Symmetric quadratic dynamical systems.

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In the talk we present results obtained recently with E.Yu.Bunkova.

We will discuss the dynamical systems wich arise in classical and modern problems of mathematics, mechanics and biology.

For such systems we introduce the notion of algebraical integrability. We will describe a wide class of algebraically integrable systems.

This class contains the Euler top, Kovalevskaya system, Lotka-Volterra type systems, Darboux-Halphen systems and modern generalizations of this systems.

Quadratic dynamical systems.

In the space $\mathbb{C}^n$ (or $\mathbb{R}^n$) with coordinates $\xi = (\xi_1, \ldots, \xi_n)^\top$ the general homogeneous quadratic dynamical system is

$$\xi_k'(t) = A_k^{i,j} \xi_i(t) \xi_j(t), \quad A_k^{i,j} = A_k^{j,i} = const, \quad k = 1, \ldots, n. \quad (1)$$

Here and below we use the Einstein summation convention.

We can identify the space of quadratic dynamical systems with the $\frac{1}{2}n^2(n+1)$-dimensional linear space of tensors

$$A = (A_k^{i,j})\colon A_k^{i,j} = A_k^{j,i} = const.$$

System (1) is homogeneous with

$$\deg t = 4, \quad \deg \xi_i = -4, \quad \deg A_{ij} = 0.$$

Classical examples.

The Euler top

$$\xi_1' = \xi_2\xi_3,$$
$$\xi_2' = \xi_3\xi_1,$$
$$\xi_3' = \xi_1\xi_2.$$

This system has integrals $\alpha_1\xi_1^2 + \alpha_2\xi_2^2 + \alpha_3\xi_3^2$ where $\alpha_1 + \alpha_2 + \alpha_3 = 0$.

The generalized Euler top for $n > 3$ and $2 \leqslant i < j \leqslant n$

$$\xi_1' = \xi_2\xi_3 + \cdots + \xi_i\xi_j + \cdots + \xi_{n-1}\xi_n,$$
$$\xi_2' = \xi_3\xi_4 + \cdots + \xi_{i+1}\xi_{j+1} + \cdots + \xi_n\xi_1,$$

and for $\xi_3', \ldots, \xi_n'$ the cyclic permutation of indexes.

The Darboux-Halphen system

$$\xi_1' = \xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3,$$
$$\xi_2' = \xi_3\xi_1 - \xi_2\xi_3 - \xi_2\xi_1,$$
$$\xi_3' = \xi_1\xi_2 - \xi_3\xi_1 - \xi_3\xi_2$$

is appeared (1878) in Darboux's analysis of triply orthogonal surfaces.

This system was solved by Halphen (1881).

The generalized Darboux-Halphen system

$$\xi_1' = \xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3 + \tau^2,$$
$$\xi_2' = \xi_3\xi_1 - \xi_2\xi_3 - \xi_2\xi_1 + \tau^2,$$
$$\xi_3' = \xi_1\xi_2 - \xi_3\xi_1 - \xi_3\xi_2 + \tau^2,$$

where

$$\tau^2 = a^2(\xi_1 - \xi_2)(\xi_3 - \xi_1) + \\ + b^2(\xi_2 - \xi_3)(\xi_1 - \xi_2) + c^2(\xi_3 - \xi_1)(\xi_2 - \xi_3).$$

This system was solved by Halphen (1881).

Let $T_1(z), T_2(z), T_3(z)$ be $n \times n$ matrix-valued meromorphic functions in a complex variable z.

The Nahm equations are a system of matrix differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} T_1(t) = [T_2(t), T_3(t)],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} T_2(t) = [T_3(t), T_1(t)],$$

$$\frac{\mathrm{d}}{\mathrm{d}t} T_3(t) = [T_1(t), T_2(t)].$$

These equations admit Lax representation (Hitchin, 1983)

$$\frac{\mathrm{d}A}{\mathrm{d}z} = [A, M]$$

with

$$A = A_{-1}\zeta^{-1} + A_0 + A_1\zeta, \qquad M = \frac{1}{2}A_0 + A_1\zeta$$

and

$$A_{-1} = T_1 + \imath T_2, \qquad A_0 = -2\imath T_2, \qquad A_1 = T_1 - \imath T_2$$

where $\imath = \sqrt{-1}$.

In the special case $n = 2$

$$T_m(z) = -\frac{\imath}{2}\xi_m(z)\sigma_m$$

where σ_m are Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $\xi_m(z)$ are functions in complex variable z.

