicity via the Bott-Taubes approach

Let $\mathfrak{E} = \mathfrak{Embed}(f : \Omega \hookrightarrow \mathbf{R}^{2k+1})$ be the space of embeddings of $\Omega \hookrightarrow \mathbf{R}^{2k+1}$. Write down the (trivial) bundle Now construct the closed $(4k+2)$-form Φ:

$$\begin{array}{ccc} \Phi = \alpha_x \wedge \alpha_y \wedge g_f^* \omega & \xleftarrow{g_f^*} & \omega = \mathrm{dVol} \\ C_2[\Omega] \times \mathfrak{E} & \xrightarrow{g_f} & S^{2k} \\ \big\downarrow I & & \\ \mathfrak{E} & & \end{array}$$

Integrate this form over the fiber $C_2[\Omega]$ in this bundle to get helicity as a 0-form $H(f)$ on $\mathfrak{Embed}$.

$$dH(f) = \int_{C_2[\Omega]} d\Phi - \int_{\partial C_2[\Omega]} \Phi = 0.$$

So helicity is constant on connected components of $\mathfrak{Embed}$,
i.e., on homotopy classes of diffeomorphisms.

Definition #3: Helicity in terms of a potential

Fix the first coordinate x in $C_2[\Omega]$ to form $C_{2,1}[\Omega]$.

$$\begin{array}{ccc} C_{2,1}[\Omega] & \xrightarrow{\;i\;} & C_2[\Omega] \\ & & \big\downarrow \pi_x \\ & & \Omega \end{array}$$

where π_x is the projection where $(x, y) \mapsto x$.

Definition

The *Biot-Savart operator for forms* maps $(k+1)$-forms to k-forms via

$$\mathrm{BS}(\alpha) = \frac{1}{\mathrm{Vol}(S^{2k})} \int_{C_{2,1}[\Omega]} \alpha_y \wedge g^* \mathrm{Vol}.$$

Helicity in terms of a potential

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The *Biot-Savart operator for forms* maps $(k+1)$-forms to k-forms via

$$\mathrm{BS}(\alpha) = \frac{1}{\mathrm{Vol}(S^{2k})} \int_{C_{2,1}[\Omega]} \alpha_y \wedge g^* \mathrm{Vol}.$$

Proposition

If α is closed Dirchlet, then $d(\mathrm{BS}(\alpha)) = \alpha$. Furthermore,

$$\mathrm{H}(\alpha) = \int_\Omega \alpha \wedge \mathrm{BS}(\alpha).$$

Change of helicity under a diffeomorphism

Let $f\colon\ \Omega \to \Omega'$ be an orientation-preserving diffeomorphism.
Let α be a closed Dirichlet form on Ω. Then its pullback $\alpha' = \left(f^{-1}\right)^* \alpha$ is a closed Dirichlet form on Ω'.

Question

How does the helicity of α change under this diffeomorphism?

$$H(\alpha') - H(\alpha) = ???$$

$$\begin{aligned} \mathrm{H}(\alpha) &= \int_\Omega \alpha \wedge \mathrm{BS}(\alpha) \\ \mathrm{H}(\alpha') &= \int_{\Omega'=f(\Omega)} \alpha' \wedge \mathrm{BS}(\alpha') \\ &= \int_\Omega \alpha \wedge f^* \mathrm{BS}\left(\left(f^{-1}\right)^* \alpha\right) \end{aligned}$$

Change of helicity under a diffeomorphism

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Observation

Both $\mathrm{BS}(\alpha)$ and $f^* \, \mathrm{BS}(\alpha')$ are primitives for α.

But they are not equal.

