Modelling Hopf solitons with elastic rods

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The Skyrme-Faddeev model

$$\varphi : \mathbb{R}^3 \to S^2$$

$$\varphi = (\varphi_1, \varphi_2, \varphi_3) \quad \varphi\cdot\varphi = 1$$

$$E = \int \partial_i\varphi\cdot\partial^i\varphi + (\varphi\cdot\partial_i\varphi\times\partial_j\varphi)(\varphi\cdot\partial^i\varphi\times\partial^j\varphi) + m^2(1-\varphi_3)\, d^3x$$

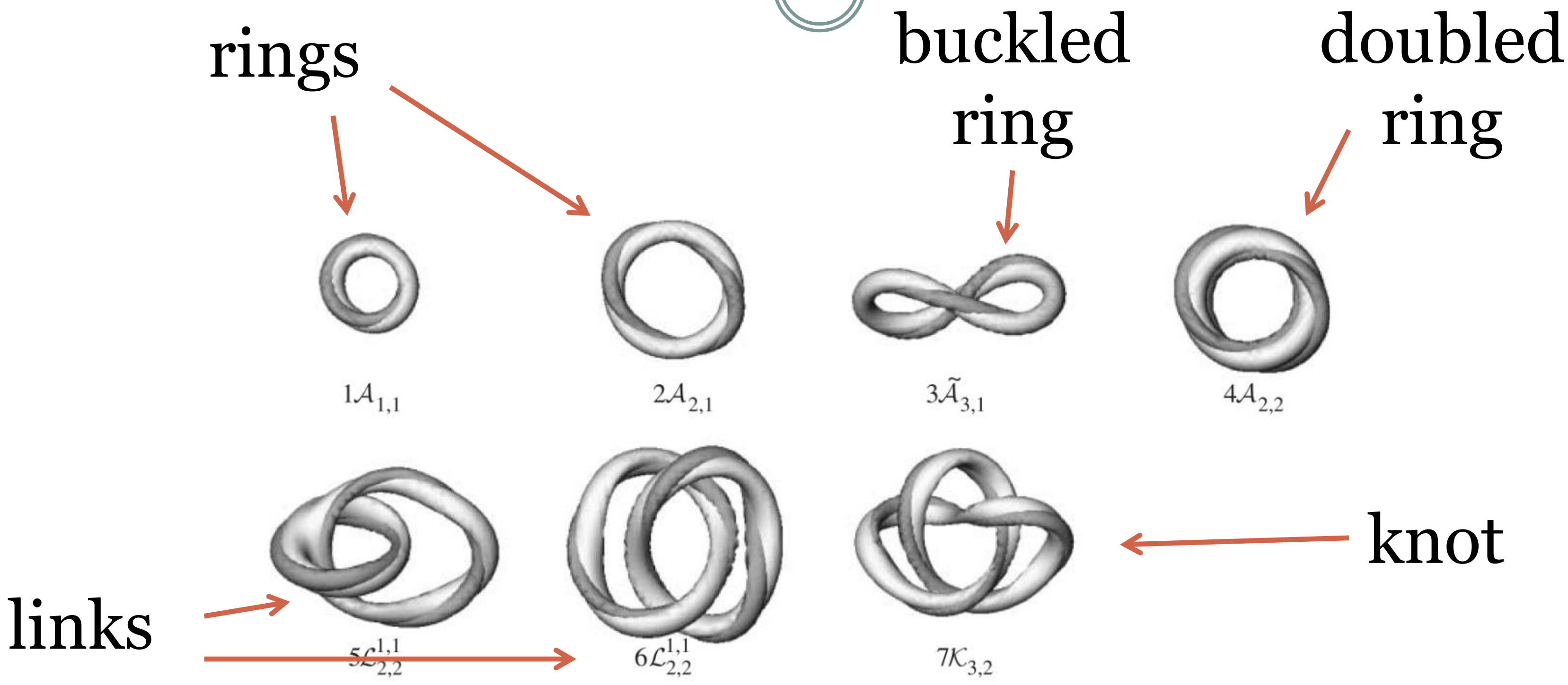
sigma-model term

additional term stabilises solitons

mass term (optional)

