Fermi surface topology and topological numbers in conductivity of normal metals.

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We consider the geometry of the quasiclassical electron trajectories on complicated Fermi surfaces in the presence of a strong magnetic field. Using rigorous topological theorems we present the classification of different topological types of open electron trajectories on the Fermi surfaces and consider the corresponding conductivity regimes in the limit $B \to \infty$. It is shown that the presence of the regular non-closed electron trajectories leads always to the presence of "Topological numbers" observable in the conductivity behaviour. From the other hand, the appearance of unstable non-closed electron trajectories may lead to rather interesting behaviour of conductivity, in particular, to the freezing of the longitudinal conductivity in the limit $B \to \infty$. The full picture of different regimes of the magneto-conductivity behaviour can be considered as an important characteristic of the dispersion relation in metals.

Topics related to the geometry of the Fermi surface in the theory of galvanomagnetic phenomena is connected with the research of the semiclassical electron trajectories on the Fermi surface, started by the school of I.M. Lifshitz (Lifshitz, M. Azbel, M.I. Kaganov, V.G. Peschanski) in 1950. Specifically, as shown in [1], the behaviour of the magneto-resistance in normal metals in the limit of strong magnetic field depends strongly on the geometry of quasi-classical electronic trajectories defined by the system

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\left[\mathbf{v}_{gr}(\mathbf{p}) \times \mathbf{B}\right] = \frac{e}{c}\left[\nabla\epsilon(\mathbf{p}) \times \mathbf{B}\right] \qquad (1)$$

in $\mathbf{p}$-space. The function $\epsilon(\mathbf{p})$ defines the dispersion relation for the fixed band, and we will assume, in addition, that the direction of the axis z is chosen along the direction of the magnetic field.

The energy $\epsilon(\mathbf{p})$ and the projection p_z of the quasimomentum $\mathbf{p}$ on the direction of $\mathbf{B}$ give the conservation laws of system (1). Trajectories of system (1) are defined by the intersections of surfaces of constant energy $\epsilon(\mathbf{p}) = const$ with the planes orthogonal to the magnetic field $\mathbf{B}$.

Each trajectory is traversed with the "speed" $ev_{gr}B/c$ in $\mathbf{p}$-space, which gives a semiclassical evolution of the electronic states in a crystal in the presence of a magnetic field.

The form of the xy-projection of the semiclassical electron motion in the $\mathbf{x}$-space is obtained by the rotation of the trajectory in the $\mathbf{p}$-space by $\pi/2$ in the plane orthogonal to $\mathbf{B}$.

As usually in the case of normal metals, only the energy levels close to ϵ_F are important for consideration. It is easy to see that the shape of the Fermi surface (as well as the direction of $\mathbf{B}$) plays a crucial role for the geometry of the electron trajectories (Fig. 1).

Note that the Fermi surface can be represented here either as a periodic surface in the $\mathbf{p}$-space, or as a compact surface embedded in three-dimensional torus $\mathbb{T}^3$, obtained after the factoring with respect to the reciprocal lattice vectors.

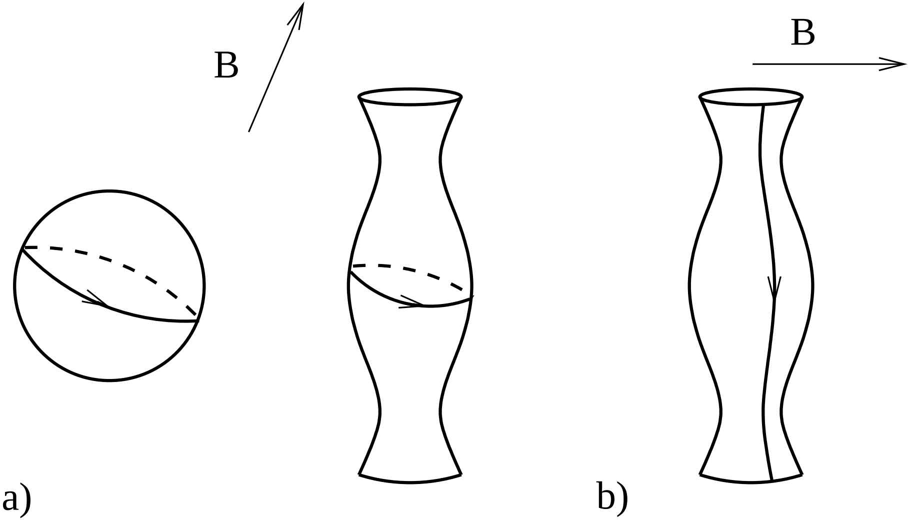

Fig. 1. Compact (a) and periodic (b) electron trajectories on different Fermi surfaces.