The change of helicity formula

$$\begin{aligned} \mathrm{H}(\alpha') - \mathrm{H}(\alpha) &= \int_\Omega \alpha \wedge \left(f^* \mathrm{BS}(\alpha') - \mathrm{BS}(\alpha)\right) \\ &= \int_\Omega \alpha \wedge (\tilde{\beta} - \beta). \end{aligned}$$

where β and $\tilde{\beta}$ are primitives for α. This integrand is exact:

$$\int_\Omega \alpha \wedge (\tilde{\beta} - \beta) = \begin{cases} \int_{\partial\Omega} \beta \wedge \tilde{\beta} & \text{if } k \text{ is odd,} \\ \frac{1}{2} \int_{\partial\Omega} \tilde{\beta} \wedge \tilde{\beta} - \beta \wedge \beta & \text{if } k \text{ is even.} \end{cases}$$

Helicity is trivial in the latter case.

Notice β and $\tilde{\beta}$ are closed on $\partial\Omega$, so this change is a cup product on $\partial\Omega$ of cohomology classes $[\beta]$ and $[\tilde{\beta}]$

The homology of domains in $\mathbf{R}^{2k+1}$

Our change-of-helicity result requires some info about homology of $\Omega, \partial\Omega$:

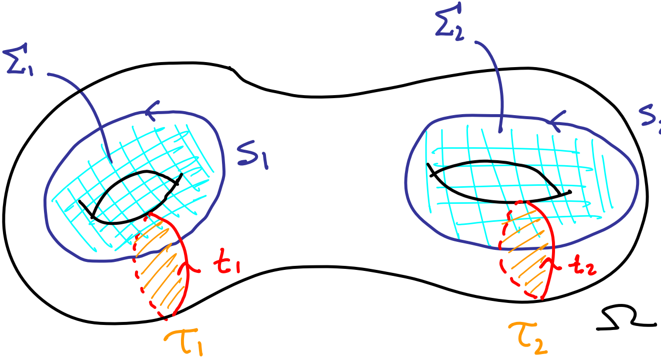
$$H_k(\partial\Omega) = H_k(\Omega) \oplus H_k(\mathbf{R}^{2k+1} \setminus \Omega).$$

We can pick generators (called the *Alexander basis*) $\{s_i\}$ for $H^k(\Omega)$ and $\{t_i\}$ for $H^k(\mathbf{R}^{2k+1} \setminus \Omega)$ on $\partial\Omega$ s.t.:

1. t_i bounds a surface $\tau_i \in H_{k+1}(\Omega, \partial\Omega)$;
   $\{\tau_i\}$ generate this group,
2. s_i bounds a surface $\sigma_i \in H_{k+1}(\mathbf{R}^{2k+1} \setminus \Omega, \partial\Omega)$;
   $\{\sigma_i\}$ generate this group,
3. $t_i^* \cup s_j^* = \delta_{ij}[\partial\Omega]^*$ in the cup product on $H^k(\partial\Omega)$,
   while $s_i^* \cup s_j^* = 0$ and $t_i^* \cup t_j^* = 0$.

Alexander basis example

Ω = solid 2-holed torus



Effect of a map on k-th homology of boundary

Similarly, we can pick generators $\{s_i'\}$ for $H_k(\Omega')$ and $\{t_i'\}$ for $H_k(\mathbf{R}^{2k+1} - \Omega')$, so that $f(t_i) = t_i'$, but $f(s_i) = s_i' +$ some t_j' terms .

In terms of the basis $\langle s_1, \ldots, s_n, t_1, \ldots t_n \rangle$,

$$f_* : H_k(\partial\Omega) \to H_k(\partial\Omega') = \left[\begin{array}{c|c} I & 0 \\ \hline (c_{ij}) & I \end{array} \right],$$

-each block represents a $(d \times d)$ matrix
-the $\left(c_{ij}\right)$ matrix is symmetric.