The Lax representation leads to the Euler top dynamical system

$$\xi_1' = \xi_2\xi_3,$$

$$\xi_2' = \xi_3\xi_1,$$

$$\xi_3' = \xi_1\xi_2.$$

A reduction of Self-Dual Yang–Mills equation.

Consider a vector bundle $E \to \mathbb{R}^4$ with fiber isomorphic, as linear space, to the gauge Lie algebra $\mathcal{G}$ of the gauge Lie group G.

SDYM equation is

$$F = *F$$

where $F = dA + A \wedge A$ is the curvature form of the connection A and $*$ is the Hodge star-operator.

M.J.Ablowitz, S.Chakravarty and R.Halburd obtained (1998) the generalized Darboux–Halpen system as a reduction of SDYM, where $\mathcal{G}$ is the Lie algebra of vector fields on S^3.

This reduction gives the quadratic dynamical system

$$\begin{aligned} \xi_i' &= \xi_j\xi_k - \xi_i(\xi_j + \xi_k) + \tau^2, \\ \tau_i' &= -\tau_i(\xi_j + \xi_k), \end{aligned} \tag{2}$$

where $1 \leqslant i \neq j \neq k \leqslant 3$ and $\tau^2 = \tau_1^2 + \tau_2^2 + \tau_3^2$.

The system (2) has the integrals

$$\tau_i^2 = \alpha_i^2(\xi_i - \xi_j)(\xi_k - \xi_i), \quad 1 \leqslant i \neq j \neq k \leqslant 3.$$

Hence the system (2) is the generalized Darboux–Halpen system, where $\alpha^2 = \alpha_1^2$, $\beta^2 = \alpha_2^2$, $\gamma^2 = \alpha_3^2$.

Lotka–Volterra type system.

$$\xi_k' = \xi_k \left(\sum_{l=1}^{n} \xi_l - 2\xi_k \right), \qquad k = 1, \ldots, n.$$

For $n = 3$ this system was considered by S. Kovalevskaya.
In this case the system has two independant quadratic integrals

$$\sum_{i \neq j} a_k \xi_i \xi_j, \qquad \sum a_k = 0.$$

For $n = 4$ this system has two independant quadratic integrals

$$(\xi_1 - \xi_3)(\xi_2 - \xi_4), \qquad (\xi_1 - \xi_2)(\xi_3 - \xi_4).$$

For $n > 4$ there are no quadratic integrals.

Representation of $\mathsf{GL}(n, \mathbb{C})$.

The change of variables $\eta = B\xi$ by a matrix $B = (B_i^j)$ leads to a representation of $\mathsf{GL}(n, \mathbb{C})$ on the space of tensors $A = (A_k^{i,j})$:

$$A_k^{i,j} \mapsto A_p^{q,r} B_k^p (B^{-1})_q^i (B^{-1})_r^j.$$

We identify the symmetric group S_n with the subgroup in $\mathsf{GL}(n, \mathbb{C})$ corresponding to the action of S_n on $\mathbb{C}^n$ as a permutation of coordinates $(\xi_1, \ldots, \xi_n)$.

Definition 1.
A system (1) is called *symmetric* if any $B \in \mathsf{S}_n$ does not change this system.

The ring of symmetric polynomials.

On the space $\mathcal{M}^n = \{\xi \in \mathbb{C}^n \colon \xi_i \neq \xi_j \; \forall i, j : i \neq j\}$
there is a free action of the group S_n.
The projection on the space of orbits
is a covering induced by the universal algebraic map

$$S \colon \mathbb{C}^n \to \mathbb{C}^n \colon (\xi_1, \ldots, \xi_n) \mapsto (h_1(\xi_1, \ldots, \xi_n), \ldots h_n(\xi_1, \ldots, \xi_n)).$$

Here h_k is the kth elementary symmetric function in $\xi_1, \ldots, \xi_n$.

Denote by Sym the ring of symmetric polynomials in $\xi_1, \ldots, \xi_n$.
We have

$$\text{Sym} = \mathbb{C}[h_1, \ldots, h_n] \subset \mathbb{C}[\xi_1, \ldots, \xi_n].$$

Symmetric quadratic dynamical systems.