As was first noted I.M. Lifshitz, M. Azbel and M.I. Kaganov ([1]), contributions to the conductivity of the trajectories of various types in the limit of strong magnetic fields are very different from each other. So, for example, contributions to the conductivity of compact (Fig. 1, a) and periodic (Fig. 1, b) trajectories found in ([1]), vary greatly in form and have the form:

1) compact trajectories

$$\sigma^{ik}(B) \sim \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} & * \end{pmatrix} , \quad \omega_B\tau \rightarrow \infty \tag{2}$$

2) periodic trajectories

$$\sigma^{ik}(B) \sim \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix} , \quad \omega_B\tau \rightarrow \infty \tag{3}$$

Note that expressions (2) - (3) give only order of magnitude of $\sigma^{ik}(B)$ for $\omega_B \tau \ \rightarrow \ \infty$, and in particular, the values $*$ denote some constants of the order of 1.

As we said above, the z-axis coincides with the direction of the magnetic field **B**. In case 2, the x axis is chosen along the mean direction of the periodic orbit in the **p**-space (and orthogonal to the mean direction of the electron motion in the **x**-space).

It is easy to see a big difference in the behaviour of the conductivity in the plane orthogonal to **B** in the cases 1 and 2. Thus, we have a complete disappearance of conductivity in the plane orthogonal to **B**, in case 1, while in the case 2, the value of σ^{yy} remains finite for $B \rightarrow \infty$. Strong anisotropy of the tensor $\sigma^{\alpha\beta}$ $(\alpha, \beta \ = \ 1, 2)$ can also determine the average direction of the periodic orbits in the **p**-space in case 2, which coincides with the "kernel" of $\sigma^{\alpha\beta}$ for $B \rightarrow \infty$.

In papers [2, 3, 4] the geometry of the electron trajectories for important and nice examples of the Fermi surfaces of nontrivial structure ("fine spatial net", analytic Fermi surfaces, the Fermi surface of tin) was analysed. In the examples described in [2, 3, 4], trajectories more complex than those presented in Fig. 1 also appeared. However, the contribution of these trajectories to the conductivity at high magnetic fields can be also presented in the form (2) - (3).

The fundamental observation about the connection of the geometry of the Fermi surface and electro-magnetic phenomena in strong magnetic fields underlies the theory of the galvanomagnetic phenomena in metals.

Later (Novikov, [8]) it was understood, however, that the problems arising in the theory of electro-magnetic phenomena, can serve as a source for rather deep problems in the three-dimensional topology, theory of dynamical systems and theory of quasi-periodic functions on the plane. Thus, in particular, S.P. Novikov posed the problem of topological classification of foliations on periodic dynamic surfaces defined by system (1). Such a formulation is connected directly to the task of complete semiclassical description of the geometry of the electron trajectories in $\mathbf{p}$-space in a plane orthogonal to the magnetic field $\mathbf{B}$.
A more general formulation of the Novikov problem can be given in the following form:

Give a classification of non-closed level curves of a quasi-periodic function on the plane with a fixed number of quasi-periods.

At the same time S.P. Novikov suggested the first conjecture about the typical behaviour of open electron trajectories in the planes orthogonal to **B**. Let us formulate here this conjecture:

Fix a complex periodic surface in $\mathbb{R}^3$ and arbitrary direction of **B** (Fig. 2). Then, in generic case, any open electronic trajectory will lie in a straight strip of finite width in the plane orthogonal to **B** and go through it (Fig. 3).

We note here also that the trajectories of this type have been named later "topologically regular".

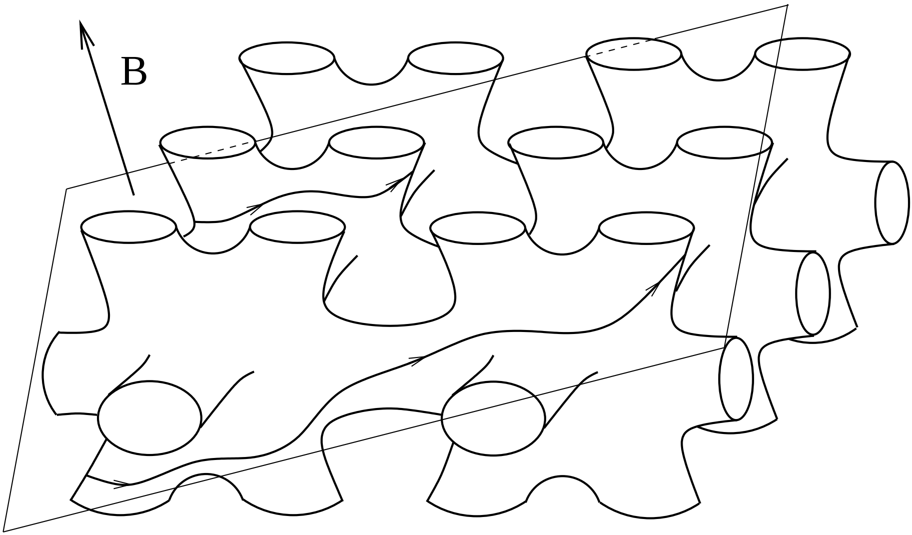

Fig. 2. Arbitrary periodic surface in **p**-space and trajectories for generic direction of **B**.

