Abelian and non-Abelian Hopfions in all odd dimensions

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Generalities

- Hopfions are finite energy “soliton like” solutions to the field equations of some scalar field system.
- The lower bound on their energies rely on the Chern-Simons density as a “topological charge density” and hence can exist only in odd space dimensions.
- The Chern-Simons density is defined in terms of a gauge connection and a gauge curvature, which means that the scalar field system must be a “complex sigma model” enabling the necessary definitions of a *composite* connection and curvature.
- The Chern-Simons density is by construction, not a total divergence, even when the sigma model constraint is taken account of. Hence, it is not a candidadte for a topological charge density as it stands.

- Only after imposition of suitable symmetries does it become a total divergence, qualifying it as a topological charge density.
- Our prescription here is to achieve this aim by imposing multi-azimuthal symmetries. This is the case with the familiar Skyrme-Fadde'ev Hopfion on $\mathbb{R}^3$.
- Multi-azimuthal symmetry in $\mathbb{R}^{2n+1}$ eliminates n azimuthal angles, each in one of the n 2-dimensional subspaces (planes).
- It follows that after imposition of such symmetry, the residual subsystems will be $(n+1)$-dimensional.
- The residual Chern-Simons density in a $(n+1)$-dimensional space can be a total divergence if the Ansatz is parametrised by $(n+1)$ independant functions of the $(n+1)$ residual 'coordinates'.

- The **residual** Chern-Simons density will be a total divergence only when parametrised by functions that fulfill the sigma model constraint. Otherwise, it will be "essentially total divergence" in the sense that only when the constraint is imposed *via* a Lagrange multiplier will the resulting Euler-Lagrange equations be *trivial*.
- Chern-Simons densities **composite** connections and curvatures present candidates for *topological charge densities* only when subjected to suitable symmetries. This is in stark contrast to the topological charge densities of solitons (*incl.* instantons, monopoles, Skyrmions, vortices, *etc.*) whose topological charge densities are defined by Chern-Pontryagin densities (and their descendants) which are total divergence without imposition of any symmetry.

Abelian Chern-Simons densities

Abelian curvature of Abelian connection A_i

$$F_{ij} = \partial_i A_j - \partial_j A_i$$

Definition of Abelian (gauge invariant) Chern-Pontryagin density in $2n+2$ dimensions

$$\begin{aligned}
\Omega_{\rm CP} &= \varepsilon_{\mu_1\nu_1\mu_2\nu_2\dots\mu_{n+1}\nu_{n+1}} F_{\mu_1\nu_1} F_{\mu_2\nu_2} \dots F_{\mu_{n+1}\nu_{n+1}} \\
&= \varepsilon_{\mu_1\nu_1\mu_2\nu_2\dots\mu_{n+1}\nu_{n+1}} \partial_{\mu_{n+1}} \left(A_{\nu_{n+1}} F_{\mu_1\nu_1} F_{\mu_2\nu_2} \dots F_{\mu_n\nu_n} \right) \\
&= \partial_{\mu_{n+1}} \Omega_{\mu_{n+1}}
\end{aligned}$$

implies definition of Abelian (gauge variant) Chern-Simons density in $2n+1$ dimensions

$$\Omega_{\rm CS} = \varepsilon_{i_1 j_1 i_2 j_2 \dots i_n j_n j_{n+1}} A_{j_{n+1}} F_{i_1 j_1} F_{i_2 j_2} \dots F_{i_n j_n}$$

$\Omega_{\rm CS}$ is not a total divergence and leads to the variational equations

$$\varepsilon_{i_1 j_1 i_2 j_2 \dots i_n j_n j_{n+1}} F_{i_1 j_1} F_{i_2 j_2} \dots F_{i_n j_n} = 0$$

which is gauge covariant.

non-Abelian Chern-Simons densities

non-Abelian curvature of Abelian connection A_i

$$F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$$

Definition of non-Abelian (gauge invariant) Chern-Pontryagin density in $2n+2$ dimensions

$$\begin{aligned}\Omega_{\rm CP} &= \varepsilon_{\mu_1\nu_1\mu_2\nu_2\dots\mu_{n+1}\nu_{n+1}} \mathrm{Tr}\, F_{\mu_1\nu_1} F_{\mu_2\nu_2} \dots F_{\mu_{n+1}\nu_{n+1}} \\ &= \partial_{\mu_{n+1}} \Omega_{\mu_{n+1}}\end{aligned}$$

which is a total divergence and likewise implies definition of non-Abelian (gauge variant) Chern-Simons density in $2n+1$ dimensions.