System (1) defines a linear operator

$$L = L(A)\colon \mathbb{C}[\xi_1, \ldots, \xi_n] \to \mathbb{C}[\xi_1, \ldots, \xi_n]$$

with $\deg L = -4$ as

$$L = A_k^{i,j} \xi_i \xi_j \frac{\partial}{\partial \xi_k}.$$

Definition 2.
A system (1) is called *symmetric* if

$$Lh(\xi_1, \ldots, \xi_n) \in \text{Sym} \qquad \text{for any} \qquad h \in \text{Sym}.$$

Lemma.
The definitions 1 and 2 are equivalent.

Corollary.
Each symmetric quadratic dynamical system has the form

$$\xi_k'(t) = \alpha\xi_k^2 + \beta\xi_k \sum_{i\neq k} \xi_i + \gamma \sum_{i\neq k} \xi_i^2 + \delta \sum_{i<j, i\neq k, j\neq k} \xi_i\xi_j.$$

with $k = 1, \ldots, n$.

Corollary.
The system

$$\xi_k' = \xi_k \left(\alpha\xi_k + \beta \sum_{l=1}^{n} \xi_l \right), \quad k = 1, \ldots, n.$$

is symmetric for any α and β.
In the case $\alpha = -2$, $\beta = 1$ this is the Lotka-Volterra type system.

Generic symmetric quadratic dynamical systems.

Let $a_1, \ldots, a_n$ be a homogeneous multiplicative basis in Sym with $\deg a_k = -4k$. Set $a_k(t) = a_k(\xi_1(t), \ldots, \xi_n(t))$.
System (1) implies the system

$$a_k'(t) = La_k(t), \qquad k = 1, \ldots, n.$$

Thus we obtain in $\mathbb{C}^n$ with coordinates $a_1, \ldots, a_n$ the homogeneous polynomial dynamical system

$$a_k'(t) = c_k a_{k+1}(t) + g_{k+1}(a_1(t), \ldots, a_k(t)), \quad k = 1, \ldots, n, \qquad (3)$$

where $c_n = 0$ and $\deg g_{k+1} = -4(k+1)$.

Definition.
A symmetric system (1) is *generic* if in the system (3) we have $c_k \neq 0$ for $k = 1, \dots, n-1$.

The property of a system being generic does not depend on the choice of multiplicative basis.

Example.
Let $a_k = N_k = \sum \xi_i^k$ be the Newton polynomials. The system

$$\xi_i'(t) = \xi_i^2, \quad i = 1, \dots, n,$$

is generic. It implies the system (3)

$$N_k' = kN_{k+1}, \quad k = 1, \dots, n-1, \quad N_n' = g_{n+1}(N_1, \dots, N_n),$$

where $\frac{1}{n} g_{n+1}(N_1, \dots, N_n)$ is the classical polynomial expressing N_{n+1} in $N_1, \dots, N_n$.

Algebraic integrability.

Consider the universal equation

$$z^n - h_1 z^{n-1} + \cdots + (-1)^n h_n \equiv 0. \qquad (4)$$

Let $\Delta \subset \mathbb{C}^n$ be the discriminant variety of (4).
It is the image under the universal algebraic map S of the configuration $\{\xi \in \mathbb{C}^n \colon \exists i \neq j,\ \xi_i = \xi_j\}$.

Definition.
System (1) is *algebraically integrable in* $U \subset \mathbb{C}$ *by a set of functions* $(h_1(t), \ldots, h_n(t))$ if $(h_1(t), \ldots, h_n(t)) \notin \Delta$ for any $t \in U$ and the set of roots $(\xi_1(t), \ldots, \xi_n(t))$ to the equation (4) is a solution to (1) for any $t \in U$.

Problem.
Find an ordinary differential equation of degree n for h and differential polynomials $h_2, \ldots, h_n$ in h such that in the neighbourhood $U \subset \mathbb{C}$ the set of functions $h_1(t) = h(t), h_2(t), \ldots, h_n(t)$ algebraically integrates system (1).

Theorem.
For each generic symmetric system (1) with the initial data $\xi(t_0) = (\xi_1(t_0), \ldots, \xi_n(t_0)) \in \mathcal{M}^n$ there is a solution to the problem of algebraic integrability in the vicinity of $\xi(t_0)$.

Consider system (3) with $a_k = h_k$ being the elementary symmetric functions.

Under the conditions of the theorem $c_k \neq 0$, $k = 1, \dots, n-1$.

Hence, $h_j(t)$ with $j = 2, \dots, n$ can be expressed as polynomials in $h_1(t), \dots, h_{j-1}(t)$ and their derivatives from the $(j-1)$th equation of system (3).