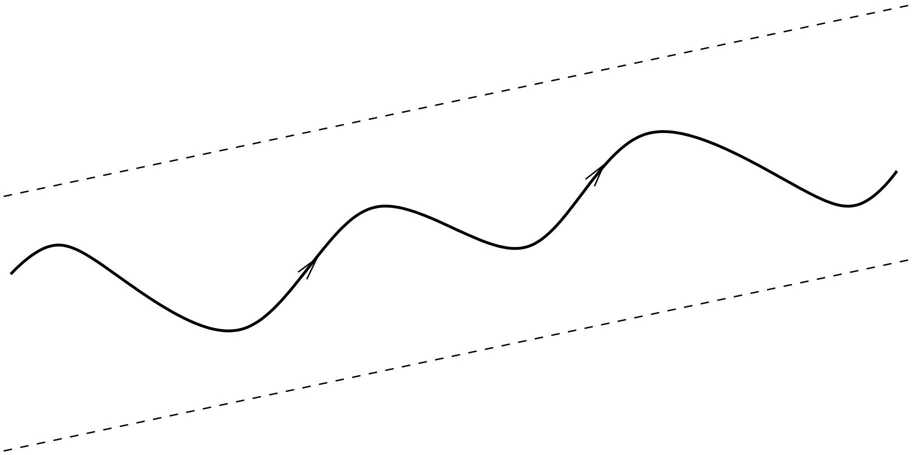

Fig. 3. "Topologically regular" open trajectory in the plane, orthogonal to **B**.

We would like to note immediately that this type of behaviour is not trivial for quasi-periodic functions. Moreover, even in the case of three quasi-periods a more sophisticated (chaotic) behaviour of unclosed trajectories in planes orthogonal to **B** can appear. However, the Novikov conjecture asserts that for the Fermi surfaces and directions of **B** in general position (i.e. except for a set of measure zero) open trajectories demonstrate the above behaviour. Note also that the Novikov conjecture in this form applies only to the case of three quasi-periods.

We shall call "topologically regular" the described above behaviour of non-closed trajectories and chaotic - any other behaviour.

The problem, posed by S.P. Novikov, was investigated on the topological seminar of S.P. Novikov (Novikov, A.V. Zorich, I.A. Dynnikov, S.P. Tsarev), and we can say that now, in principle, it is solved for the case of three quasi-periods ([9, 10, 11, 12, 13, 14, 15, 17, 18, 22, 23]).

The rigorous topological results were used by S.P. Novikov and the author for study of the asymptotic behaviour of magneto-resistance in metals in the case of $\omega_B \tau \rightarrow \infty$ ([16, 20, 21, 26, 27]). Note that the topological methods in this problem give the opportunity to introduce new topological characteristics (topological numbers) observable in the magneto-resistance in metals ([16, 21]) and to describe the new asymptotic regimes of behaviour in the magneto-resistance $\omega_B \tau \rightarrow \infty$, corresponding to the cases of chaotic behaviour of open trajectories on the Fermi surface ([20]). In addition, the complete classification of nonsingular unclosed trajectories arising on the Fermi surfaces ([23]) makes possible to give a complete description of all asymptotic regimes of the magneto-resistance in this limit ([26, 27]).

Here we give a brief description of the picture in the situation described. We omit here most topological formulations and give only a general description.

Let us fix the Fermi surface in $\mathbb{R}^3$ and consider all possible directions of **B** (Fig. 2). Let us parametrise each direction of **B** by the points on the unit sphere $\mathbb{S}^2$ and consider the "angular diagram" where different points on the sphere $\mathbb{S}^2$ will match the different situations on the Fermi surface.

First, we separate the areas on $\mathbb{S}^2$, where only closed trajectories are present on the Fermi surface. The corresponding topology of the dynamical system (1) on the Fermi surface (as in Fig. 4, b) will be called "trivial". The full set of such directions of **B** is always represented by a set of open regions on the sphere, whose complement is a closed subset of $\mathbb{S}^2$.

The corresponding behaviour of the magneto-conductivity for these areas is given by expression (2) and will not be interesting for us from topological point of view.