For $n = 1$, $D = 3$, it is

$$\Omega^{(1)}_{\rm CS} = \varepsilon_{ijk} \mathrm{Tr}\, A_k \left(F_{ij} - \frac{2}{3} F_i F_j \right) .$$

For $n = 2$, $D = 5$, it is

$$\Omega^{(2)}_{\rm CS} = \varepsilon_{ijklm} \mathrm{Tr}\, A_m \left(F_{ij} F_{kl} - F_{ij} A_k A_l + \frac{2}{5} A_i A_j A_k A_l \right) .$$

The equations of motion these result in are

$$\begin{aligned}\varepsilon_{ijk}\, F_{ij} &= 0 \\ \varepsilon_{ijklm}\, F_{ij} F_{kl} &= 0\end{aligned}$$

which are gauge covariant.

For $n \geq 3$ there are multiple distinct definitions for the CS density, each characterised by the number of traces in the definition.

For $n = 3$, $D = 7$, there are two possibilities; a definition with a single trace and another one with double trace. These are

$$\begin{aligned}\Omega^{(3)}_{\rm CS} &= \varepsilon_{ijklmnp}\mathrm{Tr}\, A_p \bigg(F_{ij} F_{kl} F_{mn} - \frac{4}{5} F_{ij} F_{kl} A_m A_n - \frac{2}{5} F_{ij} A_k F_{lm} A_n \\ &\qquad + \frac{4}{5} F_{ij} A_k A_l A_m A_n - \frac{8}{35} A_i A_j A_k A_l A_m A_n \bigg)\,, \\ \tilde{\Omega}^{(3)}_{\rm CS} &= \varepsilon_{ijklmnp}\mathrm{Tr}\, A_p \left(F_{mn} - \frac{2}{3} A_m A_n \right) \cdot (\mathrm{Tr}\, F_{ij} F_{kl})\end{aligned}$$

The equations of motion these result in are

$$\begin{aligned} \varepsilon_{ijklmnp} F_{ij} F_{kl} F_{mn} &= 0 \\ \varepsilon_{ijklmnp} \left(\mathrm{Tr}\, F_{ij} F_{kl}\right) \cdot F_{mn} &= 0 \end{aligned}$$

which are gauge covariant.

$\mathbb{C}\mathrm{P}^n$ models on $\mathbb{R}^{2n+1}$

Described by complex $n-$tuplets

$$Z = \begin{bmatrix} z_1 \\ z_2 \\ .. \\ .. \\ z_{n+1} \end{bmatrix} \equiv z_a \quad ; \quad \bar{Z} = \begin{bmatrix} \bar{z}^1 \\ \bar{z}^2 \\ .. \\ .. \\ \bar{z}^{n+1} \end{bmatrix} \equiv \bar{z}^a \quad a = 1, 2, ..., n+1\,,$$

parametrised by $2n+1$ real independant functions subject to constraint

$$Z^\dagger\, Z \equiv \bar{z}^a\, z_a = 1\,,$$

and

$$P = \left(\mathbb{1} - Z\, Z^\dagger\right) \equiv \left(\delta_a{}^b - z_a\, \bar{z}^b\right)\,.$$

is a projection operator.

This constraint in invariant under the action of the *local* Abelian transformation $g = e^{\mp i\Lambda(x)}$

$$Z \to Z\,g$$

leading to the definition of the Abelian composite connection

$$B_i = i\,Z^{\dagger}\partial_i Z \quad , \quad i = 1, 2, ..., 2n+1$$

transforming like

$$B_i \to B_i \pm \partial_i \Lambda\,.$$

There follow the definitions of the covariant derivative of Z and the *Abelian curvature*

$$\begin{aligned} D_i Z &= \partial_i\, Z + i\, B_i\, Z\,, \\ G_{ij} &= \partial_i\, B_j - \partial_j\, B_i\,, \end{aligned}$$

$D_i Z$ transforming covariantly under the action of g, and G_{ij} invariantly.