- Topological charge $Q \in \pi_3(S^2) \cong \mathbb{Z}$
- $E \sim Q^{3/4}$

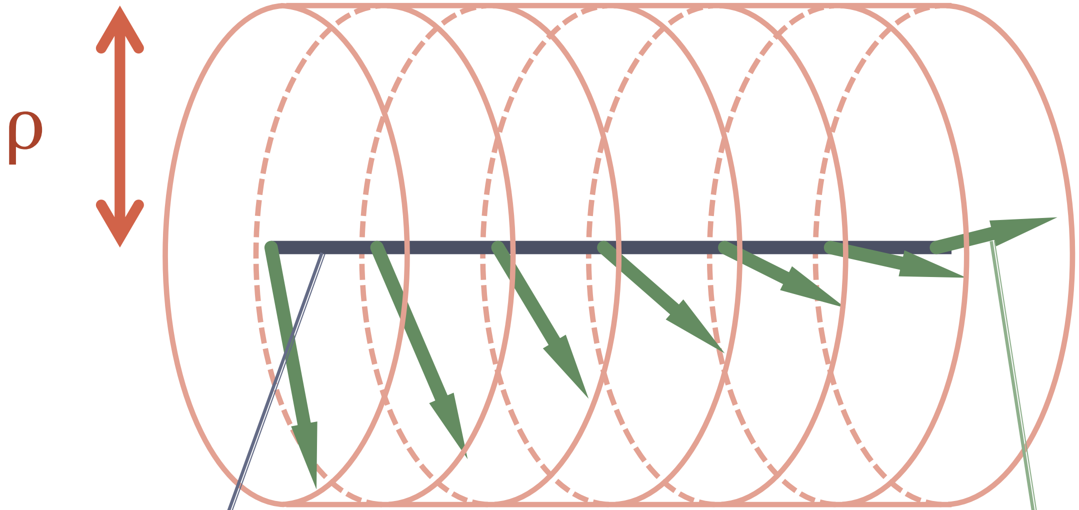
Linked and knotted solitons



- Plotted: preimages of two points in S^2
- Q = linking number

Elastic rod approximation

Modelling solitons with rods



$\mathbf{x}(\sigma)$
location curve
parametrises $\varphi^{-1}(0,0,-1)$

$\mathbf{m}(\sigma)$
frame vector
$\mathbf{m}\cdot\mathbf{x}' = 0$, $\mathbf{m}\cdot\mathbf{m} = 1$
$\mathbf{m}(\sigma) \sim \nabla\varphi_1(\mathbf{x}(\sigma))$

Elastic energy functional

$$E = \int (A + B\kappa^2 + C\theta^2)\, ds$$

Stretching energy — Bending energy — Twisting energy

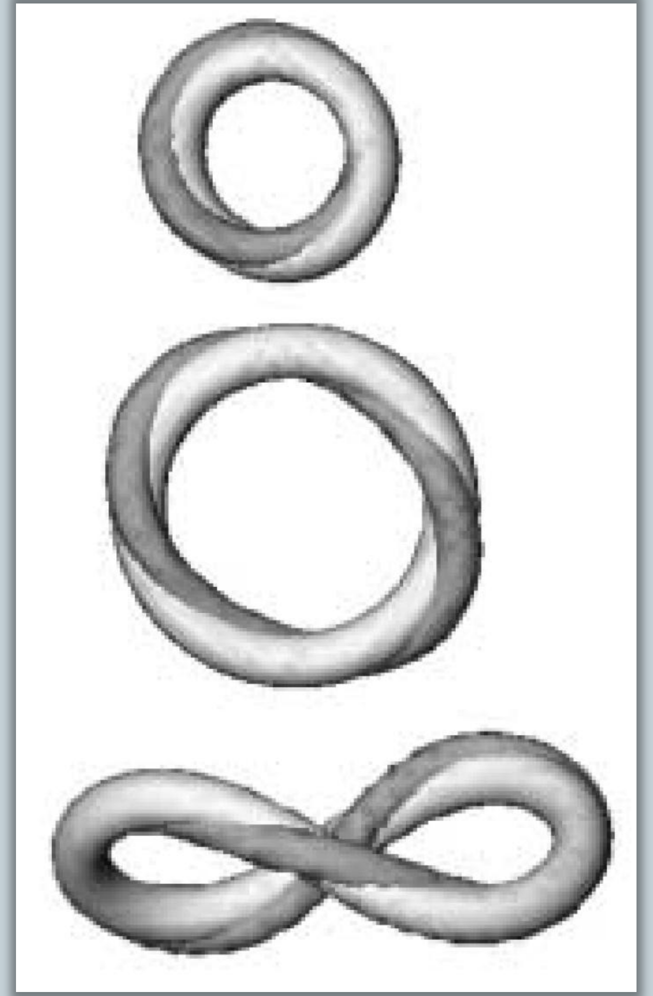
- defined using Frenet frame
- κ curvature
- $\theta = \mathbf{t} \cdot \mathbf{m}' \times \mathbf{m}$ "twist rate"
- constraint: rod cannot self-intersect
- A, B, C constants