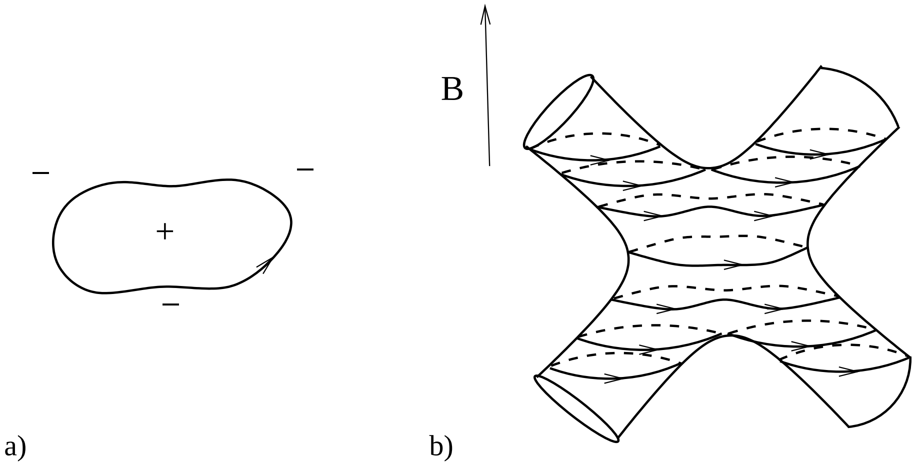

Fig. 4. (a) A closed trajectory in the plane orthogonal **B**. Signs " $+$ " and " - " correspond to domains with $\epsilon(\mathbf{p}) > \epsilon_F$ and $\epsilon(\mathbf{p}) < \epsilon_F$, respectively, (b) An example of the Fermi surface, which contains only closed trajectories.

The rest of $\mathbb{S}^2$ is the set, for which non-closed trajectories are present on the Fermi surface. It is this set, as well as the geometry of the corresponding trajectories, which will be the most interesting to us.

Let us formulate now topological results concerning this set on the unit sphere and the corresponding trajectories.

I) Let us first consider the non-closed trajectories stable with respect to the small rotations of $\mathbf{B}$. This formulation means that we are interested in situations where the non-closed trajectories exist for the whole area of the unit sphere and in a sense are "close" to each other for close directions of $\mathbf{B}$. For such trajectories the following important statements are true:

a) All non-closed trajectories on the Fermi surface, stable to small rotations of $\mathbf{B}$, are topologically regular (Fig. 3).

b) All these trajectories have the same mean direction, given by the intersection of the planes orthogonal to $\mathbf{B}$ with an integer-valued (i.e. generated by two vectors of the reciprocal lattice) plane Γ, which is fixed for a given family of non-closed trajectories.

The plane Γ can be given by the triple of integers (m_1, m_2, m_3) with the aid of the equation

$$m_1 (\mathbf{e}_1, \mathbf{r}) \; + \; m_2 (\mathbf{e}_2, \mathbf{r}) \; + \; m_3 (\mathbf{e}_3, \mathbf{r}) \; = \; 0$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ - are the basis vectors of the direct lattice.

The most important property of the plane Γ is its local stability on the angular diagram on $\mathbb{S}^2$. Integer triple (m_1, m_2, m_3) is the same for a family of open trajectories satisfying the above condition. The plane of Γ, however, may change when we cross the border of the zones on the angular diagram where stability conditions are no longer met.

We will call the regions on $\mathbb{S}^2$ corresponding to stable families of open trajectories with the same plane Γ_α - "stability zones" Ω_α on the angular diagram. Thus, we will have in general a set of different "stability zones" Ω_α with the corresponding numbers $(m_1^\alpha, m_2^\alpha, m_3^\alpha)$ on the angular diagram corresponding to a Fermi surface of general form (Fig. 5). As can be shown ([23]), each "stability zone" is a domain with piecewise smooth boundary on the sphere $\mathbb{S}^2$.

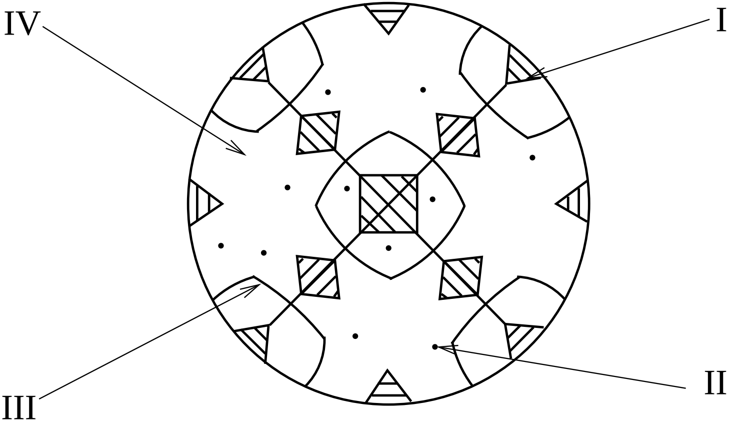

Fig. 5. Schematic representation of the angular diagram for complex enough Fermi surface. Here I - "stability zones" II - "chaotic directions", III - directions of **B**, for which unstable periodic orbits appear, IV - directions of **B**, for which only compact trajectories present on the Fermi surface.