The topological charge density of the Abelian Hopfion on $\mathbb{R}^{2n+1}$ is the Abelian Chern-Simons density on $\mathbb{R}^{2n+1}$ defined by replacing

$$A_i \to B_i \quad \text{and} \quad F_{ij} \to G_{ij}$$

$$\Omega_{\rm CS} = \varepsilon_{i_1 i_2 \dots i_{2n+1}} B_{i_{2n+1}} G_{i_1 i_2} G_{i_3 i_4} \dots G_{i_{2n-1} i_{2n}} \, .$$

Subjecting the density to variations of $Z^\dagger$ (or Z)

$$\Omega_{\rm CS} + \lambda \, (1 - Z^\dagger Z)$$

λ being the Lagrange multiplier yields the *nontrivial* gauge covariant equation

$$\varepsilon_{i_1 i_2 \dots i_{2n+1}} D_{i_{2n+1}} Z \, G_{i_1 i_2} G_{i_3 i_4} \dots G_{i_{2n-1} i_{2n}} = 0$$

which trivialises only under the appropriate symmetries, subject to which $\Omega_{\rm CS}$ becomes "essentially total divergence", and hence a candidate for a topological charge density.

CP^1 system on $\mathbb{R}^3$: mono-azimuthal (axial) symmetry

The Ansatz for Z, featuring axial symmetry in the (x_1, x_2) plane, is

$$Z = \begin{bmatrix} a + ib \\ c\, e^{in\varphi} \end{bmatrix} \equiv \begin{bmatrix} \sin\frac{f}{2}\, e^{i\alpha} \\ \cos\frac{f}{2}\, e^{in\varphi} \end{bmatrix} , \quad a^2 + b^2 + c^2 = 1$$

a, b, c, f and α are functions of

$$\rho = \sqrt{|x_\alpha|^2} \quad \text{and} \quad z \equiv x_3; \quad \alpha = 1, 2$$

n is the vorticity and φ is the azimuthal angle in the (x_1, x_2) plane.

Subject to this Ansatz, the Abelian Chern–Simons density reduces to

$$\Omega^{(3)}_{\rm CS} = -4\, \frac{n}{\rho}\, c \cdot \det \begin{vmatrix} a & b & c \\ a_{,\rho} & b_{,\rho} & c_{,\rho} \\ a_{,z} & b_{,z} & c_{,z} \end{vmatrix} .$$

This expression of $\Omega^{(3)}_{\rm CS}[a,b,c]$ is not a total divergence. Rather,

$$\Omega^{(3)}_{\rm CS}[a,b,c]+\lambda(1-a^2+b^2+c^2)$$

is "essentially total divergence" and yields *trivial* equations of motion. (λ is a Lagrange multiplier.) Only when the trigonometric parametrisation, which satisfies the sigma model constraint, does $\Omega^{(3)}_{\rm CS}[f,\alpha]$ become explicitly the "total divergence",

$$\Omega^{(3)}_{\rm CS}=\frac{4}{3}\,\frac{n}{\rho}\,(F_{,\rho}\,\alpha_{,z}-(\rho,z))\ ,$$

where $F(\rho,z)$ is the function

$$F=\cos^3\frac{f}{2}.$$

The topological charge can then be expressed as,

$$Q=\frac{4}{3}\,2\pi\,\int\Omega^{(3)}_{\rm CS}\,\rho\,d\rho\,dz=\frac{4}{3}\,2\pi\,n\,\int\,F_{[,\rho}\,\alpha_{,z]}\,d\rho\,dz$$

Denoting the coordinates in the half plane $(\rho, z) = \xi_i$, $i = 1, 2, 3$, *i.e.*,

$$\xi_i = \begin{pmatrix} r \sin\psi \\ r \cos\psi \end{pmatrix}$$

with $0 \leq \psi \leq \pi$, this integral can be rewritten as

$$\begin{aligned} Q = \frac{4}{3}\,2\pi\, n \int \varepsilon_{ij}\,\partial_i F\,\partial_j\alpha\; d^2\xi \quad &= \quad \frac{4}{3}\,2\pi\, n \int \left(F\,\hat{x}_i\,\varepsilon_{ij}\;\partial_j\alpha\right)\Big|_{r\to\infty}\hat{\xi}_i\, dS \\ &= \quad \frac{4}{3}\,2\pi \int_{\psi=0}^{\psi=\pi} F\,\partial_\psi\,\alpha|_{r\to\infty}\; d\psi \end{aligned}$$

where ψ is the polar angle in the (ρ, z) half plane.