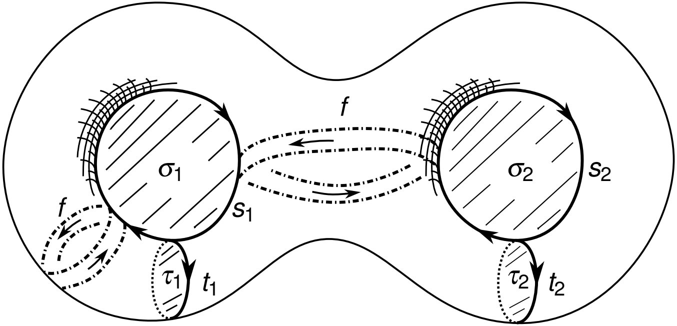
Theorem (CP)

The change in the helicity of α under f is

$$\mathrm{H}\left(\alpha'\right) - \mathrm{H}(\alpha) = \sum_{i,j} c_{ij} \cdot \mathrm{Flux}(\alpha, \tau_i)\,\mathrm{Flux}(\alpha, \tau_j)$$

Alexander basis example

Ω = solid 2-holed torus



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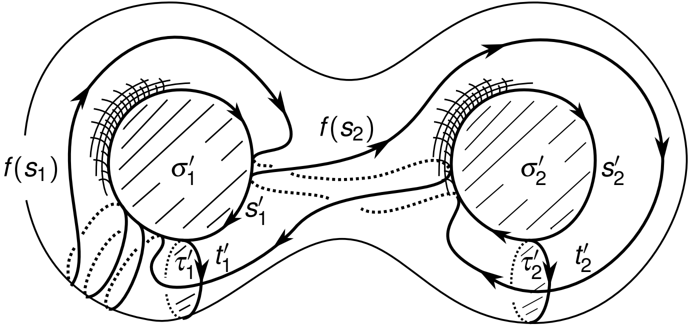


$$(c_{ij}) = \begin{pmatrix} -4 & 1 \\ 1 & -1 \end{pmatrix}$$

Proof sketch

The k-classes $\gamma = [\mathrm{BS}(\alpha)]$ and $\tilde{\gamma} = [f^* \, \mathrm{BS}(\alpha')]$ are determined by their integrals over the basis $\langle s_1, \ldots, s_n, t_1, \ldots, t_n \rangle$ for $H_k(\partial\Omega)$.

- The map f takes $\tau_i = \partial t_i$ to $\tau_i' = \partial t_i'$, so

$$\int_{t_i} \gamma = \int_{\tau_i} \alpha = \mathrm{Flux}(\alpha, \tau_i) = \int_{\tau_i'} \left(f^{-1}\right)^* \alpha = \int_{t_i'} \tilde{\gamma}.$$

- $\int_{s_i} \gamma = \int_{\sigma_i} \alpha = 0$ and $\int_{s_i'} \tilde{\gamma} = \int_{\sigma_i'} \alpha' = 0$.
- To conclude, since $f(s_i) = s_i' + \sum_j c_{ji} t_i'$,

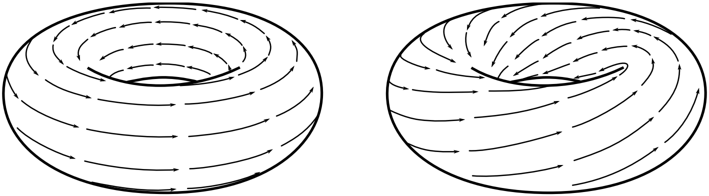
$$\int_{s_i} \tilde{\gamma} = \int_{f(s_i)} \mathrm{BS}\left(\alpha'\right) = \int_{\sigma_i'} \alpha' + \sum_j c_{ji} \int_{\tau_j'} \alpha' = 0 + \sum_j c_{ji} \, \mathrm{Flux}(\alpha, \tau_j).$$

Now, take the cup product $\gamma \cup \tilde{\gamma}$.

$$\gamma \cup \tilde{\gamma} = \sum_i \int_{s_i} \gamma \int_{t_i} \tilde{\gamma} = \sum_{i,j} c_{ji} \, \mathrm{Flux}(\alpha, \tau_j) \, \mathrm{Flux}(\alpha, \tau_i)$$

