Higher order topological invariants from the Chern-Simons action

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References

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Outline

duality

higher order gauge fields

higher order closed curves by duality

examples

Chern-Simons path integral

Wilson loop operators as observables

Intersection and linking numbers

disjoint oriented closed curves $C, C' \subset \mathbb{R}^3$
surfaces $S, S' \subset \mathbb{R}^3$ such that $\partial S = C, \partial S' = C'$

count the number of points $\#(C \cap S')$
smooth deformations of $C \subset \mathbb{R}^3 \setminus C'$ change $\#(C \cap S')$
$\#(C \cap S')$ is not invariant under the deformations

transverse intersection point $P = C \cap S'$
intersection index $I(C, S', P) \in \{-1, 1\}$ depends on the relative orientation of C and S' at P
intersection number $I(C, S') = \sum_{P \in C \cap S'} I(C, S', P)$
$I(C, S')$ is invariant under the above deformations

Intersection and linking numbers

another invariance:

$I(C,S') = I(C,\tilde{S}')$ for any $\partial S' = \partial \tilde{S}' = C'$

conclude that $I(C,S')$ depends only on C, C'

call it the linking number: $L(C,C') = I(C,S')$
check orientations: $L(C,C') = L(C',C)$

use Green functions to find the Gauss formula

$$L(C,C') = (4\pi)^{-1} \int_{C\times C'} \sum_{a,b,c} \epsilon_{abc} \frac{(x-x')^a}{\|x-x'\|^3} dx^b \wedge dx'^c$$

Duality

closed curve $C \subset \mathbb{R}^3$
a dual set (C, F), where $F \in \Lambda^2$ and $dF = 0$
existence from Poincaré duality and de Rham theorem:

$$\int_C B = \int_{\mathbb{R}^3} B \wedge F \text{ for any } B \in \Lambda^1$$

B is arbitrary $\Rightarrow$ $\operatorname{supp} F = C$
$\mathbb{R}^3$ is simply connected $\Rightarrow$ there exists $A \in \Lambda^1$ such that $F = dA$
Stokes theorem:

$$\int_S dB = \int_{\mathbb{R}^3} B \wedge dA = \int_{\mathbb{R}^3} dB \wedge A$$

Duality

a particular solution of

$$\int_S dB = \int_{\mathbb{R}^3} dB \wedge A$$

is

$$A(x) = \int_{y \in S} \delta(x-y) \sum_a dx^a * dy^a$$

a singular gauge with $\mathrm{supp} A = S$

another dual set (C', F'):

$$\int_{C'} B' = \int_{\mathbb{R}^3} B' \wedge F',\ F' = dA' \text{ for any } B' \in \Lambda^1$$

Duality

take $B = A'$, $B' = A$ and use the Stokes theorem:

$$\int_C A' = \int_{C'} A = \int_{\mathbb{R}^3} A \wedge dA' = L(C, C')$$

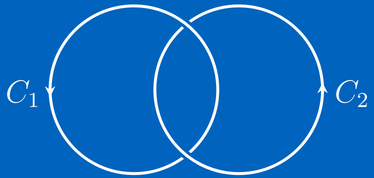
$L(C, C')$ is a topological invariant:

$L(C, C') = L(\tilde{C}, C')$ for any $C, \tilde{C} \subset \mathbb{R}^3 \setminus C'$

conclude $dA'|_{\mathbb{R}^3 \setminus C'} = 0$

if $A' = df'$, then $L(C, C') = 0$ trivially
a nontrivial case: $dA' = 0$, $A' \neq df'$

First order linking example



the Hopf link

$C_1 \cap S_2 = P$

use the singular gauge to compute

$$L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 1$$

the first order non-self linking is $\tilde{L}_1(C, C) = 2$

for $C = \cup_{1 \leq i \leq N} C_i$:

$$L_1(C, C) = \sum_{i,j} L(C_i, C_j)$$

Second order fields

$\{C_i\}_{1\leq i\leq N}$ disjoint closed curves in $\mathbb{R}^3$
first order dual sets (C_i, F_i)

second order dual sets (C_{ij}, F_{ij})

$$\int_{C_{ij}} B = \int_{\mathbb{R}^3} B \wedge F_{ij} \text{ for any } B \in \Lambda^1$$