Requiring the field configurations have the asymptotic values

$$\lim_{r\to\infty} f(r,\theta) = 0\;, \quad \lim_{r\to\infty} \alpha(r,\theta) = m\,\pi\;,$$

the results is

$$Q = \frac{8}{3}\, n\, m\, \pi^2\,.$$

CP^2 system on $\mathbb{R}^5$: bi-azimuthal symmetry

The Ansatz for Z, featuring bi-azimuthal symmetry in the (x_1, x_2) and (x_3, x_4) planes, is

$$Z = \begin{bmatrix} a + ib \\ c_1\, e^{in_1\varphi} \\ c_2\, e^{in_2\chi} \end{bmatrix} \equiv \begin{bmatrix} \sin\frac{1}{2}f\, e^{i\alpha} \\ \cos\frac{1}{2}f\ \sin g\, e^{in_1\varphi} \\ \cos\frac{1}{2}f\ \cos g\, e^{in_2\chi} \end{bmatrix}$$

a, b, c_1, c_2, f, g and α are functions of

$$\rho = \sqrt{|x_\alpha|^2}, \quad \sigma = \sqrt{|x_A|^2}, \quad \text{and} \quad z \equiv x_5; \quad \alpha = 1, 2; \quad A = 3, 4$$

n_1 and n_2 are the vorticities, and, φ and χ are the azimuthal angles in the (x_1, x_2) and (x_3, x_4) planes respectively. Subject to this Ansatz, the Abelian Chern–Simons density on $\mathbb{R}^5$ reduces to

$$\Omega_{\mathrm{CS}}^{(5)} = 32\,\frac{n_1 n_2}{\rho\sigma}\, c_1\, c_2 \cdot \det \begin{vmatrix} a & b & c_1 & c_2 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} \end{vmatrix}.$$

Using the trigonometric parametrisation of this Ansatz, in terms of the functions

$$F = \cos f + \frac{1}{2}\cos 2f \quad , \quad G = \cos 2g \, ,$$

and denoting the coordinates in the quarter sphere $(\rho, \sigma, z) = \xi_i$, $i = 1, 2, 3$, *i.e.*,

$$\xi_i = \begin{pmatrix} r \, \sin\psi \, \sin\theta \\ r \, \sin\psi \, \cos\theta \\ r \, \cos\psi \end{pmatrix}$$

with $0 \leq \psi \leq \pi$ and $0 \leq \theta \leq \frac{\pi}{2}$, the topological charge Q is given by the volume integral

$$\begin{aligned} Q &= -(2\pi)^2 \, n_1 \, n_2 \int \varepsilon_{ijk} \partial_i G \, \partial_j G \, \partial_k \alpha \, d^3\xi \\ &= -(2\pi)^2 \, n_1 \, n_2 \int \varepsilon_{ijk} \left(F \, \partial_j G \, \partial_k \alpha \right) \Big|_{r\to\infty} \hat{\xi}_i \, dS \end{aligned}$$

where $dS = r^2 \sin\psi \;\, d\psi \, d\theta$, *i.e.*

$$Q = 4\,\pi^2\, n_1\, n_2 \int_{\psi=0}^{\pi} \int_{\theta=0}^{\frac{\pi}{2}} F\, \left(\partial_\psi G\, \partial_\theta \alpha - \partial_\psi \alpha\, \partial_\theta G\right) \Big|_{r\to\infty} d\psi\, d\theta\,.$$

Here, ψ is the polar angle between the (ρ, σ)-plane and z.