Example

Suppose f is a single Dehn twist in a solid torus:



For inner radius R (and any core radius), then we compute

$$H(\alpha) = 0 \qquad H(\alpha') = \pi^2 R^4$$

Here, $f(s) = s' + t'$, so $c = 1$. Thus,

$$H(\alpha') - H(\alpha) = 1 \cdot \mathrm{Flux}^2(\alpha) = (\pi R^2)^2$$

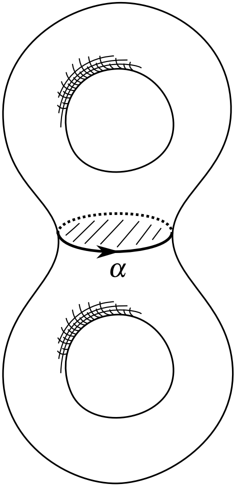
Example preserving helicity not in SDiff

$\Omega =$ a solid double torus

f: Dehn twist around α on $\partial\Omega$; extend twist to interior.

f_*: identity on the homology of Ω and $\partial\Omega$

Therefore, f_* preserves helicity, but $f \notin$ SDiff.



Application: Alternate integral formulas for helicity

Recall our cohomology class definition of helicity

$$[\alpha_x \wedge \alpha_y] \cup [g^* \mathrm{dVol}] = \frac{\mathrm{H}(\alpha)}{\mathrm{Vol}(\Omega)^2} \left[\mathrm{dVol}_{C_2[\Omega]}\right].$$

It depends only on the cohomology classes of $[\alpha_x \wedge \alpha_y]$ and $[g^* \mathrm{dVol}_{S^{2k}}]$.

Thus, we may define the helicity integrand using any unit volume form on S^{2k}. We obtain alternate integral formulas for helicity.

motivation: linking number formula by ± 1 crossings, which can be derived from Gauss integral formula by concentrating sphere's mass at the north pole.

4-dim integral for helicity

One particularly nice integral formula (good for numerics?):

Proposition

Let V be a divergence-free vector field in $\mathbf{R}^3$, tangent to $\partial\Omega$, is given by the 4-dim. integral

$$\mathrm{H}(V) = \frac{1}{4\pi} \int_{x \in \Omega} \int_{y \in x^+(\Omega)} V(x) \times V(y) \cdot (0,0,1) \,\mathrm{dVol}_x \, dy_3,$$

where $x = (x_1, x_2, x_3)$ and $x^+(\Omega) = \{(x_1, x_2, y_3) : y_3 > x_3\}$.

4-dim integral for helicity

Proposition

$$H(V) = \frac{1}{4\pi} \int_{x \in \Omega} \int_{y \in x^+(\Omega)} V(x) \times V(y) \cdot (0,0,1) \, \mathrm{dVol}_x \, dy_3,$$

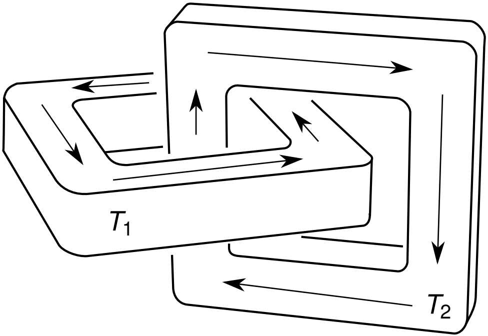
Proof:

- Consider sequence of 2-forms on S^2 converging to the δ-form (concentrated at north pole) where each has integral 4π over the entire sphere.
- they are cohomologous on S^2 to the standard area form, so their pullbacks $g^*\gamma$ are cohomologous 2-forms on $C_2[\Omega]$.
- helicities derived from these forms all equal standard helicity.
- their limit is the formula above.