Let us now divide the directions of $\mathbf{B}$ into three "categories":

1) Rational directions of $\mathbf{B}$.
We call the direction of $\mathbf{B}$ - rational if the plane, orthogonal to $\mathbf{B}$, contains two linearly independent vectors of the reciprocal lattice.

2) Directions of $\mathbf{B}$ of irrationality 2.
We call the direction of $\mathbf{B}$ - direction of irrationality 2 if the plane, orthogonal to $\mathbf{B}$, contains only one (up to a factor) reciprocal lattice vector.

3) Directions of $\mathbf{B}$ of irrationality 3.
We call the direction of $\mathbf{B}$ - direction of irrationality 3, if the plane, orthogonal to $\mathbf{B}$, does not contain reciprocal lattice vectors.

It is easy to see that directions of irrationality 3 represent generic directions, while the other two types have measure zero on the unit sphere.

It is easy to see that for rational directions of **B** the corresponding picture in the plane, orthogonal to **B**, is periodic. All non-closed trajectories in this case are also periodic and it is easy to show that they always lie in straight strips of finite width and pass them through. The only remark here is that periodic orbits are not necessarily stable to small rotations of **B**, and therefore, do not exhibit fully the properties that we have described for the stable topologically regular trajectories. In addition, these trajectories can have different mean directions in different planes orthogonal to **B**. In the latter case, by summing the contributions of (3) in different coordinate systems, we obtain the magneto-conductivity tensor which has the full rank for $\omega_B \tau \rightarrow \infty$ for this direction of **B**.

The comments above exhaust, generally speaking, all the possibilities for nonsingular unclosed trajectories in the case of rational directions of **B**.

We now describe the situation in the case of directions of **B** of irrationality 2. Note first that the unstable periodic trajectories may also exist in this case. The simplest example of this situation is the same picture Fig. 1 b, where periodic orbits always arise if **B** is orthogonal to the cylinder axis. However, the situation, when the periodic orbits in different planes, orthogonal to **B**, have different mean directions, in this case is not possible. The total contribution of all such trajectories in magneto-conductivity so has the form (3) in an appropriate coordinate system and the general conductivity tensor always has a strong anisotropy for $\omega_B \tau \rightarrow \infty$ in the plane, orthogonal to **B**.

However, the unstable periodic orbits do not exhaust all possibilities for unstable trajectories for directions of **B** of irrationality 2. Thus, more complex "chaotic" trajectories can also occur in this situation. The corresponding example was built by S.P. Tsarev and was the first example of a chaotic trajectory on the Fermi surface. We will not describe here in detail the example of Tsarev and say only that the Tsarev chaotic trajectory still have an asymptotic direction in the plane orthogonal to **B**. Thus, in these cases, we still have regular "drift" of the electronic states along some direction in **p**-space, but "oscillations" in the orthogonal direction can be thus arbitrarily large for large segments of the trajectory. As can be shown ([18]), this situation always takes place for the chaotic trajectories in the case of directions of **B** of irrationality 2. We will call the trajectories of this type the Tsarev chaotic trajectories.

Contribution to magneto-conductivity given by chaotic trajectories of Tsarev for $\omega_B \tau \rightarrow \infty$ also has strong anisotropy in the plane orthogonal to $\mathbf{B}$, and reminds contribution of (3) with a slightly different dependence on the value of B. Asymptotic direction of the Tsarev trajectories (in $\mathbf{p}$-space) coincides with the direction of "zero conductivity" in the plane orthogonal to $\mathbf{B}$, and can be measured experimentally.

We now note that the above cases exhaust all opportunities for non-singular open electron trajectories in the case of magnetic field directions of irrationality 2.

We now turn to the most general case of the directions of **B** of irrationality 3. Let us note that no periodic orbits may exist in this case and, therefore, we can not have unstable periodic orbits. However, the chaotic trajectories, showing a rather complex behaviour in planes orthogonal to **B**, can occur in this situation. Examples of such trajectories were built by I.A. Dynnikov ([18]) and represent the most complex type of trajectories arising on the Fermi surfaces. The trajectories of this type exhibit a "random walk" in the plane orthogonal to **B**, covering up larger and larger area (Figure 6).