B is arbitrary $\Rightarrow$ $\operatorname{supp} F_{ij} = C_{ij}$

Second order fields

a general 2-form in terms of A_i, A_j:

$$F_{ij} = f_i dA_j - f_j dA_i + gA_i \wedge A_j$$

determine arbitrary functions f_i, f_j, g from $dF_{ij} = 0$

$$dF_{ij}|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0 \Leftarrow dg|_{\mathbb{R}^3 \setminus (C_i \cup C_j)} = 0 \Leftarrow g = 1$$

$$dF_{ij}|_{C_i} = 0 \Rightarrow (df_j - A_j)|_{C_i} = 0 \Rightarrow L(C_i, C_j) = 0$$

second order field F_{ij} can be defined only if $L(C_i, C_j) = 0$

Second order fields

does $dF_{ij} = 0$ imply $F_{ij} = dA_{ij}$?

yes in $\mathbb{R}^3$
no in $\mathbb{R}^3 \setminus \{C_i \cup C_j\}$

seek a solution $A_{ij} = \frac{1}{2}(\gamma_i A_j - \gamma_j A_i)$

require $F_{ij} = dA_{ij}$ and find

$$
\begin{aligned}
(d\gamma_i - A_i)|_{\mathbb{R}^3\setminus(C_i \cup C_j)} &= 0 \\
d\gamma_i|_{C_i} &= 0 \\
(d\gamma_i - 2A_i)|_{C_j} &= 0 \\
(\gamma_i - 2f_i)|_{C_j} &= 0
\end{aligned}
$$

plus the same expressions with $i \longleftrightarrow j$

Second order fields

conclude

[nonlocal quantity] $\gamma_i|_{\mathbb{R}^3\setminus(\cup_k C_k)} = \int_{\Gamma_i} A_i$, where $\Gamma_i \subset \mathbb{R}^3 \setminus (\cup_k C_k)$

now find $\gamma_i|_{C_j}$

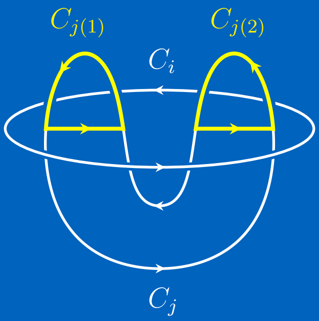
two cases

(1) $S_i \cap C_j = \emptyset \;\Rightarrow\; d\gamma_i|_{C_j} = 0 \;\Rightarrow\; \gamma_i|_{C_j} = \text{const}$

(2) $S_i \cap C_j \neq \emptyset \;\Rightarrow\; S_i \cap S_j = \cup_m \big(S_i \cap S_j\big)_{(m)}$ [disjoint segments]

Second order fields

(2) $S_i \cap C_j \neq \varnothing \;\Rightarrow\; S_i \cap S_j = \cup_m (S_i \cap S_j)_{(m)}$ [disjoint segments]



$C_{j(m)}$ is the segment of C_j which closes the curve $(S_i \cap S_j)_{(m)}$ and agrees with its orientation

the resulting closed curve is
$C_{ij(m)} = C_{j(m)} \cup (S_i \cap S_j)_{(m)}$

define $C'_j = \cup_m C_{j(m)}$ and its complement C''_j in C_j

Second order fields

recall $F_{ij} = dA_{ij} \Rightarrow (d\gamma_i - 2A_i)|_{C_j} = 0$

conclude

$\gamma_i|_{C'_j} = \text{const},\ \gamma_i|_{C''_j} = \text{const},\ \gamma_i|_{C'_j} - \gamma_i|_{C''_j} = 2$