C/B dimensionless

Hopf solitons on $\mathbb{R}^3$

D. Harland, J.M. Speight & P.M. Sutcliffe,
Phys. Rev. D85 (2011) 065008

Michell's instability



- Michell 1889: loop with Q twists buckles if

$$Q > Q_c = (B/C)\sqrt{3}$$

- $2 < Q_c < 3$ implies

$$0.58 < C/B < 0.87$$

- Fix $C/B = 0.85$

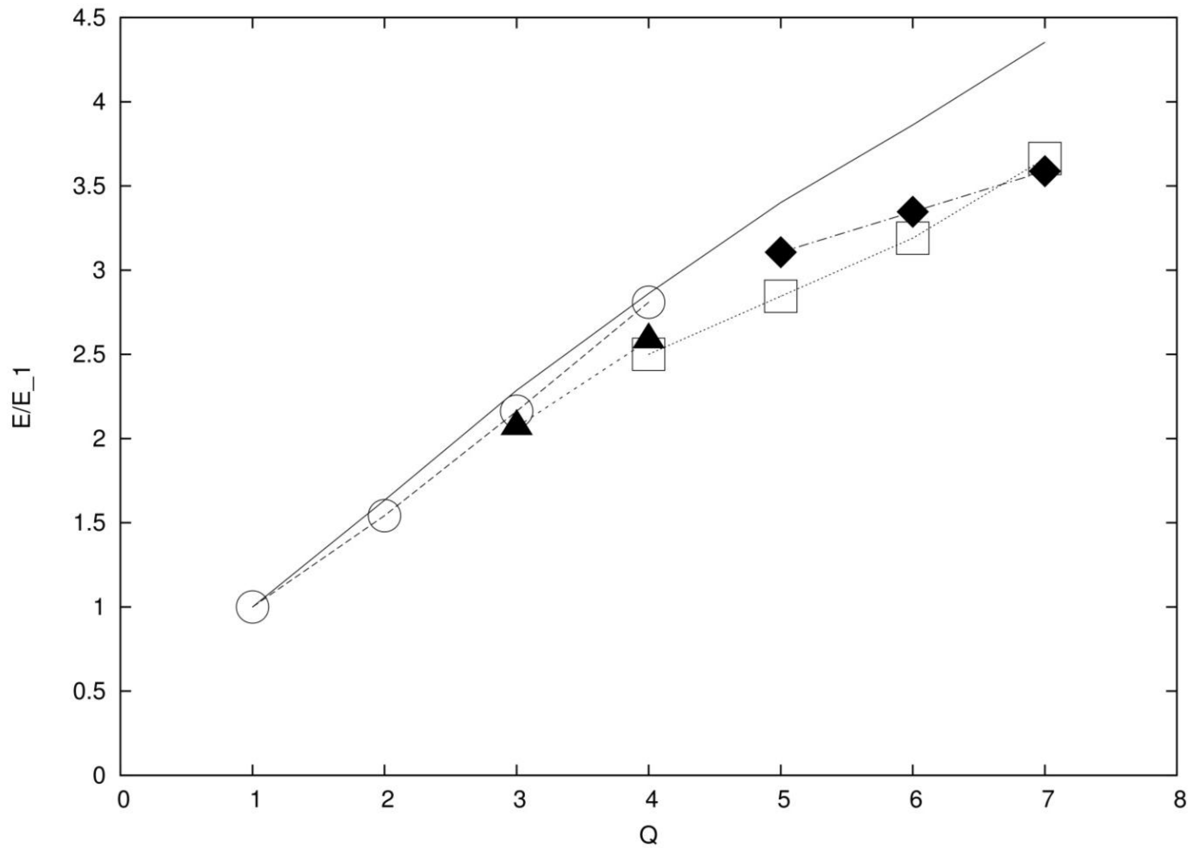
circle: flat ring

triangle: buckled ring

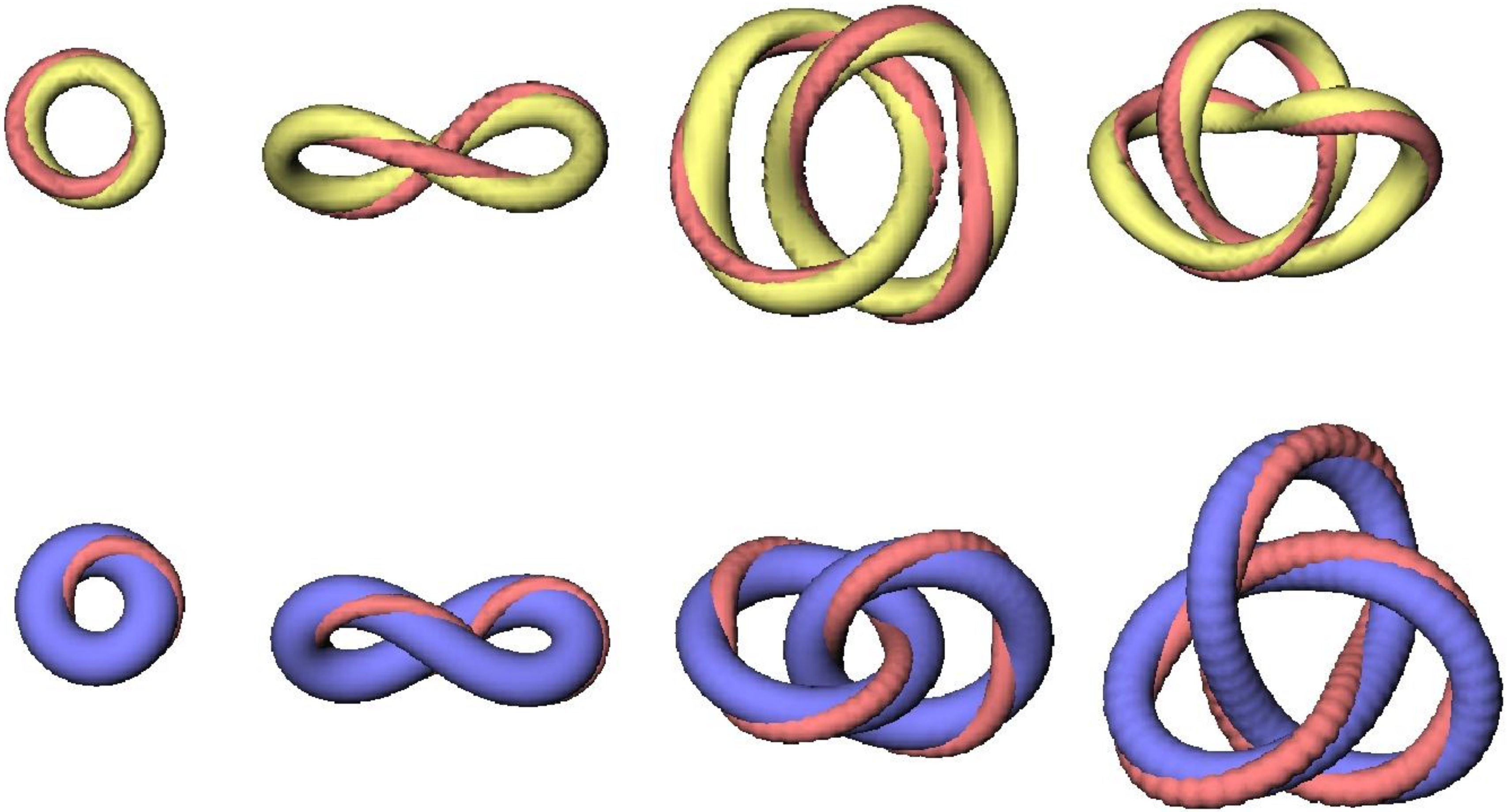