Thus, the equation

$$a_n'(t) = g_{n+1}(a_1(t), \dots, a_n(t)),$$

gives a homogeneous differential equation for $h_1(t)$:

$$h^{(n)} + \alpha h h^{(n-1)} + \dots + \gamma h^n = 0 \qquad (5)$$

with constant coefficients $\alpha, \dots, \gamma$.

Therefore, $(h_1(t), \ldots, h_n(t))$ algebraically integrates system (3).

The initial conditions in (5) are as follows:

$$h_1(t_0) = \xi_1(t_0) + \cdots + \xi_n(t_0)$$

and

$$h_1^{(k)}(t_0) = (L^k h_1)(t_0).$$

Thus we have reduced the problem of integrability of a symmetric quadratic dynamical system to the question of solving an ordinary differential equation of the form

$$h^{(n)} + \alpha h h^{(n-1)} + \cdots + \gamma h^n = 0 \tag{6}$$

with constant coefficients $\alpha, \ldots, \gamma$.

Problem.
Classify the non-linear ordinary differential equations (6) obtained from generic quadratic dynamical systems.

Definition.
The system (1) is said to be *quasi-symmetric* with respect to $B \in GL(n, \mathbb{C})$ if this system in the coordinates $\eta = B\xi$ is symmetric.

A quasi-symmetric system is *generic* if the symmetric system in the coordinates $\eta = B\xi$ is generic.

Corollary.
For each generic quasi-symmetric system (1) there is a solution to the problem of algebraic integrability.

Two-dimensional systems.

The general two-dimensional symmetric quadratic dynamical system has the form

$$\xi_1' = \alpha\xi_1^2 + \beta\xi_1\xi_2 + \gamma\xi_2^2, \qquad (7)$$
$$\xi_2' = \gamma\xi_1^2 + \beta\xi_1\xi_2 + \alpha\xi_2^2.$$

It is generic for $\beta \neq \alpha + \gamma$.

In the coordinates $\eta = B\xi$ where $B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ we obtain

$$\eta_1' = (\alpha + \gamma + \beta)\eta_1^2 + (\alpha + \gamma - \beta)\eta_2^2, \qquad \eta_2' = (\alpha - \gamma)\eta_1\eta_2.$$

Therefore, B establishes a one-to-one correspondence between quasi-symmetric quadratic dynamical systems

$$\eta_1' = a\eta_1^2 + b\eta_2^2, \qquad (8)$$
$$\eta_2' = c\eta_1\eta_2.$$

for constant a, b, c and symmetric quadratic dynamical systems.

The normalizer of dynamical systems of the form (8) is the diagonal torus in $\mathrm{GL}(2,\mathbb{C})$.

The conjugation by the matrix

$$B = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

takes this torus to the group of matrices of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$.

This group brings the space of two-dimensional symmetric quadratic dynamical systems into itself.

In the generic case system (7) is algebraically integrable by the set of functions $(h_1(t), h_2(t))$ where h_1 is a solution to the equation

$$h'' - \lambda_1 h' h + \lambda_2 h^3 = 0 \qquad (9)$$

with $\lambda_1 = (3\alpha + \beta - \gamma)$, $\lambda_2 = (\alpha - \gamma)(\alpha + \beta + \gamma)$, and

$$h_2 = \frac{1}{2(\beta - \alpha - \gamma)} h_1' - \frac{\alpha + \gamma}{2(\beta - \alpha - \gamma)} h_1^2.$$

The initial conditions for h_1 in (9) corresponding to the generic case are $(h_1(t_0), h_1'(t_0))$ where

$$(\beta + \alpha + \gamma) h_1^2(t_0) \neq 2h_1'(t_0).$$

Special cases.

The general solution to (9) is

For $\lambda_1 = 0$: $$h(t) = k_2 \mathrm{sn}\left(\left(\sqrt{\frac{\lambda_2}{2}}t + k_1\right)k_2, i\right).$$

For $\lambda_2 = 0$: $$h(t) = \frac{\sqrt{2k_1}}{\sqrt{\lambda_1}}\tanh\left(\sqrt{\frac{k_1\lambda_1}{2}}(t + k_2)\right).$$

For $\lambda_1^2 = 9\lambda_2$: $$h(t) = \frac{6(k_1 t + k_2)}{\lambda_1(k_1 t^2 + 2k_2 t + 2)}.$$

Here k_1, k_2 are constants, sn is the Jacobi sine and $i = \sqrt{-1}$.

Problem.