set

$\gamma_i|_{C'_j} = 2,\ \gamma_i|_{C''_j} = 0$

now the duality

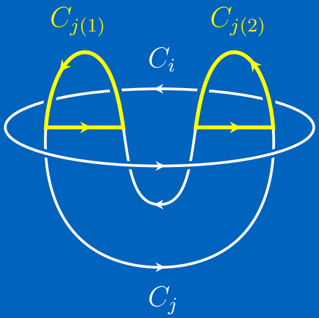
$$\int_{C_{ij}} B = \int_{\mathbb{R}^3} B \wedge F_{ij} \text{ for any } B \in \Lambda^1$$

becomes

$$\int_{C_{ij}} B = -\int_{C'_i} B + \int_{C'_j} B + \int_{S_i \cap S_j} B \text{ for any } B \in \Lambda^1$$

Second order fields

$$\int_{C_{ij}} B = -\int_{C_i'} B + \int_{C_j'} B + \int_{S_i \cap S_j} B \text{ for any } B \in \Lambda^1$$



conclude
$C_{ij} = C_i'^{-1} \cup C_j' \cup (S_i \cap S_j)$

$C_{ij} = [a_i, a_j] = a_i a_j a_i^{-1} a_j^{-1}$
is a 2 order commutator path

agrees with
$\operatorname{supp} F_{ij} = C_{ij}$

use Stokes theorem $\int_{S_{ij}} dA = \int_{\mathbb{R}^3} dA \wedge A_{ij}$ for $\partial S_{ij} = C_{ij}$ to find
$\operatorname{supp} A_{ij} = S_{ij}$

Third order fields

similar to

$$dA_{ij}|_{\mathbb{R}^3\setminus(C_i\cup C_j)} = (A_i \wedge A_j)|_{\mathbb{R}^3\setminus(C_i\cup C_j)} \text{ for } L(C_i, C_j) = 0$$

define

$$dA_{ijk}|_{\mathbb{R}^3\setminus(C_i\cup C_j\cup C_k)} = (A_i \wedge A_{jk} + A_{ij} \wedge A_k)|_{\mathbb{R}^3\setminus(C_i\cup C_j\cup C_k)}$$

for $L(C_{ij}, C_k) = 0$ for all (i, j, k)

extend A_{ijk} to the whole $\mathbb{R}^3$
topological obstruction to extensions

integrability requires
$L(C_i, C_j) = 0,\ L(C_i, C_k) = 0,\ L(C_j, C_k) = 0,\ L(C_i, C_j, C_k) = 0$

$C_{ijk} = [a_i, [a_j, a_k]]$ is a 3 order commutator path

Higher order fields

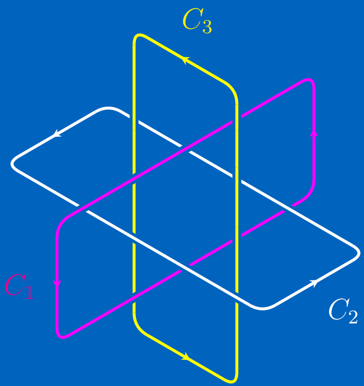
recursively define dual sets

$$(C_{I_p}, dA_{I_p}), \quad I_p = (i_1, i_2, \ldots, i_p), \quad p = 1, 2, \ldots, n$$

If all linkings of order q, where $1 \leq q \leq p-1$, vanish, then we can similarly construct dual sets (C_{I_p}, dA_{I_p}), where $I_p = (i_1, i_2, \ldots, i_p)$. The quantities