Requiring that the field configurations in question satisfy the asymptotic values

$$\lim_{r\to\infty} f = 0 \quad , \quad \lim_{r\to\infty} g = \theta \quad , \quad \lim_{r\to\infty} \alpha = m\,\pi\,,$$

the result is

$$\Omega_{\rm CS}^{(5)} = -12\, n_1\, n_2\, m\, \pi^3\,.$$

CP^3 system on $\mathbb{R}^7$: tri-azimuthal symmetry

The Ansatz for Z, featuring tri-azimuthal symmetry in the (x_1, x_2), (x_3, x_4) and (x_5, x_6) planes, is

$$Z = \begin{bmatrix} a + ib \\ c_1\, e^{in_1\varphi} \\ c_2\, e^{in_2\chi} \\ c_3\, e^{in_3\zeta} \end{bmatrix} = \begin{bmatrix} \sin\frac{1}{2}f\, e^{i\alpha} \\ \cos\frac{1}{2}f\ \sin g\ \cos h\, e^{in_1\varphi} \\ \cos\frac{1}{2}f\ \sin g\ \sin h\, e^{in_2\chi} \\ \cos\frac{1}{2}f\ \cos g\, e^{in_3\zeta} \end{bmatrix}$$

a, b, c_1, c_2, c_3, f, g, h and α are functions of

$$\rho = \sqrt{|x_\alpha|^2}, \quad \sigma = \sqrt{|x_A|^2}, \quad \tau = \sqrt{|x_a|^2}, \quad \text{and} \quad z \equiv x_7;$$

with $\alpha = 1, 2$; $A = 3, 4$; $a = 5, 6$. n_1, n_2 and n_3 are the vorticities, and, φ, χ and ζ are the azimuthal angles in the (x_1, x_2), (x_3, x_4) and (x_5, x_6) planes respectively.

Subject to this Ansatz the Abelian Chern–Simons density on $\mathbb{R}^7$ reduces to

$$\Omega^{(7)}_{\rm CS} = 96\,\frac{n_1\,n_2\,n_3}{\rho\sigma\tau}\,c_1\,c_2\,c_3\cdot\det\left|\begin{array}{ccccc} a & b & c_1 & c_2 & c_3 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} & c_{3,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} & c_{3,\sigma} \\ a_{,\tau} & b_{,\tau} & c_{1,\tau} & c_{2,\tau} & c_{3,\tau} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} & c_{3,z} \end{array}\right| .$$

Using the trigonometric parametrisation of this Ansatz, in terms of the functions

$$F = \cos^6\frac{f}{2}\quad , \quad G = \frac{1}{4}(1-\cos 2g)^2\quad , \quad H = \cos 2h\,,$$

and denoting the coordinates in the sextant of the hypersphere $(\rho,\sigma,\tau,z) = \xi_i$, $i = 1,2,3,4$, *i.e.*,

$$\xi_i = \left(\begin{array}{c} r\,\sin\psi\,\sin\theta_1\,\sin\theta_2 \\ r\,\sin\psi\,\sin\theta_1\,\cos\theta_2 \\ r\,\sin\psi\,\cos\theta_1 \\ r\,\cos\psi \end{array}\right)$$

with $0\le\psi\le\pi$, $0\le\theta_1\le\frac{\pi}{2}$ and $0\le\theta_2\le\frac{\pi}{2}$,

$$
\begin{aligned}
Q &= -4\,(2\pi)^3\, n_1\, n_2\, n_3 \int \varepsilon_{ijkl}\partial_i F\, \partial_j G\, \partial_k H\, \partial_z \alpha\, d^4\xi \\
&= -4\,(2\pi)^3\, n_1\, n_2\, n_3 \int \varepsilon_{ijkl}\left(F\, \partial_j G\, \partial_k H \partial_l \alpha\right)\Big|_{r\to\infty} \hat{\xi}_i\, dS
\end{aligned}
$$

where $dS = r^3 \sin^2\psi \sin\theta_1 \; d\psi \; d\theta_1 \; d\theta_2$.

Requiring the field configuration in question take the asymptotic values

$$
\lim_{r\to\infty} g = \theta_1 \quad , \quad \lim_{r\to\infty} h = \theta_2 \quad , \quad \lim_{r\to\infty} \alpha = m\,\pi\,,
$$

the result is

$$
\Omega^{(5)}_{\rm CS} = 64\, n_1\, n_2\, n_3\, m\, \pi^4\,.
$$

Grassmannian models on $\mathbb{R}^{2n+1}$

These Grassmannian models are described by complex valued fields

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where z_1 and z_2 are complex $2n \times 2n$ matrices subject to the constraint

$$Z^\dagger Z = \mathbb{1}_{2n\times 2n} \,.$$

and the $4n \times 4n$ quantity

$$\Pi = \left(\mathbb{1} - Z\,Z^\dagger\right)$$

is a projection operator.