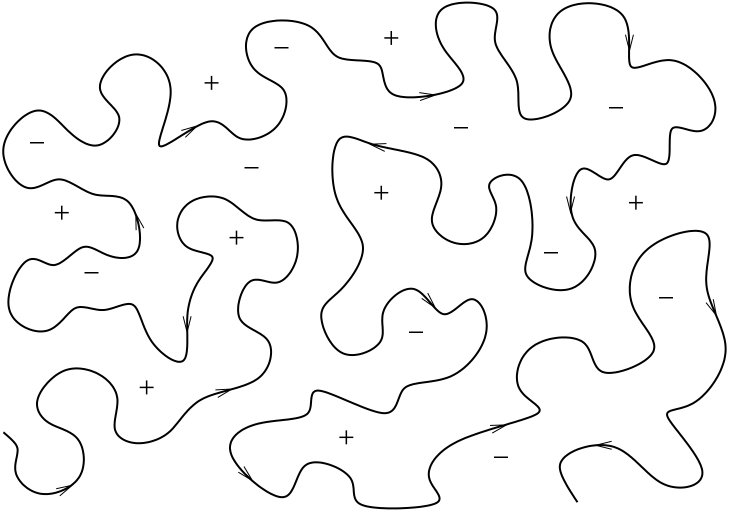

Fig. 6. Dynnikov chaotic trajectory in the plane orthogonal to **B**. Signs " + " and " - " correspond to areas with $\epsilon(\mathbf{p}) > \epsilon_F$ and $\epsilon(\mathbf{p}) < \epsilon_F$, respectively.

As in the case of Tsarev, no other type of non-closed trajectories can exist on the Fermi surface in the presence of the Dynnikov trajectory. As was shown in [23], the measure of the corresponding directions of **B** on the unit sphere is equal to zero for the Fermi surface in general position. However, this situation is also possible in the experiment for metals with rather complicated Fermi surfaces, and specially chosen direction of **B**.

The contribution of such trajectories to magneto-conductivity is very different from those given by (2) - (3). The most interesting thing is that the trajectory of this type do not give a contribution to the magneto-conductivity both in directions orthogonal to **B**, and in the direction along **B** in the limit $\omega_B \tau \rightarrow \infty$ ([20]). As a consequence of this fact, "longitudinal" conductance is given in this situation only by the contribution of closed trajectories (if they exist) on the Fermi surface. It can be expected, therefore, that the "longitudinal" conductivity must have sharp local minima for these directions of magnetic field.

The form of the corresponding contribution to the conductivity tensor was proposed in [20]:

$$\sigma^{ik}(B) \;\sim\; \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{2\beta-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{\beta+\delta-2} \\ (\omega_B\tau)^{-1} & (\omega_B\tau)^{2\alpha-2} & (\omega_B\tau)^{\alpha+\delta-2} \\ (\omega_B\tau)^{\beta+\delta-2} & (\omega_B\tau)^{\alpha+\delta-2} & (\omega_B\tau)^{2\delta-2} \end{pmatrix} \qquad (4)$$

$\omega_B\,\tau \;\rightarrow\; \infty$ where α, β, δ are some parameters characterising the trajectory $(0 < \alpha, \beta, \delta < 1)$. [1]

One can see from (4), that the conductivity decreases both in the plane orthogonal to **B** (but more slowly than in (2)) and in the direction along **B**. Recall that the contribution of (4) must be, in general, complex, with the contribution of (2) from closed trajectories.

[1]In general, σ^{zz} also has a nonvanishing correction of order T^2/ϵ_F^2, which is very small for most normal metals.

Types of trajectories described above exhaust all possibilities now for nonsingular open trajectories on the Fermi surface.

We note here that we have not assumed that the Fermi surface is connected. The results stated thus hold in the case where the Fermi surface consists of several disconnected components. However, the condition of the non-self-intersecting of the Fermi surface is important here. So, we always assume that the Fermi surface does not contain intersecting components. Let us add that there are also some topological conditions of general position, which are also assumed to be satisfied for derivation of the above results. We will not discuss here these conditions in detail and assume that they are always satisfied for physical Fermi surfaces. A typical form of the angular diagrams for Fermi surface of the general form shown in Fig. 5.

Let us describe here also all possible asymptotic regimes of the behaviour of conductivity in the limit $\omega_B \tau \rightarrow \infty$ ([21, 26, 27]).