dA_{I_p} are related to Massey products of cohomology groups

Second order linking example



the Borromean rings

choose $C_i \cap S_j = \varnothing$ for $i \neq j$

use the singular gauge to compute

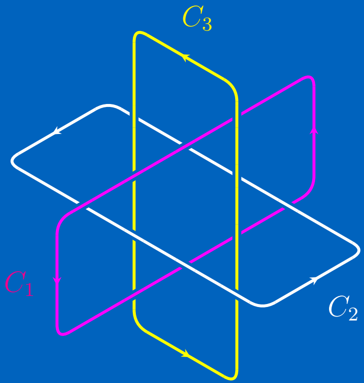
$$L(C_i, C_j) = \int_{\mathbb{R}^3} dA_i \wedge A_j = 0 \text{ for } i \neq j$$

take $C = C_1 \cup C_2 \cup C_3$

first order non-self linking:

$$\tilde{L}_1(C, C) = 2L(C_1, C_2) + 2L(C_2, C_3) + 2L(C_3, C_1) = 0$$

Second order linking example



choose
$S_1 \cap S_2 \cap S_3 = P$
$C_i \cap S_k = \varnothing$
$C_j \cap S_k = \varnothing$

use the singular gauge to compute

$$L(C_{ij}, C_k) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_k - \frac{1}{2}\int_{C_i} \gamma_j A_k + \frac{1}{2}\int_{C_j} \gamma_i A_k = \epsilon_{ijk}$$

Second order linking example

similarly

$$\tilde{L}(C_{ij}, C_j) = \int_{\mathbb{R}^3} A_i \wedge A_j \wedge A_j - \frac{1}{2}\int_{C_i} \gamma_j A_j + \frac{1}{2}\int_{C_j} \gamma_i A_j = 0$$

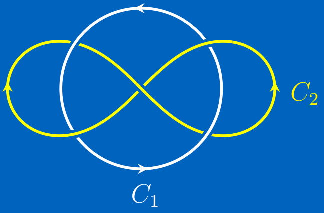
for any $i \neq j$

take $C = C_1 \cup C_2 \cup C_3 \cup C_{12} \cup C_{23} \cup C_{31}$

the second order non-self linking

$$\tilde{L}_2(C, C) = 2L(C_{12}, C_3) + 2L(C_{23}, C_1) + 2L(C_{31}, C_2) = 6$$

Third order linking example



the Whitehead link

choose $C_1 \cap S_2 = P_1 \cup P_2$

use the singular gauge to compute

$$L(C_1, C_2) = \int_{\mathbb{R}^3} dA_1 \wedge A_2 = 0$$

take $C = C_1 \cup C_2$

first order non-self linking: $\tilde{L}_1(C, C) = 0$

Third order linking example

similarly

$$L(C_{12}, C_k) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_k - \frac{1}{2}\int_{C_1} \gamma_2 A_k + \frac{1}{2}\int_{C_2} \gamma_1 A_k = 0, \; k \in \{1, 2\}$$

take $C = C_1 \cup C_2 \cup C_{12}$

second order non-self linking: $\tilde{L}_2(C, C) = 0$

Third order linking example

choose S_1 and S_2 such that
$S_{12} = S_{12(1)} \cup S_{12(2)}$
$S_1 \cap S_2 \cap S_{12} = P$

use the singular gauge again
third order self linking

$$L(C_{12}, C_{12}) = \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 2$$

Third order linking example

third order curves C_{121} and C_{212}

$$L(C_{121}, C_2) = \int_{\mathbb{R}^3} (A_1 \wedge A_{21} + A_{12} \wedge A_1) \wedge A_2 = 2 \int_{\mathbb{R}^3} A_1 \wedge A_2 \wedge A_{12} = 4$$

$$L(C_{212}, C_1) = 4$$

take $C = C_1 \cup C_2 \cup C_{12} \cup C_{121} \cup C_{212}$

third order non-self linking

$$\tilde{L}_3(C, C) = 2L(C_{121}, C_2) + 2L(C_{212}, C_1) = 16$$

Path integral

Chern-Simons field theory action $S(B) = \int_{\mathbb{R}^3} B \wedge dB$

necessary topological properties:

$S(B)$ does not depend on the choice of metric
$S(B)$ can be related to linking numbers

choose a closed curve Γ_α such that $dB|_{\Gamma_\alpha} = dB_\alpha = \text{const}$

