Topological Solitons from Geometry

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- Atiyah, Manton, Schroers. Geometric models of matter. arXiv:1111.2934. Proc. R. Soc. Lond **A** (2012).
  MD. Abelian vortices from Sinh–Gordon and Tzitzeica equations. arXiv:1201.0105. Phys. Lett. B. **710** (2012).
  MD. Skyrmions from gravitational instantons. arXiv:1206.0016. Proc. R. Soc. Lond. **A** (2013).

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- Particles: mass, localisation. Solitons, classical non–linear field equations, topological charges, stability.

Solitons: Non–singular, static, finite energy solutions of the field equations.

$$L = \int_{\mathbb{R}} \Big(\frac{1}{2}\phi_t^2 - \frac{1}{2}\phi_x^2 - U(\phi) \Big) dx, \quad \text{where} \quad \phi : \mathbb{R}^{1,1} \to \mathbb{R}.$$

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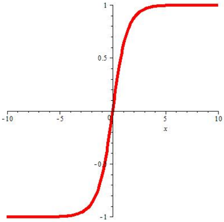

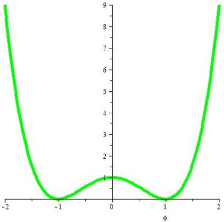

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- Skyrmions: $U : \mathbb{R}^{3,1} \to SU(2)$.
  - Static+finite energy. $U : S^3 \to SU(2)$.
  - Topological baryon number $\pi_3(S^3) = \mathbb{Z}$.

$$B = -\frac{1}{24\pi^2} \int_{\mathbb{R}^3} \mathrm{Tr}[(U^{-1}dU)^3].$$

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- Taubes equation: $z = x + iy,\ g = \Omega(z,\bar{z})\, dzd\bar{z}$.
  Set $\phi = \exp(h/2 + i\chi)$, solve for A.

$$\Delta_0 h + \Omega - \Omega e^h = 0, \quad \text{where} \quad \Delta_0 = 4\partial_z\partial_{\bar{z}}.$$

Vortex from geometry. $\Delta_0 h + \Omega - \Omega e^h = 0.$

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$$N = \frac{1}{2\pi}\int_\Sigma B \, \mathsf{vol}_\Sigma, \quad h \sim 2N \log|z - z_0| + const + \dots \, .$$

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- Solitons from geometry: $L = T\Sigma, \quad SO(2) \cong U(1)$,
  Chern number=Euler characteristic.

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- One more integrable case: Tzitzeica equation and affine spheres.

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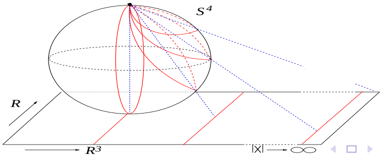

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  where $\theta \in [0, \pi)$, $\phi \in [0, 2\pi)$, $\psi \in [0, 4\pi)$.
  - Small r: $g \sim$ flat metric on $\mathbb{R}^4$.
  - Large r: Hopf bundle $S^3 \to S^2$, Chern class $= 1$.
  - Asymptotically locally flat (ALF): $M \sim S^1$ bundle over S^2 at ∞.
- Particles = gravitational instantons. (Atiyah, Manton, Schroers).
  - Charged particles = ALF instantons. Charge = Chern number of asymptotic S^1 bundle.
  - Neutral particles = compact instantons.
  - Proton, neutron, electron, neutrino. Atiyah–Hitchin, $\mathbb{CP}^2$, Taub–NUT, S^4.

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Skyrmions from gravitational instantons

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- AH, Taub–NUT, $\mathbb{CP}^2$ are $SO(3)$ invariant. Pick $SO(2) \subset SO(3)$.

$$U(r, \psi, \theta) = \exp\Big(-\pi \sum_{j=1}^{3} f_j(r) n_j \mathbf{t}_j \Big),$$

where $\mathbf{n} = (\cos\psi \sin\theta, \sin\psi \sin\theta, \cos\theta)$ and

$$f_1 = f_2 = -\frac{r}{r+m}, \quad f_3 = \frac{r(r+2m)}{(r+m)^2}.$$

- Skyrmion on three–space $(\mathcal{B}, h) = M/SO(2) \cong \mathbb{H}^2 \times S^1$

$$h = h_{\mathbb{H}^2} + R^2 d\psi^2, \quad R : \mathbb{H}^2 \to \mathbb{R}.$$

Geometry of Taub–NUT skyrmion

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- Upper half plane: $h = \frac{dx^2+dy^2}{y^2}, R^2 = \frac{x^2+y^2}{y^2(m^{-1}\sqrt{x^2+y^2}+1)^2+x^2}$.

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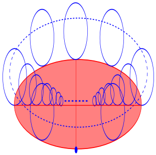

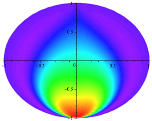

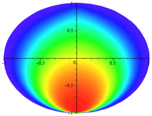

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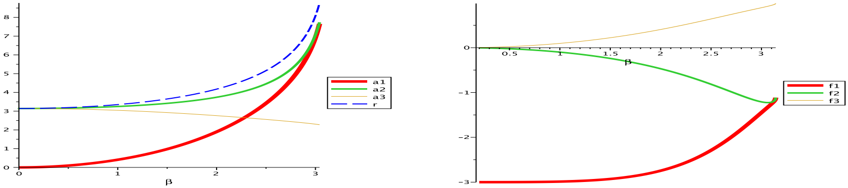
  2. $B_{AH} = 1$. Set $r = 2\int_0^{\pi/2} \sqrt{1 - \sin(\beta/2)^2 \sin(\tau)^2}^{\,-2} d\tau$.



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Thank You.