Topological Solitons from Geometry

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- Waves: Dispersion, diffraction, superpositions. QFT: Fourier transform, infinities, regularisation, . . . , string theory.
- Particles: mass, localisation. Solitons, classical non-linear field equations, topological charges, stability.
Solitons: Non–singular, static, finite energy solutions of the field equations.

\[
L = \int_{\mathbb{R}} \left( \frac{1}{2} \phi_t^2 - \frac{1}{2} \phi_x^2 - U(\phi) \right) dx, \quad \text{where} \quad \phi : \mathbb{R}^{1,1} \to \mathbb{R}.
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  - Nontrivial topology
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  - Complete integrability

Static vortices: Abelian gauge potential $+ Higgs$ field $\phi: \mathbb{R}^2 \rightarrow \mathbb{C}$. Boundary conditions: $F = 0$ and $|\phi| = 1$ as $r \rightarrow \infty$.

$\phi: S^1 \rightarrow S^1$. Vortex number $\pi_1(S^1) = \mathbb{Z}$.

Skyrmions: $U: \mathbb{R}^3, 1 \rightarrow SU(2)$. Static+finite energy. $U: S^3 \rightarrow SU(2)$. Topological baryon number $\pi_3(S^3) = \mathbb{Z}$.

$B = -\frac{1}{24} \pi^2 \int_{\mathbb{R}^3} Tr[(U - 1) dU]^3$. 

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$$B = -\frac{1}{24\pi^2} \int_{\mathbb{R}^3} \text{Tr}[(U^{-1}dU)^3].$$
2D Kähler manifold \((\Sigma, g, \omega)\). Hermitian line bundle \(L \to \Sigma\) with \(U(1)\) connection \(A\) and a global section \(C^\infty\) section \(\phi : \Sigma \to L\).
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\[
E[A, \phi] = \frac{1}{2} \int_\Sigma (|D\phi|^2 + |F|^2 + \frac{1}{4} (1 - |\phi|^2)^2) \text{vol}_\Sigma, \quad F = dA
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**Taubes equation:** \(z = x + iy, \quad g = \Omega(z, \bar{z}) \, dzd\bar{z}\).

Set \(\phi = \exp(h/2 + i\chi)\), solve for \(A\).

\[
\Delta_0 h + \Omega - \Omega e^h = 0, \quad \text{where} \quad \Delta_0 = 4\partial_z \partial_{\bar{z}}.
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Vortex from geometry. $\Delta_0 h + \Omega - \Omega e^h = 0$.

- Vortex number

$$N = \frac{1}{2\pi} \int_{\Sigma} B \text{vol}_{\Sigma}, \quad h \sim 2N \log |z - z_0| + \text{const} + \ldots.$$
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\Delta_0 (h/2) = \sinh (h/2), \quad \text{Sinh–Gordon equation.}
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- Solitons from geometry: \( L = T\Sigma \), \( SO(2) \cong U(1) \), Chern number = Euler characteristic.
Geometry of Sinh–Gordon vortex

- $g = e^{-h/2} dz d\bar{z}$. Constant mean curvature surface $\Sigma \subset \mathbb{R}^{2,1}$.
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Large $r$: Modified Bessel equation, $h \sim 8\lambda K_0(r)$ for a constant $\lambda$.

Small $r$: Liouville equation, $h = 4\sigma \log r + ...$ for a constant $\sigma$.

In general, Painlevé III equation.

Theorem (MD, 2012)

1. $\sigma(\lambda) = 2\pi - \frac{1}{2} \arcsin \left( \frac{\pi \lambda}{\sqrt{\lambda^2 - 1}} \right)$ if $0 \leq \lambda \leq \pi - 1$.

2. There exists a one–vortex solution with strength $4\sqrt{2}/\pi$.

3. CMC surface with deficit angle $\pi$ near $r = 0$.

Uses isomonodromy theory, and connection formulae (Kitaev).

One more integrable case: Tzitzeica equation and affine spheres.
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Skyrmions

- Skyrme model of Baryons. Nonlinear pion field $U : \mathbb{R}^{3,1} \rightarrow SU(2)$. 

Energy of static skyrmions. Set $R_j = U - 1 \partial_j U \in su(2)$. 

$$E[U] = -\frac{1}{2} \int_{\mathbb{R}^3} \left( \text{Tr} \left( R_j R_j \right) + \frac{1}{8} \left( \text{Tr} \left[ \left[ R_i, R_j \right] \left[ R_i, R_j \right] \right) \right) \right) d^3x > \frac{12}{\pi^2} |B|.$$ 

Finite energy, $U : S^3 \rightarrow S^3$ and $B = \text{deg}(U) \in \mathbb{Z}$. 

Skyrmions from instanton holonomy (Atiyah + Manton, 1989). 

$U(x) = P \exp \left( -\int R A^4(x, \tau) d\tau \right)$. 

Yang–Mills potential $A \in su(2)$ for a Yang–Mills field $F = dA + A \wedge A$ on $\mathbb{R}^4$. Baryon number = instanton number.
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**Particles as Gravitational Instantons**

- **Gravitational instantons** = Riemannian, Einstein four manifolds \((M, g)\) whose curvature is concentrated in a finite region of a space-time.

\[
\begin{align*}
g &= \frac{1}{r^2} \left( \frac{1}{r} dr^2 + (r^2 + mr)(d\theta^2 + \sin^2 \theta d\phi^2) + r^2 d\psi^2 \right), \\
\theta &\in [0, \pi), \quad \phi \in [0, 2\pi), \quad \psi \in [0, 4\pi).
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- Small \(r\): \(g \sim \) flat metric on \(\mathbb{R}^4\).
- Large \(r\): Hopf bundle \(S^3 \to S^2\), Chern class = 1.
- Asymptotically locally flat (ALF): \(M \sim S^1\) bundle over \(S^2\) at \(\infty\).

Charged particles = ALF instantons. Charge = Chern number of asymptotic \(S^1\) bundle.

Neutral particles = compact instantons. Proton, neutron, electron, neutrino. Atiyah–Hitchin, \(\mathbb{CP}^2\), Taub–NUT, \(S^4\).
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- **Taub–NUT metric**

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g = \frac{r + m}{r} dr^2 + (r^2 + mr) (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{rm^2}{r + m} (d\psi + \cos \theta d\phi)^2,
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where \(\theta \in [0, \pi), \phi \in [0, 2\pi), \psi \in [0, 4\pi)\).
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AH, Taub–NUT, $\mathbb{C}P^2$ are $\text{SO}(3)$ invariant. Pick $\text{SO}(2) \subset \text{SO}(3)$.

$$U(r, \psi, \theta) = \exp \left( -\pi \sum_{j=1}^{3} f_j(r)n_j t_j \right),$$

where $n = (\cos \psi \sin \theta, \sin \psi \sin \theta, \cos \theta)$ and

$$f_1 = f_2 = -\frac{r}{r + m}, \quad f_3 = \frac{r(r + 2m)}{(r + m)^2}.$$
Skyrmion on three–space \((B, h) = M/SO(2) \cong \mathbb{H}^2 \times S^1\)

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Geometry of Taub–NUT Skyrmion

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Poincare disc \(\mathbb{D} \to \mathbb{H}^2, x + iy = \frac{z-i}{iz-1}, \quad S^1 \to \mathcal{B} \to \mathbb{D}\)
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1. $B_{TN} = 2$ (integration).

2. $B_{AH} = 1$. Set $r = 2 \int_0^{\pi/2} \sqrt{1 - \sin(\beta/2)^2 \sin(\tau)^2}^{-2} d\tau$. 
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Thank You.