square: link

diamond: trefoil knot

C/B=0.85, $\rho=((B+C)/A)^{1/2}$



Energies of Skyrme-Faddeev solitons vs elastic rods

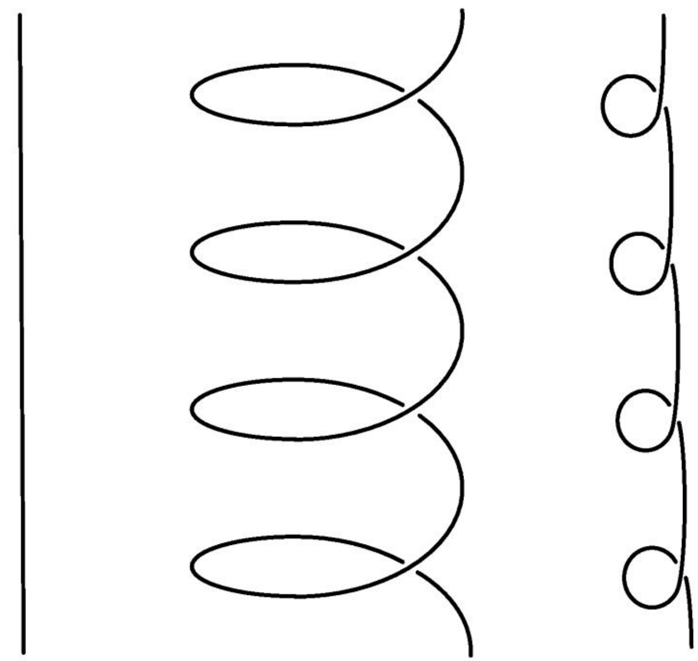


Helical buckling