1) According to the above angular diagram, we can distinguish first the directions of **B**, where only closed trajectories are present and thus the complete conductivity tensor has the asymptotic form (2)

$$\sigma^{ik}(B) \;\; \sim \;\; \frac{ne^2\tau}{m^*} \left(\begin{array}{ccc} (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-2} & (\omega_B \tau)^{-1} \\ (\omega_B \tau)^{-1} & (\omega_B \tau)^{-1} & * \end{array} \right) \;\; , \quad \omega_B \tau \;\; \rightarrow \;\; \infty$$

2) There are stability zones Ω^{α}, corresponding to conductivity tensor (3)

$$\sigma^{ik}(B) \;\sim\; \frac{ne^2\tau}{m^*} \begin{pmatrix} (\omega_B\tau)^{-2} & (\omega_B\tau)^{-1} & (\omega_B\tau)^{-1} \\ (\omega_B\tau)^{-1} & * & * \\ (\omega_B\tau)^{-1} & * & * \end{pmatrix} \;, \quad \omega_B\tau \;\rightarrow\; \infty$$

The only direction along which the conductivity vanishes at $\omega_B\tau \;\rightarrow\; \infty$ coincides with the mean direction of open trajectories in **p**-space and always belongs to some integral plane Γ^{α}, generated by two reciprocal lattice vectors. The plane Γ^{α} is the same for all points of the stability zone Ω^{α} and can be given by a triple of integers $(m^1_{\alpha}, m^2_{\alpha}, m^3_{\alpha})$, so that for all mean directions $\boldsymbol{\eta}$ of topologically regular trajectories in the stability zone, we have the relation

$$m^1_{\alpha}(\boldsymbol{\eta}, \mathbf{e}_1) \;+\; m^2_{\alpha}(\boldsymbol{\eta}, \mathbf{e}_2) \;+\; m^3_{\alpha}(\boldsymbol{\eta}, \mathbf{e}_3) \;=\; 0$$

where $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ - are the basis direct lattice vectors.

3) Along the lines of the angular diagram, where there are additional open trajectories, their contribution is still is given by (3), but unstable with respect to small rotations of **B** taking away from these lines. In addition, these lines may exist in regions IV and I of the angular diagram Fig. 5. In the second case, the mean direction of additional periodic open trajectories coincide with the direction of the stable open trajectories in the stability zone and the total conductivity tensor is also given by (3), the coefficients of which, however, have jumps on the corresponding lines.

4) For directions of **B**, corresponding to chaotic Tsarev cases, the asymptotic behaviour of the conductivity tensor is close to (3) with a slightly different dependence on $\omega_B \tau$.

5) For directions of **B**, corresponding to chaotic Dynnikov cases, the asymptotic behaviour of the conductivity tensor has an anomalous form, since the contribution of the corresponding chaotic trajectories decreases in all directions in the limit $\omega_B \tau \rightarrow \infty$. In this case we expect decreasing of conductivity in all directions in the plane orthogonal to **B**, and finite conductivity along **B**, given by the contribution of compact trajectories. However, it is expected that the conductivity along **B** will have a sharp minimum at the corresponding points of the angular diagram because of the vanishing of the contribution of the part of the Fermi surface, swept by the chaotic trajectory.

6) We should note also one more possibility. That is, let us call the direction $m^1_\alpha \mathbf{e}_1 + m^2_\alpha \mathbf{e}_2 + m^3_\alpha \mathbf{e}_3$ - a special rational direction if it belongs to the corresponding stability zone Ω^α.

Generally speaking, such directions give several possibilities for the asymptotic behaviour of the conductivity tensor for $\omega_B \tau \rightarrow \infty$.

We would like to describe here, however, also another useful approach the considered problem mentioned above, based on the consideration of the whole dispersion relation $\epsilon(\mathbf{p})$ ([19, 21, 23]). Let us consider the dispersion relation $\epsilon(\mathbf{p})$, such that $\epsilon_{min} \leq \epsilon(\mathbf{p}) \leq \epsilon_{max}$ for the conductivity band. Consider all the energy levels $\epsilon(\mathbf{p}) = c$, where $\epsilon_{min} \leq c \leq \epsilon_{max}$ and the corresponding electron trajectories for a fixed direction of $\mathbf{B}$.

We can state the same theorems for open electron trajectories at each energy level $\epsilon(\mathbf{p}) = c$ in the same form as previously formulated them for the Fermi energy ϵ_F. The most important topological statement in this analysis is that the angle diagrams for different levels of energy "do not compete" with each other.

This means that if we have open trajectories of some type (i.e. "topologically regular", periodic, "chaotic" etc.) at some energy level, then we can have open trajectories only of the same type or just the closed trajectories at all other energy surfaces for this direction of **B**. (In fact, in the case of chaotic trajectories, open trajectories exist only on one energy level ([23]) as we shall see below). This fact allows us to define the "stability zones or "chaotic" directions of **B** for the whole dispersion relation. You can then introduce the angular diagram for the whole dispersion relation $\epsilon(\mathbf{p})$, which will show the type of open trajectories at all energy levels in the range $\epsilon_{min} \leq \epsilon \leq \epsilon_{max}$. In this case, the corresponding picture has the following properties (see [23]):

1) Each point of $\mathbb{S}^2$ either belongs to some stability zone (or its boundary), or represents a chaotic direction of Tsarev or Dynnikov type.

