Fermionic quantization of knot solitons

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Joint work with Dave Auckly and Steffen Krusch

"Knot" solitons

$$\varphi : M^3 \to S^2, \quad E = \int_M |d\varphi|^2 + |\varphi^*(\omega_{area})|^2$$

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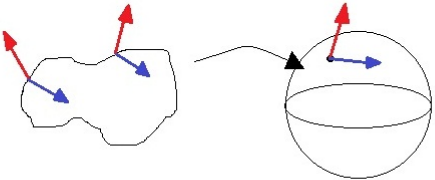

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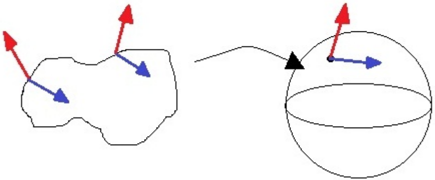

- $\varphi(\infty) = (0, 0, 1)$
- Framed cobordism class of any regular preimage

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Within the α family, classes are labelled by elements of

$$H^3(M; \mathbb{Z})/2\alpha \smile H^1(M; \mathbb{Z})$$

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- Essential maps: $\varphi^{-1}(\text{reg pt})$ wraps around a nontrivial 1-cycle in M. Not localized. Topological geons?

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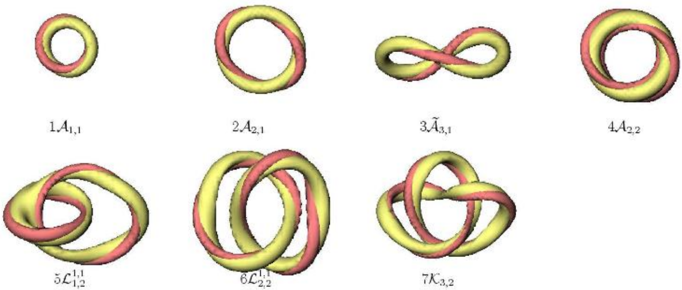

Picture credit: Battye and Sutcliffe

Are knot solitons really solitons?

- Spatially localized, but string-like core
- $E \sim Q^{\frac{3}{4}}$ (not $\sim Q$, like skyrmions)
- Can they be quantized as fermions (like skyrmions)?
- Question about the topology of $(S^2)^M_*$

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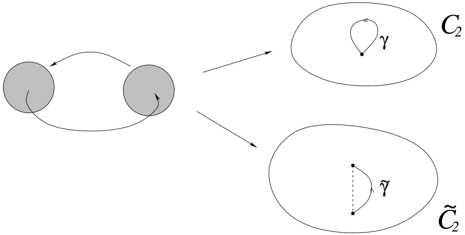
- Need exchange loop in $\mathscr{C}_2$ to be noncontractible and of even order in $\pi_1(\mathscr{C}_2)$ (...+ many consistency checks)



- Key question: what is $\pi_1(\mathscr{C})$?

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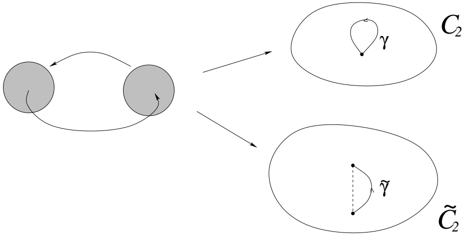

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Hopfions from Skyrmions: the Hopf fibration

$$\pi : \mathbb{C}^2 \supset S^3 \to S^2 \equiv \mathbb{C}P^1, \qquad (z_1, z_2) \mapsto [z_1 : z_2]$$

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$$U(x), \widetilde{U}(x) \mapsto U(x)\widetilde{U}(x)$$

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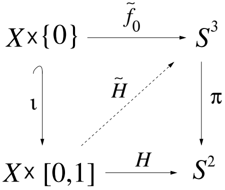
- Basic fact: $\pi_* : (SU(2))^M \to (S^2)^M_*$ surjects, and $B \mapsto Q$

Hopfions from Skyrmions: the Hopf fibration

- $\pi : S^3 \to S^2$ is a **fibration**: homotopy lifting property



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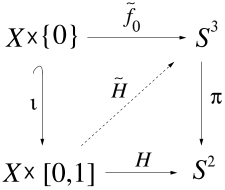

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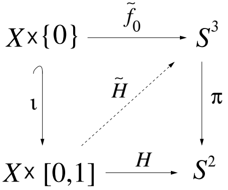

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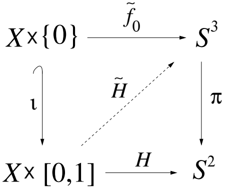

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- Our case: $\rho = \pi_*$, $E = SU(2)^M$, $B = (S^2)^M_*$, $F = U(1)^M$

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- In particular π_{**} is an isomorphism if $H^1(M;\mathbb{Z}) = 0$ (e.g. $\pi_1(M)$ is finite)

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- Demand $\Pi^*\Psi = -\Psi$, i.e.

$$\Psi(\Pi(p)) \equiv -\Psi(p)$$

Gives odd exchange statistics

Finkelstein-Rubinstein symmetry constraints

- Consider quantum ground state $\Psi : \widetilde{\mathscr{C}}_Q \to \mathbb{C}$
  What are its spin L^2, L_3 and isospin K_3 quantum numbers?
- Assume classical energy minimizer φ invariant under simultaneous spatial rotation by α about x_3 axis and isorotation by β about φ_3 axis. Then

$$\gamma_{\alpha\beta} : [0,1] \to \mathscr{C}_Q, \qquad [\gamma_{\alpha\beta}(t)](\mathbf{x}) = R(\beta t)\varphi(R(\alpha t)\mathbf{x})$$

  is a closed loop in $\mathscr{C}_Q$.
- Quantum ground state also eigenstate of spin $\hat{L}^2, \hat{L}_3$ and isospin $\hat{K}_3$
- Two points $p, \Pi(p) \in \widetilde{\mathscr{C}}_Q$ correspond to $\varphi \in \mathscr{C}_Q$. If $\gamma_{\alpha\beta}$ noncontractible, $\Psi(\Pi(p)) = -\Psi(p)$, so

$$(e^{-i\alpha\hat{L}_3}e^{-i\beta\hat{K}_3}\Psi)(p) = -\Psi(p)$$

  So either $\Psi(p) = 0$ (bizarre) or $e^{-i(\alpha L_3 + \beta K_3)} = -1$

Finkelstein-Rubinstein symmetry constraints

- Classical symmetries predict (iso)spin quantum numbers
- Assume L, L_3, K_3 take lowest values allowed by constraints

$\lvert Q \rvert$	E_Q	shape	symmetry	ground state
1	135.2	unknot	$(1,1)$	$\lvert \frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \rangle$
2	220.6	unknot	$(2,1)$	$\lvert 0,0,0 \rangle$
3	308.9	unknot	C_2^1	$\lvert \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \rangle$
4	385.5	unknot	$(2,2)$	$\lvert 0,0,0 \rangle$
5	459.8	link	—	$\lvert \frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2} \rangle$
6	521.0	link	—	$\lvert 0,0,0 \rangle$
7	589.0	knot	—	$\lvert \frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2} \rangle$

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  Have to impose FR constraints "by hand"
- Genuinely localized hopfions can be consistently fermionically quantized
  $\Rightarrow$ particle-like in this respect, at least
- Key to understanding this is algebraic topology of the Hopf fibration

Commun. Math. Phys. **263** (2006) 173-216
Commun. Math. Phys. **264** (2006) 391-410

Jobs at Leeds

- Chair/Associate Professorship in Geometry
- Postdoc on Skyrmions (numerics)