D. Foster & D. Harland,
Proc Roy Soc Lond A 469, 3172-3190

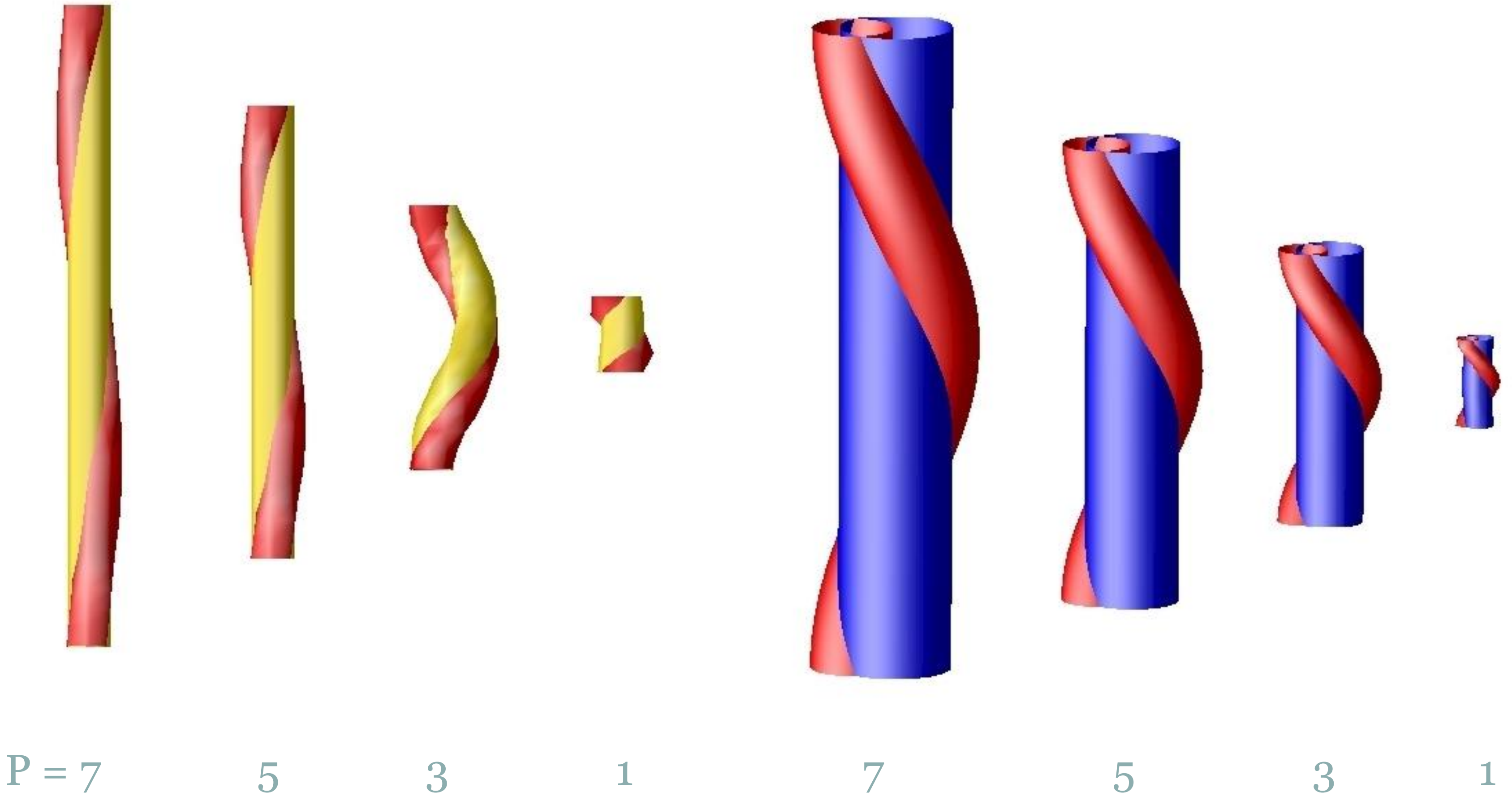
Buckling instabilities

decrease P

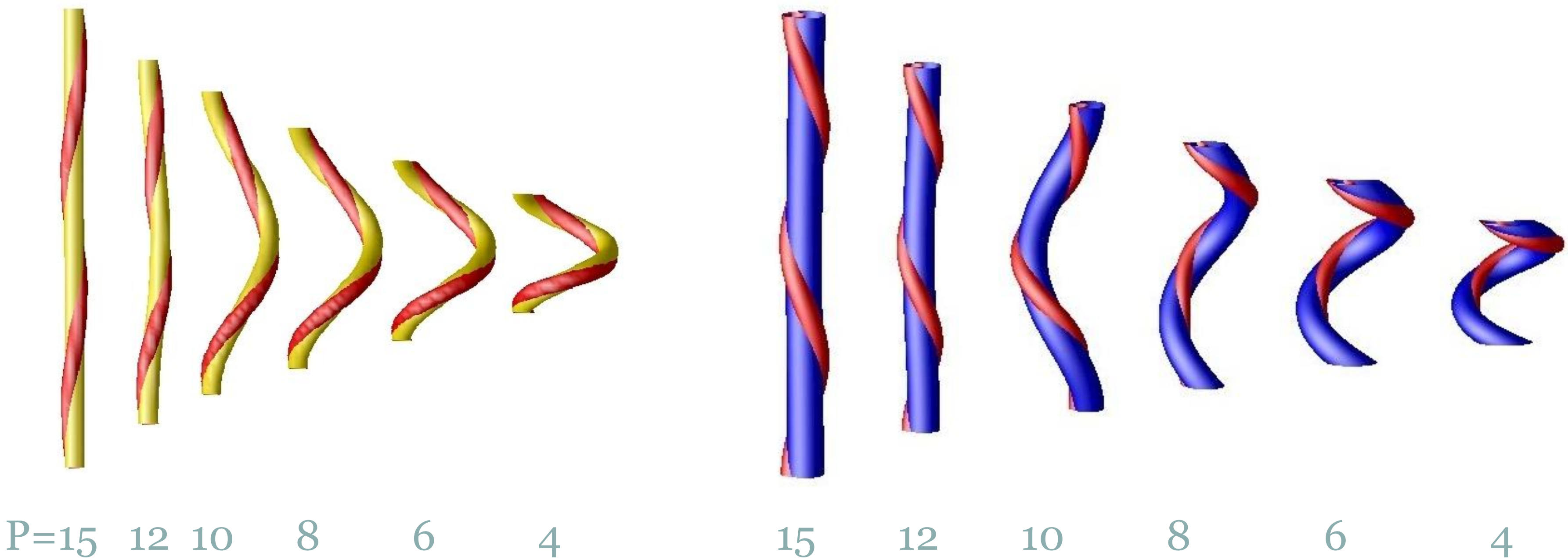


$SO(2) \times SO(2)$ $SO(2)$.