2) For any direction of **B** open trajectories exist either in a connected energy interval $\epsilon_1(\mathbf{B}) \leq \epsilon \leq \epsilon_2(\mathbf{B})$ ($\epsilon_{min} < \epsilon_1(\mathbf{B}) \leq \epsilon_2(\mathbf{B}) < \epsilon_{max}$) or only at one energy level $\epsilon(\mathbf{p}) = \epsilon_0(\mathbf{B})$.

3) In all the cases where we have the situation of finite energy interval containing open trajectories ($\epsilon_2(\mathbf{B}) > \epsilon_1(\mathbf{B})$) we always have the "topologically regular" case and the corresponding direction belongs to some stability zone on $\mathbb{S}^2$.

In other words, we can not have directions of **B**, for which there is no open trajectories on any energy level. Moreover, open trajectories always exist in a connected energy interval that can be reduced to just one energy level. Chaotic trajectories can appear only in the latter case, so they disappear at any other energy level for the corresponding direction of **B**.

We can call the situation $\epsilon_2(\mathbf{B}) > \epsilon_1(\mathbf{B})$ the generic situation and the situation $\epsilon_2(\mathbf{B}) = \epsilon_1(\mathbf{B})\epsilon_0(\mathbf{B})$ - degenerate situation. "Chaotic" trajectories (of Tsarev or Dynnikov type) can arise only in the "degenerate" situations according to this terminology. (Note that we also have $\epsilon_2(\mathbf{B}) = \epsilon_1(\mathbf{B})$ at the boundaries of "stability zones", but in this case the trajectories are topologically regular).

The general structure of the set of "stability zones" and "chaotic directions" on the sphere is quite nontrivial. In general the angular diagram contains an infinite number of stability zones with increasing "topological numbers". [2]. The stability zones form an everywhere dense subset of $\mathbb{S}^2$. However, an infinite number of "chaotic directions" also presents on the angular diagram in this situation. The union of these sets ("stability zones" and "chaotic directions") covers the entire sphere $\mathbb{S}^2$. Each "chaotic direction" can be approximated by a sequence of regular directions on $\mathbb{S}^2$, but the form of the corresponding regular trajectories becomes an increasingly complex, giving in the limit the chaotic trajectory. The structure of the set of "chaotic directions" on $\mathbb{S}^2$ resembles the structure of the Cantor sets in general.

[2] In fact there are only two possibilities: (1) only one "stability zone", filling unit sphere $\mathbb{S}^2$, (2) an infinite number of stability zones on $\mathbb{S}^2$ (see [23]).

Let us formulate here the second conjecture of S.P, Novikov concerning the Hausdorff dimension of the set of "chaotic directions" on $\mathbb{S}^2$:

Novikov conjecture.

The Hausdorff dimension of the set of "chaotic directions" on $\mathbb{S}^2$ is strictly less than 2.

The second Novikov conjecture was confirmed in numerical experiments (see [25]). However, the rigorous proof of the second Novikov conjecture is still unknown.

Let us add finally that the directions of **B** of irrationality 1 and 2, for which there are unstable periodic trajectories, can also be defined for the dispersion relation and are also present on the total angular diagram. Fig. 7 shows a typical angular diagram for the general dispersion relation $\epsilon(\mathbf{p})$.

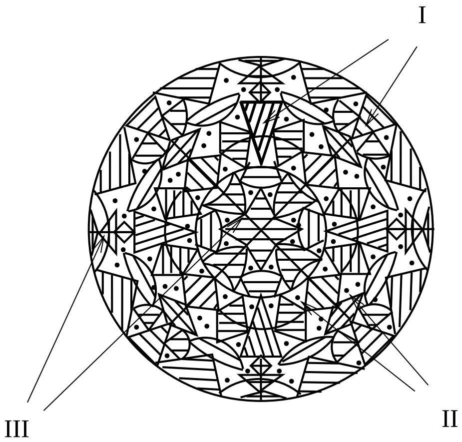

Fig. 7. Typical angular diagram for the general dispersion relation (very schematically). Here I - the set of stability zones; II - the set of random directions; III - set where additional unstable periodic orbits appear.

Note that the angular diagram for normal metals Fig. 5, reveals in this case only the part diagram in Fig. 7, where the open electronic trajectory fall to the Fermi level (ie, the condition $\epsilon_1(\mathbf{B}) \leq \epsilon_F \leq \epsilon_2(\mathbf{B})$ or $\epsilon_F = \epsilon_0(\mathbf{B})$) is satisfied.

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