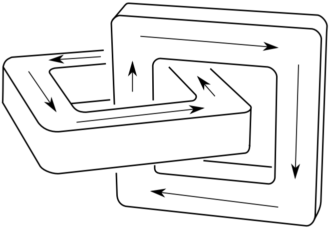
An example

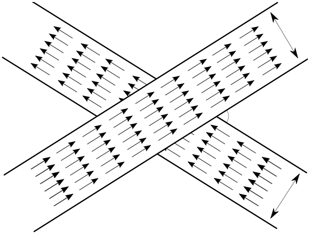
Theorem. [Moffatt; Berger-Field] Given 2 linked tubes T_1, T_2 and divergence-free V tangent to boundary of tubes.

$$\mathrm{H}(V) = \mathrm{H}_1(V) + \mathrm{H}_2(V) + \mathrm{Lk}(T_1, T_2)\,\mathrm{Flux}_1\,\mathrm{Flux}_2$$



An example

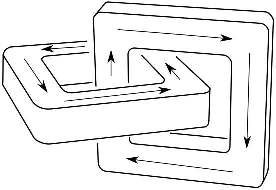




Tubes have width w, height h, unit length fields parallel to walls.
Assume at overcrossing, tubes are rectangular boxes running in parallel planes.

- We can arrange the tubes so that for any $y \in x^{+}(\Omega)$ with x and y in same tube, the vectors $V(x)$ and $V(y)$ are collinear.
- Integrand vanishes for these pairs
- We may further arrange the tubes so that there are only two regions where $y \in x^{+}(\Omega)$ for x and y in different tubes.

An example



Now we integrate over these pairs.

- Since $V(x)$ and $V(y)$ lie in horizontal planes, the integrand is constant $\sin\theta$.

- domain of integration (for x) is a prism of height h, base a parallelogram of length $w/\sin\theta$ and width w; the domain of integration for y is a line segment above each x of length h.
- Total (4-dim) volume of integration is $w^2 h^2/\sin\theta$
- So integral formula gives $\mathrm{H}(V) = w^2 h^2$.
- Moffatt/Berger-Field: Flux $= wh$ for both tubes, $\mathrm{Lk} = 1$.
  Thus, $\mathrm{H}(V) = w^2 h^2$.

Obstructions to Diff invariants

We may view helicity as the *4-dimensional flux* of a “doubling” of the initial form over a particular spanning surface in the six-dimensional domain $C_2[\Omega]$.

$$\mathrm{H}(\alpha) = \int_{C_2[\Omega]} \underbrace{\alpha_x \wedge \alpha_y}_{\text{“doubling” of } \alpha} \wedge \underbrace{g^* \mathrm{dVol}_{S^2}}_{\text{a 2d “curve”}}$$
$$= \underbrace{[\alpha_x \wedge \alpha_y] \cup [g^* \mathrm{dVol}_{S^2}](C_2[\Omega])}_{\text{a 4d “flux”}}.$$

Conclusion

If $\Omega = D^3$, then any invariant integral in the “helicity form”

$$\int_{\Omega \times \Omega} V(x) \times V(y) \cdot W(x,y) \, \mathrm{dVol}_x \, \mathrm{dVol}_y$$

where W is a curl-free field in x and y is either (a scalar multiple) of helicity or zero.

Proof.

Only one nontrivial period $\implies$ only one nontrivial flux. □

Generalized helicity: taking more points

Goal

Find other SDiff *invariants that work on arbitrary domains.*

The "doubling" construction generalizes easily to constructing a $2n$-form on the space $C_n[D^3]$ of configurations of n points in D^3. Can we construct a new "higher helicity" by taking n points and pairing with a n-form β?

Theorem

No. Any invariant integral in the form

$$\int_{C_n[D^3]} \alpha_{x_1} \wedge \cdots \wedge \alpha_{x_n} \wedge \beta$$

where β is a closed n-form is either zero or proportional to a power of helicity.

Proof.

F. Cohen calculated the cohomology of $C_n[D^3]$ & showed that it is generated by cup products of 2-dimensional classes corresponding to pairwise Gauss maps. □

The Theorem of Baader and Marché

In some sense, this is a weak version of a theorem of Baader and Marché (2008).