This constraint is invariant under the action of the local *unitary* non-Abelian gauge transformation g acting on Z

$$Z \to Z\,g \quad , \quad Z^\dagger \to g^\dagger\,Z^\dagger \,.$$

Here, the unitary matrix g is chosen to be an element of $SO(2n+2)$ in the $2n \times 2n$ chiral Dirac matrix representation.

The invariance of this constraint leads to the definition of the non-Abelian (anti-Hermitian) *composite* connection

$$B_i = Z^\dagger \partial_i Z \,.$$

transforming like

$$B_i \to g^{-1} B_i \, g + g^{-1} \partial_i \, g \,.$$

There follow the definitions of the covariant derivative of Z and the (composite) non-Abelian curvature

$$\begin{aligned} D_i Z &= \partial_i Z - Z\, B_i \\ G_{ij} &= \partial_{[i} B_{j]} + [B_i, B_j] \end{aligned}$$

which under the action of g transform covariantly as

$$\begin{aligned} D_i Z &\to D_i Z\, g \,, \\ G_{ij} &\to g^{-1} G_{ij}\, g \,. \end{aligned}$$

The non-Abelian Chern-Simons densities on $\mathbb{R}^{2n+1}$, up to $n=3$, are

$$\Omega_{\rm CS}^{(1)} = \varepsilon_{ijk} \mathrm{Tr}\, B_k \left(G_{ij} - \frac{2}{3} B_i B_j \right) .$$

$$\Omega_{\rm CS}^{(2)} = \varepsilon_{ijklm} \, \mathrm{Tr}\, B_m \left(G_{ij} G_{kl} - G_{ij} B_k B_l + \frac{2}{5} B_i B_j B_k B_l \right) .$$

$$\begin{aligned} \Omega_{\rm CS}^{(3)} &= \varepsilon_{ijklmnp} \mathrm{Tr}\, B_p \Big(G_{ij} G_{kl} G_{mn} - \frac{4}{5} G_{ij} G_{kl} B_m B_n - \frac{2}{5} G_{ij} B_k G_{lm} B_n \\ &\qquad + \frac{4}{5} G_{ij} B_k B_l B_m B_n - \frac{8}{35} B_i B_j B_k B_l B_m B_n \Big) , \\ \tilde{\Omega}_{\rm CS}^{(3)} &= \varepsilon_{ijklmnp} \mathrm{Tr}\, B_p \left(G_{mn} - \frac{2}{3} B_m B_n \right) \cdot \left(\mathrm{Tr}\, G_{ij} G_{kl} \right) . \end{aligned}$$

Subjecting these densities to variations of $Z^\dagger$ (or Z)

$$\Omega_{\rm CS}^{(n)} + \Lambda\,(\mathbb{1} - Z^\dagger\, Z)$$

Λ being the (now matrix valued) Lagrange multiplier, the resulting *nontrivial* gauge covariant equations in the above examples are

$$\begin{aligned}
\varepsilon_{ijk}\, D_k Z\, G_{ij} &= 0 \\
\varepsilon_{ijklm}\, D_m Z\, G_{ij}\, G_{kl} &= 0 \\
\varepsilon_{ijklmnp}\, D_p Z\, G_{ij}\, G_{kl}\, G_{mn} &= 0 \\
\varepsilon_{ijklmnp}\, (\mathrm{Tr}\, G_{ij} G_{kl}) \cdot D_p Z\, G_{mn} &= 0
\end{aligned}$$

which trivialise only under the appropriate symmetries, subject to which $\Omega_{\rm CS}^{(n)}$, rendering them candidates for topological charge densities.

2×4 Grassmannian on $\mathbb{R}^3$: mono-azimuthal symmetry

The Anstaz we for the field Z on $\mathbb{R}^3$ is

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a\,\mathbb{1} + 2\,b\,\Sigma_{34} \\ c\,n^\alpha\,\tilde{\Sigma}_\alpha \end{bmatrix},$$

the functions (a, b, c) depending on (ρ, z) as in the Abelian case and with the unit vector n^α

$$n^\alpha = \begin{pmatrix} \cos\, n\,\varphi \\ \sin\, n\,\varphi \end{pmatrix}$$

n being the winding (vortex) number in the (x_1, x_2) plane.