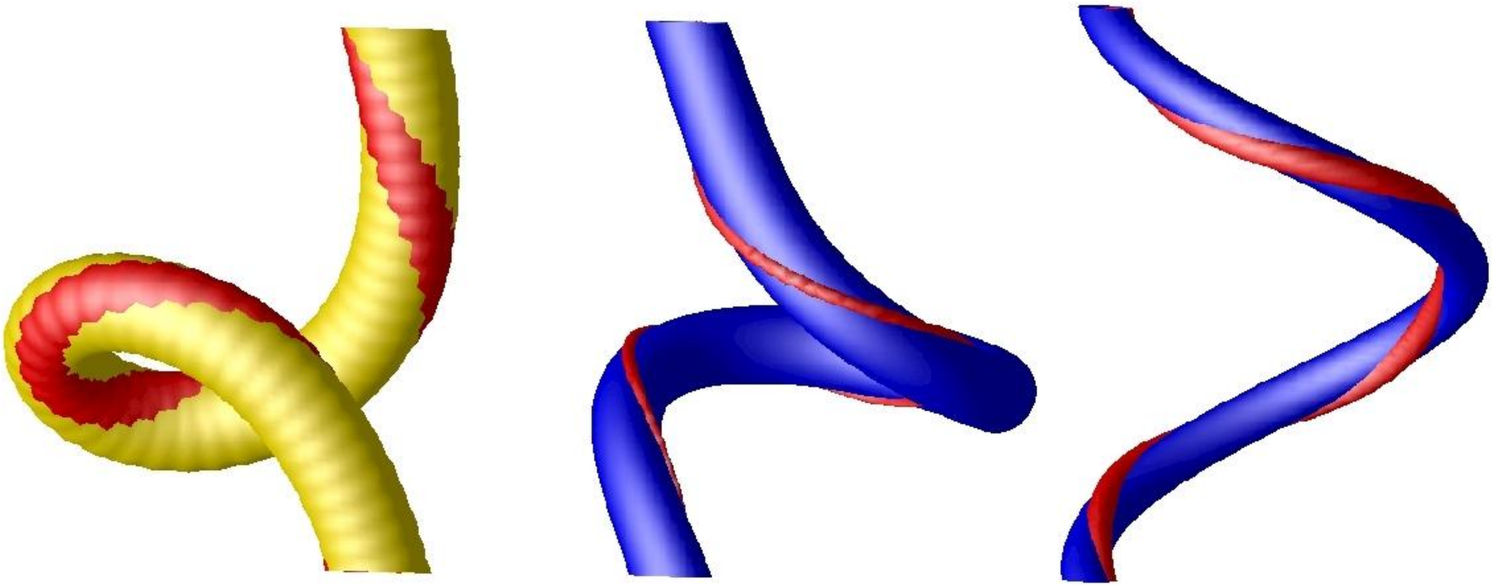
Rods vs solitons: Q=1



Rods vs solitons: Q=2



Rods vs solitons: Q=3



P = 6

Dynamics

ONGOING

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Dynamical elastic rod model

- Unique Lorentz-invariant dynamics
- $x : \Sigma^2 \rightarrow \mathbb{R}^{3,1}$
- m^μ unit normal vector to worldsheet