For given equation

$$h'' - \lambda_1 h' h + \lambda_2 h^3 = 0$$

find a system of the form (7). We have

$$3\alpha + \beta = \lambda_1 + \gamma; \qquad \alpha = \lambda_{2,1} + \gamma; \qquad \alpha + \beta = \lambda_{2,2} - \gamma,$$

where $\lambda_{2,1}\,\lambda_{2,2} = \lambda_2$. Thus

$$\lambda_{2,2} + 2\lambda_{2,1} = \lambda_1; \qquad \lambda_{2,1}\,\lambda_{2,2} = \lambda_2.$$

Hence $\lambda_{2,2} = \frac{\lambda_2}{\lambda_{2,1}}$ if $\lambda_2 \neq 0$ and $\lambda_{2,1}$ is a root to the equation

$$z^2 - \frac{1}{2}\lambda_1 z + \frac{1}{2}\lambda_2 = 0.$$

We obtain

$$\alpha = \lambda_{2,1} + \gamma; \qquad \beta = \lambda_{2,2} - \lambda_{2,1} - 2\gamma.$$

In the case $\lambda_2 = 0$ we have

$$\lambda_{2,1} = 0, \quad \text{then } \lambda_{2,2} = \lambda_1 \; \text{ and } \alpha = \gamma, \; \beta = \lambda_1 - 2\gamma,$$

or

$$\lambda_{2,2} = 0, \quad \text{then } \lambda_{2,1} = \frac{1}{2}\lambda_1 \; \text{ and } \alpha = \frac{1}{2}\lambda_1 + \gamma, \; \beta = -\frac{1}{2}\lambda_1 - 2\gamma.$$

Let $\lambda_1 = 0$ and $\lambda_2 \neq 0$. Then

$$\lambda_{2,1} = \sqrt{-\frac{1}{2}\lambda_2}\,, \qquad \lambda_{2,2} = -2\lambda_{2,1};$$

or

$$\lambda_{2,1} = -\sqrt{-\frac{1}{2}\lambda_2}\,, \qquad \lambda_{2,2} = -2\lambda_{2,1}.$$

Hence

$$\alpha = \lambda_{2,1} + \gamma, \qquad \beta = -3\lambda_{2,1} - 2\gamma,$$

where $\lambda_{2,1}^2 = -\frac{1}{2}\lambda_2$.

Let $\lambda_2 = 0$ and $\lambda_1 \neq 0$. Then $\lambda_{2,1}\left(\lambda_{2,1} - \frac{1}{2}\lambda_1\right) = 0$.

In the case $\lambda_{2,1} = 0$ we obtain $\lambda_{2,2} = \lambda_1$ and

$$\alpha = \gamma; \qquad \beta = \lambda_1 - 2\gamma.$$

Let $\lambda_{2,1} \neq 0$ then $\lambda_{2,2} = \frac{1}{2}\lambda_1$ and

$$\alpha = \frac{1}{2}\lambda_1 + \gamma; \qquad \beta = -\frac{3}{2}\lambda_1 - 2\gamma.$$

Let $\lambda_1^2 = 9\lambda_2 \neq 0$. Then

$$\lambda_{2,1}^2 - \frac{1}{2}\lambda_1\lambda_{2,1} + \frac{1}{18}\lambda_1^2 = 0; \qquad \lambda_{2,2} = \lambda_1 - 2\lambda_{2,1}.$$

We obtain $\lambda_{2,1} = \frac{1}{3}\lambda_1$ or $\lambda_{2,1} = \frac{1}{6}\lambda_1$ and

$$\alpha = \lambda_{2,1} + \gamma, \qquad \beta = \lambda_1 - 3\lambda_{2,1} - 2\gamma.$$

Let $\lambda_{2,1} = \frac{1}{3}\lambda_1$, $\lambda_{2,2} = \frac{1}{3}\lambda_1$. Then

$$\alpha = \frac{1}{3}\lambda_1 + \gamma; \qquad \beta = -2\gamma.$$

Let $\lambda_{2,1} = \frac{1}{6}\lambda_1$, $\lambda_{2,2} = \frac{2}{3}\lambda_1$. Then

$$\alpha = \frac{1}{6}\lambda_1 + \gamma; \qquad \beta = \frac{1}{2}\lambda_1 - 2\gamma.$$

In the case $\gamma = 0$ we obtain the system

$$\xi_1' = \frac{1}{6}\lambda_1(\xi_1^2 + 3\xi_1\xi_2),$$
$$\xi_2' = \frac{1}{6}\lambda_1(\xi_2^2 + 3\xi_1\xi_2).$$

Three-dimensional systems.