[closed Γ_α may require using $\mathbb{S}^3$ instead of $\mathbb{R}^3$]
[avoid field configurations with sources (monopoles)]

Path integral

$\Gamma = \cup_\alpha \Gamma_\alpha$ densely fills $\mathbb{R}^3 \Rightarrow$

$B = \sum_\alpha B_\alpha$ for any $B \in \Lambda^1$

$\Rightarrow$

$$S(B) = \sum_{\alpha,\beta} L(\Gamma_\alpha, \Gamma_\beta)$$

[choice of measure for Γ?]

$S(B) = L(\Gamma, \Gamma)$ self-linking of the set of closed field lines of dB

Path integral

$dB = 0$ is the classical equation of motion for $S(B)$

disjoint closed curves $C = \cup_\alpha C_\alpha$

$\int_C B$ is invariant with respect to deformations of $C \iff dB = 0$

$S(B)$ is a topological observable (at least in the semiclassical limit)

similarly $$\int_C B = \sum_{\alpha,\beta} L(\Gamma_\alpha, C_\beta) = L(\Gamma, C)$$

Path integral

path integral measure $\exp\left[iS(B)\right]$

Wilson loop operator $W(C,B) = \exp\left[i\int_C B\right]$ as an observable

expectation value $Z(C) = \int DB \exp\left[iS(B)\right] W(C,B)$

use duality (C, dA) to find

$$Z(C) = \int DB \exp\left[i\int_{\mathbb{R}^3} B \wedge d(B+A)\right]$$

Path integral

change the variable $B = B' - \frac{1}{2}A$ to find

$$Z(C) = \int DB \exp\left[i \int_{\mathbb{R}^3} B \wedge d(B + A) \right]$$
$$= \int DB' \exp\left[i \int_{\mathbb{R}^3} B' \wedge dB' - \frac{1}{2} i \int_{\mathbb{R}^3} d(B' \wedge A) - \frac{1}{4} i \int_{\mathbb{R}^3} A \wedge dA \right]$$

ignore the boundary term:

$$Z(C) = Z(\varnothing) \exp\left[-\frac{1}{4} i (L(C, C)) \right]$$

$$Z(\varnothing) = \int DB' \exp\left[i \int_{\mathbb{R}^3} B' \wedge dB' \right]$$

$$L(C, C) = \int_{\mathbb{R}^3} A \wedge dA$$

Path integral

C_{I_p} is a curve of order p
$C = \cup_p \cup_{I_p} C_{I_p}$

a product of Wilson loop operators as a single Wilson loop operator for the union of the corresponding loops

$$\prod_p \prod_{I_p} W(C_{I_p}, B) = W(C, B)$$

$\Rightarrow$

$$L(C, C) = \sum_{p,q} \sum_{I_p, J_q} L(C_{I_p}, C_{J_q})$$

$L(C_{I_p}, C_{J_q})$ is the first order linking of a p-curve and q-curve
$L(C_{I_p}, C_{J_q})$ is the linking of order $p + q - 1$ of the curves $\{C_i\}_{1 \le i \le N}$

Path integral

linkings of various orders

$$L(C,C) = \sum_r L_r(C,C)$$

$$L_1(C,C) = \sum_{i,j} L(C_i, C_j)$$

$$L_2(C,C) = \sum_{i,j,k} L(C_{ij}, C_k)$$

$$L_3(C,C) = L_{3,1}(C,C) + L_{3,2}(C,C)$$

$$L_{3,1}(C,C) = \sum_{i,j,k,l} L(C_{ijk}, C_l)$$

$$L_{3,2}(C,C) = \sum_{i,j,k,l} L(C_{ij}, C_{kl})$$

Summary

higher order gauge fields

higher order closed curves by duality

Wilson loop operators as observables in Chern-Simons gauge theory

non-abelian case

apply higher order fields to link polynomials