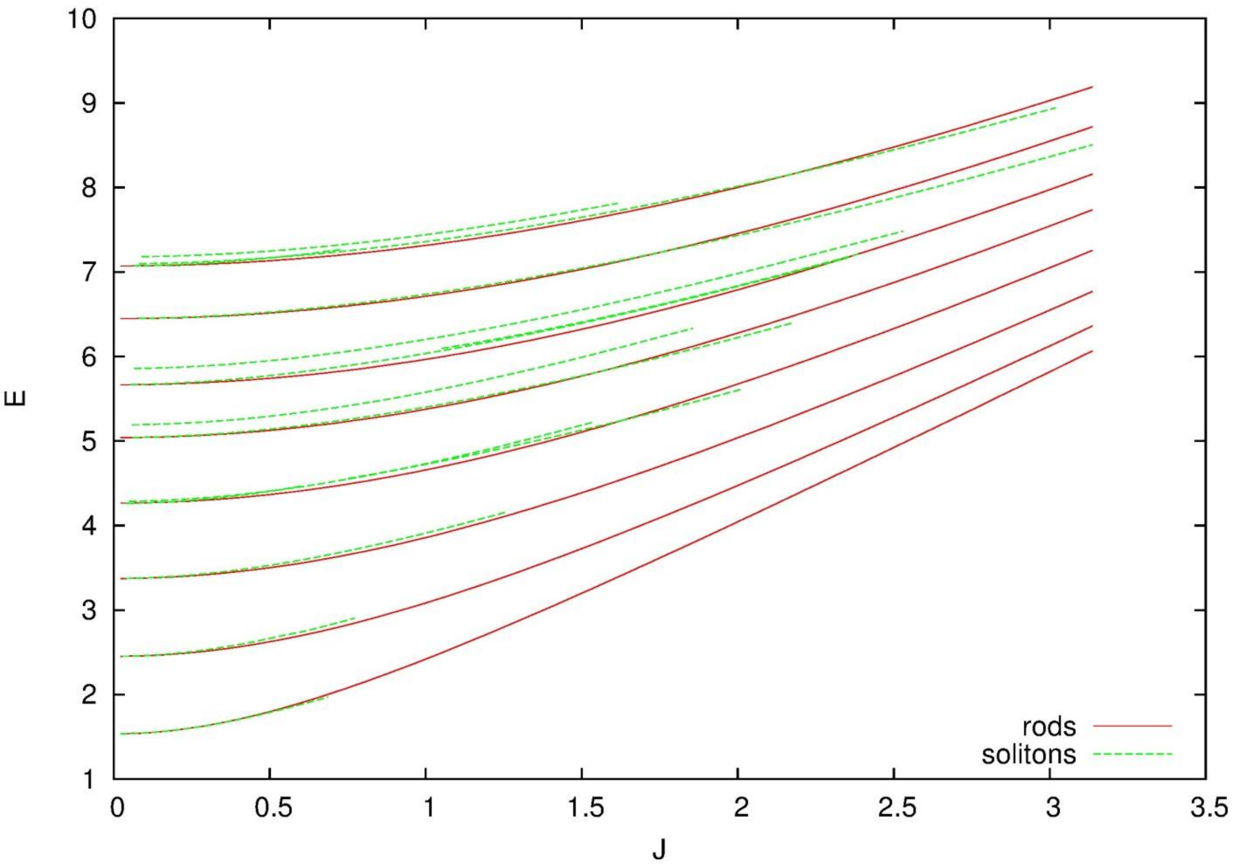
$$S[x,m]=\int_\Sigma (A + B\, \alpha^\mu{}_{ab}\alpha_\mu{}^{ab} + C\, \theta_\mu\theta^\mu)\, \sqrt{(-\det \gamma)}\, d\sigma^1 d\sigma^2$$

- γ induced metric on Σ
- $\alpha^\mu{}_{ab} = \partial_a\partial_b x^\nu\, (\delta_\nu{}^\mu - \gamma^{cd}\partial_c x_\nu \partial_d x^\mu)$
- $\theta_\mu = (-\gamma)^{-1/2}\varepsilon_{\kappa\lambda\nu\rho}\partial_\mu m^\kappa m^\lambda \partial_1 x^\nu \partial_2 x^\rho$

Prediction:

E(J)=

√(E²+ (C/A)J²)



Energy of solitons as a function of charge J

Conclusions

- Effective theory for the Skyrme-Faddeev model based on elastic rods
- Simple
- Many successes
- Tool for searching soliton “landscape”
- Explain $Q^{3/4}$?
- Model other solitons with elasticity theory?