The general three-dimensional symmetric quadratic dynamical system has the form

$$\begin{aligned} \xi_1' &= \alpha\xi_1^2 + \beta\xi_1(\xi_2+\xi_3) + \gamma(\xi_2^2+\xi_3^2) + \delta\xi_2\xi_3, \\ \xi_2' &= \alpha\xi_2^2 + \beta\xi_2(\xi_3+\xi_1) + \gamma(\xi_3^2+\xi_1^2) + \delta\xi_3\xi_1, \\ \xi_3' &= \alpha\xi_3^2 + \beta\xi_3(\xi_1+\xi_2) + \gamma(\xi_1^2+\xi_2^2) + \delta\xi_1\xi_2. \end{aligned} \tag{10}$$

It is generic for $2\alpha - 2\beta + 4\gamma - \delta \neq 0$ and $\alpha - \beta - \gamma + \delta \neq 0$.

In the coordinates $\eta = B\xi$ where

$$B = \begin{pmatrix} 1 & 1 & 1 \\ \varepsilon & \varepsilon^2 & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix}, \quad \varepsilon^3 = 1, \varepsilon \neq 1,$$

we obtain

$$\begin{aligned} \eta_1' &= k_1\eta_1^2 + k_2\eta_2\eta_3, \\ \eta_2' &= k_3\eta_3^2 + k_4\eta_1\eta_2, \\ \eta_3' &= k_3\eta_2^2 + k_4\eta_1\eta_3, \end{aligned} \tag{11}$$

where

$$k_1 = \frac{1}{3}(\alpha + 2\beta + 2\gamma + \delta), \qquad k_2 = \frac{1}{3}(2\alpha - 2\beta + 4\gamma - \delta),$$

$$k_3 = \frac{1}{3}(\alpha - \beta - \gamma + \delta), \qquad k_4 = \frac{1}{3}(2\alpha + \beta - 2\gamma - \delta).$$

Therefore, the matrix B establishes a one-to-one correspondence between systems of the form (10) and (11).

The maximal subgroup of $\mathrm{GL}(3,\mathbb{C})$ that takes the space of systems of the form (11) into itself is generated by matrices

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}, \quad b^3 = c^3, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Therefore, the subgroup of matrices in $\mathrm{GL}(3,\mathbb{C})$ obtained from this subgroup by conjugation by the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ \varepsilon & \varepsilon^2 & 1 \\ \varepsilon^2 & \varepsilon & 1 \end{pmatrix}, \quad \varepsilon^3 = 1, \quad \varepsilon \neq 1,$$

brings the space of three-dimensional symmetric quadratic dynamical systems into itself.

Darboux-Halphen system.

The classical Darboux-Halphen system

$$\xi_1' = \xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3,$$

$$\xi_2' = \xi_3\xi_1 - \xi_2\xi_3 - \xi_2\xi_1,$$

$$\xi_3' = \xi_1\xi_2 - \xi_3\xi_1 - \xi_3\xi_2$$

is symmetric and generic.

It implies the system (3) of the form

$$h_1' = -h_2,$$

$$h_2' = -6h_3,$$

$$h_3' = -4h_1h_3 + h_2^2.$$

Theorem.
The Darboux-Halphen system is algebraically integrable by (h_1, h_2, h_3) where $-2h_1$ is a solution to the Chazy-3 equation

$$y''' = 2yy'' - 3(y')^2,$$

and

$$h_2 = -h_1',$$

$$h_3 = -\frac{1}{6}h_2' = \frac{1}{6}h_1''.$$

General Darboux-Halphen system.

$$\xi_1' = a\,(\xi_2\xi_3 - \xi_1\xi_2 - \xi_1\xi_3) + b\xi_1^2,$$

$$\xi_2' = a\,(\xi_3\xi_1 - \xi_2\xi_3 - \xi_2\xi_1) + b\xi_2^2,$$

$$\xi_3' = a\,(\xi_1\xi_2 - \xi_3\xi_1 - \xi_3\xi_2) + b\xi_3^2.$$

This system is symmetric and generic for $a \neq -2b$, $2a \neq b$.
For $a \neq b$ the function

$$y = -2(a-b)(\xi_1 + \xi_2 + \xi_3)$$

is a solution to the equation

$$y''' = 2yy'' - 3(y')^2 + c(6y' - y^2)^2 \quad \text{with} \quad c = \frac{-b^2}{4(a+2b)(a-b)}.$$

For $c = 0$ this is the Chazy-3 equation.