The spin matrices $\tilde{\Sigma}_\alpha = (\tilde{\Sigma}_1, \tilde{\Sigma}_2)$, are the first two of the chiral Dirac matrices in 4 dimensions, the last two being $(\tilde{\Sigma}_3, \tilde{\Sigma}_4)$. The chiral representations of the $SO(4)$ algebra are

$$\Sigma_{ij} = -\frac{1}{4}\,\Sigma_{[i}\tilde{\Sigma}_{j]},$$

with $i, j = \alpha, 3, 4$.

It turns out that this Ansatz leads to an Abelian composite connection $B_i = (B_\alpha, B_z)$, so our prescription cannot supply a non-Abelian Hopfion in three dimensions.

$$
\begin{aligned}
B_\alpha &= 2\left[(a\,b_{,\rho} - b\,a_{,\rho})\,\hat{x}_\alpha + \frac{n}{\rho}\,c^2\,(\hat{x}\varepsilon)_\alpha\right]\Sigma_{12}\,,\\
B_z &= 2\,(a\,b_{,z} - b\,a_{,z})\,\Sigma_{12}\,,
\end{aligned}
$$

whose commutators

$$[B_i, B_j] = [B_i, B_z] = 0.$$

The composite curvature G_{ij} is then Abelian, and coincides with the previously constructed Abelian case.

4×8 Grassmannian on $\mathbb{R}^5$: bi-azimuthal symmetry

The Anstaz we use for the field on $\mathbb{R}^5$ is

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a\,\mathbb{1} + 2\,b\,\Sigma_{56} \\ c_1\,n_1^\alpha\,\tilde{\Sigma}_\alpha + c_2\,n_2^A\,\tilde{\Sigma}_A \end{bmatrix},$$

The functions (a, b, c_1, c_2) depending on (ρ, σ, z) as in the corresponding Abelian case, with $\alpha = 1, 2$, $A = 3, 4$ and $z \equiv x_5$. φ and χ are the azimuthal angles in the (x_1, x_2) and (x_3, x_4) planes respectively, such that the unit vectors n^α and n_2^A are parametrised as

$$n_1^\alpha = \begin{pmatrix} \cos\ n_1\,\varphi \\ \sin\ n_1\,\varphi \end{pmatrix} \quad n_2^A = \begin{pmatrix} \cos\ n_2\,\chi \\ \sin\ n_2\,\chi \end{pmatrix},$$

(n_1, n_2) being the winding (vortex) numbers in each of the two planes respectively.

The spin matrices $(\tilde{\Sigma}_\alpha, \tilde{\Sigma}_A, \tilde{\Sigma}_5, \tilde{\Sigma}_6)$, are the (left) chiral Dirac matrices in 6 dimensions, in terms of which the chiral representations of the $SO(6)$ algebra are constructed.

Subject to the bi-azimuthally symmetric Ansatz the Chern-Simons density reduces to

$$\Omega_{\rm CS}^{(5)} = 96\, i\, \frac{n_1 n_2}{\rho\sigma}\, c_1\, c_2\, [-4 + 3(c_1^2 + c_2^2)] \cdot \det \begin{vmatrix} a & b & c_1 & c_2 \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} \end{vmatrix} .$$

Note that this non-Abelian density differs qualitatively from the corresponding Abelian one due to the appearance of the prefactor

$$[-4 + 3(c_1^2 + c_2^2)]\,.$$

Using the trigonometric parametrisation of this Ansatz, in terms of the functions

$$F = 5\cos f + \cos^2 f - \cos^3 f \quad , \quad G = \cos 2g\,,$$

and denoting the coordinates in the quarter sphere $(\rho, \sigma, z) = \xi_i$, $i = 1, 2, 3$, *i.e.*, as before in the Abelian case on $\mathbb{R}^5$ the topological charge Q is given by the volume integral