Definition

Let $K(p, T)$ be the (straight-line) closure of the orbit of V from p integrated for time T.

Theorem (Baader and Marché, 2008)

If V is an ergodic vector field on a domain Ω in $\mathbf{R}^3$ that does not charge any periodic orbit and $v(K)$ is a Vassiliev invariant of knots, then for almost all $p \in \Omega$,

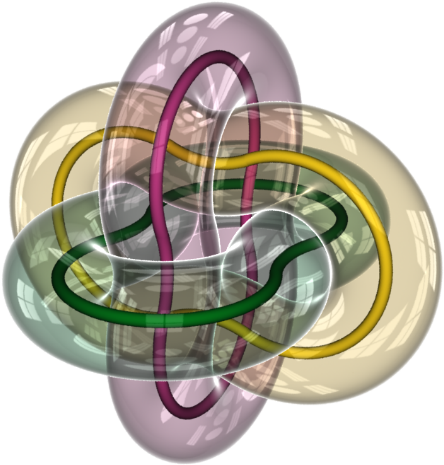
$$\lim_{T \to \infty} \frac{v(K(p, T))}{T^2} = \lambda H(V)^n$$

for some constant λ and some power n.

So where should we look for higher helicities?

Answer

In a different domain?



So where should we look for higher helicities?

Theorem (Monastyrsky-Retakh, Berger, Hornig+, Komendarcyzk, . . .)

If V is divergence-free and tangent to the boundary of three unlinked tubes T_1, T_2, T_3 in $\mathbf{R}^3$, there is a nontrivial third-order helicity integral

$$H_{123}(V)$$

Theorem (Monastyrsky-Retakh, Berger, Hornig+, Komendarcyzk, . . .)

If V is divergence-free and tangent to the boundary of three unlinked tubes T_1, T_2, T_3 in $\mathbf{R}^3$, there is a nontrivial third-order helicity integral

$$H_{123}(V) = \mathrm{Flux}(V, T_1)\,\mathrm{Flux}(V, T_2)\,\mathrm{Flux}(V, T_3)\overline{\mu}_{123}(T_1, T_2, T_3).$$

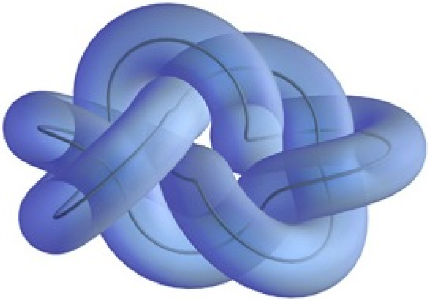
Objection

If Ω is three unlinked tubes, a version of $C_3[\Omega]$ certainly contains a cohomology class to integrate against. But one is measuring the topology of the domain, not of the vector field . . .

So where should we look for higher helicities?

Answer

By cutting and pasting parts of $C_k(\Omega)$, for special Ω?



Conjecture

If Ω is a tube, we can cut and paste integrals over parts of the tube to define a helicity-like integral. For instance, in analogy to the X and Y integral formulation of the second finite-type invariant. Again, one is measuring the topology of the tube, not the field . . .

Maybe not SDiff???

Maybe we're being too geometric and not physical enough?

Question

Maybe we should restrict our diffeos further?

Invariants of a field V evolving along a diffeomorphism $f \in \mathrm{SDiff}$ may be too broad; they're certainly applicable to plasma flows.

Could we impose extra conditions on the diffeomorphism . . . take $X \subsetneq \mathrm{Diff}$ that model plasma flows, and ask for X-invariants???

cf. Hornig-Yeates

Thank you!

Parting question

Are there physical settings where an invariant $(k+1)$-form in $2k+1$ dimensions occurs (for $k > 1$)?

Thank you for listening.

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`arxiv:0903.1465`