For $c = -\frac{4}{k^2-36}$ this is the Chazy-12 equation.

The generalized Darboux-Halphen system

$$\eta_1' = \eta_2\eta_3 - \eta_1\eta_2 - \eta_1\eta_3 + \tau^2,$$
$$\eta_2' = \eta_3\eta_1 - \eta_2\eta_1 - \eta_2\eta_3 + \tau^2,$$
$$\eta_3' = \eta_1\eta_2 - \eta_3\eta_1 - \eta_3\eta_2 + \tau^2,$$

where

$$\tau^2 = \alpha^2(\eta_1-\eta_2)(\eta_3-\eta_1)+\beta^2(\eta_2-\eta_3)(\eta_1-\eta_2)+\gamma^2(\eta_3-\eta_1)(\eta_2-\eta_3)$$

is symmetric if and only if $\alpha^2 = \beta^2 = \gamma^2$ and in this case generic for $\alpha^2 \neq \frac{1}{4}$ and $\frac{1}{9}$.

It is the case $b = a - 1$ of the general Darboux-Halphen system in coordinates

$$\eta_i = a\xi_i - \frac{1}{2}(a-1)(\xi_j + \xi_k), \quad i \neq j \neq k$$

with $\alpha^2 = (a-1)^2/(3a-1)^2$.

The general Darboux-Halphen system implies the system (3) of the form

$$h_1' = -(a + 2b)h_2 + bh_1^2,$$

$$h_2' = -3(2a + b)h_3 + bh_1 h_2,$$

$$h_3' = -(4a - b)h_1 h_3 + ah_2^2.$$

Four-dimensional systems.

The general four-dimensional symmetric quadratic dynamical system has the form

$$\xi_1' = \alpha\xi_1^2 + \beta\xi_1(\xi_2 + \xi_3 + \xi_4) + \gamma(\xi_2^2 + \xi_3^2 + \xi_4^2) + \delta(\xi_2\xi_3 + \xi_2\xi_4 + \xi_3\xi_4),$$

$$\xi_2' = \alpha\xi_2^2 + \beta\xi_2(\xi_3 + \xi_4 + \xi_1) + \gamma(\xi_3^2 + \xi_4^2 + \xi_1^2) + \delta(\xi_3\xi_4 + \xi_3\xi_1 + \xi_4\xi_1),$$

$$\xi_3' = \alpha\xi_3^2 + \beta\xi_3(\xi_4 + \xi_1 + \xi_2) + \gamma(\xi_4^2 + \xi_1^2 + \xi_2^2) + \delta(\xi_4\xi_1 + \xi_4\xi_2 + \xi_1\xi_2),$$

$$\xi_4' = \alpha\xi_4^2 + \beta\xi_4(\xi_1 + \xi_2 + \xi_3) + \gamma(\xi_1^2 + \xi_2^2 + \xi_3^2) + \delta(\xi_1\xi_2 + \xi_1\xi_3 + \xi_2\xi_3).$$

In the case of four-dimensional Lotka-Volterra type system

$$\xi_k' = \xi_k \left(\sum_{l=1}^{4} \xi_l - 2\xi_k \right), \qquad k = 1, \dots, 4,$$

we obtain the equation

$$h'''' - h'''h + 5h''h' - 4h''h^2 - 8(h')^2 h + 4h'h^3 = 0$$

and differential polynomials

$$h_2 = \frac{1}{4}(h' + h^2),$$

$$h_3 = \frac{1}{24}(h'' + 2h'h),$$

$$h_4 = \frac{1}{192}(h''' + h''h + 2(h')^2 - 2h'h^2).$$

The system

$$\xi_i' = a(\xi_j\xi_k + \xi_j\xi_l + \xi_k\xi_l) - 2a\xi_i(\xi_j + \xi_k + \xi_l) + b\xi_i^2 \qquad (12)$$

where the indices (i, j, k, l) run over the four cyclic permutations of $(1, 2, 3, 4)$ is symmetric and generic for $3a \neq b$, $a \neq -b$.

The function

$$h = \frac{3a - b}{4}(\xi_1 + \xi_2 + \xi_3 + \xi_4)$$

is a solution to the equation

$$h'''' + 20hh''' - 24h'h'' + 96h^2h'' - 144h(h')^2 + \\ + c(h' + h^2)(h'' + 6hh' + 4h^3) = 0 \qquad (13)$$

with $c = \frac{-64b^2}{(a+b)(3a-b)}$.