$$
\begin{aligned}
Q &= -\frac{3}{2}\, i\,(2\pi)^2\, n_1\, n_2 \int \varepsilon_{ijk}\partial_i G\, \partial_j G\, \partial_k \alpha\, d^3\xi \\
&= -\frac{3}{2}\, i\,(2\pi)^2\, n_1\, n_2 \int \varepsilon_{ijk}\,(F\,\partial_j G\,\partial_k \alpha)\Big|_{r\to\infty} \hat{\xi}_i\, dS
\end{aligned}
$$

where $dS = r^2 \sin\psi \; d\psi\, d\theta$, and where we have applied Gauss' Theorem, and

$$
Q = 6\, i\, \pi^2\, n_1\, n_2 \int_{\psi=0}^{\pi} \int_{\theta=0}^{\frac{\pi}{2}} F\; (\partial_\psi G\, \partial_\theta \alpha - \partial_\psi \alpha\, \partial_\theta G)\Big|_{r\to\infty} d\psi\, d\theta\,.
$$

Requiring the field configurations in question have the asymptotic values

$$
\lim_{r\to\infty} f = 0 \quad , \quad \lim_{r\to\infty} g = \theta \quad , \quad \lim_{r\to\infty} \alpha = m\,\pi\,,
$$

the result is

$$
\Omega_{\rm CS}^{(5)} = 60\, i\, n_1\, n_2\, m\, \pi^3\,.
$$

8×16 Grassmannian on $\mathbb{R}^7$: tri-azimuthal symmetry

The Anstaz we use for the field on $\mathbb{R}^7$ is

$$Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} a\,\mathbb{1} + 2\,b\,\Sigma_{78} \\ c_1\,n_1^\alpha\,\tilde{\Sigma}_\alpha + c_2\,n_2^A\,\tilde{\Sigma}_A + c_3\,n_3^A\,\tilde{\Sigma}_a \end{bmatrix},$$

The functions (a, b, c_1, c_2, c_3) depending on (ρ, σ, τ, z) as in the corresponding Abelian case, with $\alpha = 1, 2$, $A = 3, 4$, $a = 5, 6$ and $z \equiv x_7$. The azimutal angles in the (x_1, x_2), (x_3, x_4) and (x_5, x_6) planes are denoted by φ, χ and ζ respectively, such that the unit vectors n_1^α, n_2^A and n_3^a are parametrised as

$$n_1^\alpha = \begin{pmatrix} \cos\ n_1\,\varphi \\ \sin\ n_1\,\varphi \end{pmatrix} \quad n_2^A = \begin{pmatrix} \cos\ n_2\,\chi \\ \sin\ n_2\,\chi \end{pmatrix}, \quad n_3^a = \begin{pmatrix} \cos\ n_3\,\zeta \\ \sin\ n_3\,\zeta \end{pmatrix},$$

(n_1, n_2, n_3) being the winding (vortex) numbers in each of the three planes respectively. The spin matrices $(\tilde{\Sigma}_\alpha, \tilde{\Sigma}_A, \tilde{\Sigma}_a, \tilde{\Sigma}_7, \tilde{\Sigma}_8)$, are the (left) chiral Dirac matrices in 8 dimensions

Subject to the tri-azimuthally symmetric Ansatz in the Chern-Simons density reduces to

$$\Omega_{\rm CS}^{(7)} \simeq \frac{n_1\, n_2\, n_3}{\rho\sigma\tau}\, c_1\, c_2\, c_3\, \Theta(c_1, c_2, c_3) \cdot \det \begin{vmatrix} a & b & c & d & e \\ a_{,\rho} & b_{,\rho} & c_{1,\rho} & c_{2,\rho} & c_{3,\rho} \\ a_{,\sigma} & b_{,\sigma} & c_{1,\sigma} & c_{2,\sigma} & c_{3,\sigma} \\ a_{,\tau} & b_{,\tau} & c_{1,\tau} & c_{2,\tau} & c_{3,\tau} \\ a_{,z} & b_{,z} & c_{1,z} & c_{2,z} & c_{3,z} \end{vmatrix}$$

where the prefactor $\Theta(c_1, c_2, c_3)$ is not yet calcuated. Using the trigonometric parametrisation of this Ansatz and requiring the appropriate asymptotic behaviours, as before, the resulting topological charge is calculated

$$\Omega_{\rm CS}^{(5)} \simeq n_1\, n_2\, n_3\, m\, \pi^4\,.$$