For $c = 64$ this equation possesses the Painlevé property.
This corresponds to the cases $a = 0$ or $3a + 2b = 0$.

In the case $a = 0$ and $b = 1$ system (12) becomes the system

$$\xi_k'(t) = \xi_k^2, \quad k = 1, \ldots, n,$$

considered above.
Therefore the general solution to (13) in this case has the form

$$h(t) = \frac{1}{4}\left(\frac{1}{t-a_1} + \frac{1}{t-a_2} + \frac{1}{t-a_3} + \frac{1}{t-a_4}\right).$$

In the case $a = 2$, $b = -3$ system (12) becomes

$$\xi_i' = 2(\xi_j\xi_k + \xi_j\xi_l + \xi_k\xi_l) - 4\xi_i(\xi_j + \xi_k + \xi_l) - 3\xi_i^2.$$

The linear change $\eta_i = -3(\xi_j + \xi_k + \xi_l)$, $i \neq j \neq k \neq l$ brings this system to the system

$$\eta_k'(t) = \eta_k^2, \quad k = 1, \dots, n,$$

of the previous case.

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ADDENDUM

Fuchs–Poincaré problem.

Definition.
A point is called a *singularity* of a function if this function is not analytic (possibly not defined) in that point.

For example, the point $t = 0$ is singularity for

$$f_1(t) = \frac{1}{t} \quad \text{and} \quad f_2(t) = \sqrt{t}.$$

Definition.
A singularity of a function is called *a critical singularity* if going around this singularity changes the value of the function.

A point $t = 0$ is not a critical singularity for $\frac{1}{t}$, but is a critical singularity for $\sqrt{t}$.

L. Fuchs remarked that differential equation solutions can have movable singularities, that is singularities whose *location depends* on the initial conditions of the solution.

In 1884 L. Fuchs and H. Poincaré have considered the problem of integrating differential equations and came to the conclusion that it is closely connected to the problem of defining new functions by means of non-linear ordinary differential equations.

Consider the first-order explicit differential equation

$$y' = F(t, y)$$

with F being a rational function of y and a locally-analytical function of t.

L. Fuchs proved that among such equations only the Riccati equation

$$y' = P_0(t) + P_1(t)y + P_2(t)y^2$$

does not have movable critical singularities.

All first-order algebraic differential equations with such property can be transformed into the Riccati equation or the Weierstrass equation

$$(y')^2 = 4y^3 - g_2 y - g_3.$$

Both this equations are integrable in terms of previously known special functions.

In 1888 S. Kovalevskaya solved the classical precession of a top under the influence of gravity problem.

S. Kovalevskaya's approach to the problem is based on finding solutions with no movable critical singularities.

She proved that there exists only three cases with such solutions. Two of them are the famous Euler and Lagrange tops.

In the third case (now named Kovalevskaya top) she found new solutions and thus was first to discover the advantages of solving differential equations whose solutions have no movable critical singularities.

Painlevé property.

The property of a differential equation that its solutions have no movable critical singularities is well known now as the *Painlevé property*.

The general solution to equations with Painlevé property lead to the single-valued function.

All *linear* ordinary differential equations have the Painlevé property, but it turns out that this property is rare for *non-linear* differential equations.

Around 1900, P. Painlevé studied second order explicit non-linear differential equations

$$y'' = F(t, y, y')$$

with F being a rational function of y and y' and a locally-analytical function of t.

It turned out that among such equations up to certain transformations only fifty equations have the Painlevé property, and among them six are not integrable in terms of previously known functions.

P. Painlevé and B. Gambier have introduced new special functions, now known as *Painlevé transcendents*, as the general solutions to this equations.

Chazy equations.

In 1910 J. Chazy considered the problem of classification of all third-order differential equations of the form

$$y''' = F(t, y, y', y''),$$

where F is a polynomial in y, y', and y'' and locally analytic in t, having the Painlevé property.

The most known autonomous Chazy equations are

Chazy-3 equation: $y''' = 2yy'' - 3(y')^2,$

Chazy-12 equation: $y''' = 2yy'' - 3(y')^2 - \dfrac{4}{k^2 - 36}(6y' - y^2)^2,$

where $k \in \mathbb{N}$, $k > 1$, $k \neq 6$.