Fermions coupled to Hopf Solitons

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Quantised Flux in Tightly Knotted and Linked Systems
based on hep-th/0304264, hep-th/0908.2114 and work in progress

7 December, 2012

Outline

- Motivation
- Skyrmions and Hopf solitons on S^3
- Fermions on S^3
- Results
- Outlook

Motivation

- Similarities between Skyrmions and Hopf solitons
- Fermionic Hopf solitons in topological superconductors (Ran, Hosur, Vishwanath (2010))
- QCD with Adjoint Quarks (Bolognesi, Shifman (2007))
- Fermion number in knotted soliton background (Freyhult, Niemi (2002))

Skyrmions

The Skyrme Lagrangian Density

$$\mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2} \left(\partial_\mu \phi \cdot \partial^\mu \phi \right)^2 + \frac{1}{2} \left(\partial_\mu \phi \cdot \partial_\nu \phi \right) \left(\partial^\mu \phi \cdot \partial^\nu \phi \right)$$
$$+ \lambda \left(\phi \cdot \phi - 1 \right),$$

where $\phi = (\phi_0, \phi_1, \phi_2, \phi_3)$, and $\phi(\infty) = (1, 0, 0, 0)$.

- Topological charge $B \in \pi_3(S^3)$.
- Solutions are localised (particle-like, Derrick's theorem)
- Energy bound: $E \geq 12\pi^2 |B|$ (is of Bogomolny type)
- Bogomolny equations can be solved for vacuum $\phi = (1, 0, 0, 0)$ and on the 3-sphere of radius 1. Then ϕ is the identity map.
- The identity map is a global minimum in the charge 1 sector if the radius of the 3-sphere is less than $\sqrt{2}$ (Manton (1987))

Hopf solitons

The Faddeev-Skyrme Lagrangian Density

$$\mathcal{L} = \partial_\mu \phi \cdot \partial^\mu \phi - \frac{1}{2}\left(\partial_\mu \phi \cdot \partial^\mu \phi\right)^2 + \frac{1}{2}\left(\partial_\mu \phi \cdot \partial_\nu \phi\right)\left(\partial^\mu \phi \cdot \partial^\nu \phi\right)$$
$$+\lambda\left(\phi \cdot \phi - 1\right),$$

where $\phi = (\phi_1, \phi_2, \phi_3)$, and $\phi(\infty) = (0, 0, 1)$.

- Topological charge $Q \in \pi_3(S^2)$.
- Solutions are localised (string-like, Derrick's theorem)
- Energy bound: $E \geq const. |Q|^{\frac{3}{4}}$
- Vacuum solution $\phi = (0, 0, 1)$ has $Q = 0$ and Hopf map has $Q = 1$.
- The Hopf map is a global minimum in the charge 1 sector if the radius of the 3-sphere is less than $\sqrt{2}$ (Ward (1998))

Fermions on S^3

- Dirac equation in curved background:

$$\mathcal{L}_{fermion} = \bar{\psi}(i\gamma^\alpha e_\alpha{}^\kappa(\partial_\kappa + \Omega_\kappa))\psi.$$

where Ω is the spin connection and $e_\alpha{}^\kappa$ are frame fields.

- Let $S^3 = \{x_i^2 + w^2 = 1\}$. With stereographic coordinates

$$X_i = \frac{x_i}{1 - w}$$

the metric on $\mathbb{R} \times S^3$ is

$$g_{\mathbb{R}\times S^3} = \mathrm{diag}\left(1, -\frac{4}{(1+R^2)^2}, -\frac{4}{(1+R^2)^2}, -\frac{4}{(1+R^2)^2}\right).$$

- Lagrangian density:

$$\mathcal{L}_{fermion} = \bar{\psi}(X_i, t)\left(i\gamma^0\partial_t - i\gamma^i\left(\frac{1+R^2}{2}\partial_{X_i} - X_i\right)\right)\psi(X_i, t).$$

Interaction terms

- Interaction term for Skyrmions:

$$\mathcal{L}_{int} = -g\bar{\psi}(\phi_0 + i\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\phi})\psi,$$

  chirally coupled.

- On S^3 we can write the Skyrme field $\phi = (\phi_0, \boldsymbol{\phi})$ as

$$\phi = (\cos f(\chi), \sin f(\chi)\mathbf{e}_\chi)\,.$$

  where χ is the "radial" angle on S^3 and $\mathbf{e}_\chi$ is the "radial" unit vector. Setting $f(\chi) = B\chi$ gives exact solutions for $B = 0$ and $B = 1$.

- Interaction terms for Hopf solitons:

$$\mathcal{L}_{int} = -ig\bar{\psi}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\phi}\psi$$

  where $\phi = (0, 0, 1)$ for the vacuum and the Hopf map for $Q = 1$.

Time independent Schrödiger equation for Skyrmions

- Set $\psi(\mathbf{x}, t) = \mathrm{e}^{iEt}\psi(\mathbf{x})$ and obtain

$$E\psi = \begin{pmatrix} g\cos f(\chi) & \boldsymbol{\sigma}\cdot\boldsymbol{p} + ig\boldsymbol{e}_\chi\cdot\boldsymbol{\tau}\sin f(\chi) \\ \boldsymbol{\sigma}\cdot\boldsymbol{p} - ig\boldsymbol{e}_\chi\cdot\boldsymbol{\tau}\sin f(\chi) & -g\cos f(\chi) \end{pmatrix}\psi =: H\psi.$$

- Here

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

  is a doublet of Dirac spinors. $\psi_{1,2}$ are 2×2 matrices where spin $\hat{\mathbf{S}} = \frac{1}{2}\boldsymbol{\sigma}$ acts on the left and isospin $\hat{\mathbf{I}} = \frac{1}{2}\boldsymbol{\tau}$ acts on the right.

- Also,

$$\boldsymbol{p} = -i\boldsymbol{e}_\chi\left(\partial_\chi + \frac{\sin\chi}{1-\cos\chi} + \frac{1}{\sin\chi}2\hat{\boldsymbol{S}}\cdot\hat{\mathbf{L}}\right),$$

  where $\hat{\mathbf{L}}$ is the angular momentum operator.

- H commutes with grand spin $\hat{\mathbf{G}} = \hat{\mathbf{L}} + \hat{\mathbf{S}} + \hat{\mathbf{I}}$ and parity $\hat{P} : \mathbf{x} \mapsto -\mathbf{x}$.

Eigenstates

- Can decompose wave function ψ into eigenstates of $\hat{\mathbf{G}}$ and parity, e.g. parity $(-1)^G$ state with quantum numbers G and M is

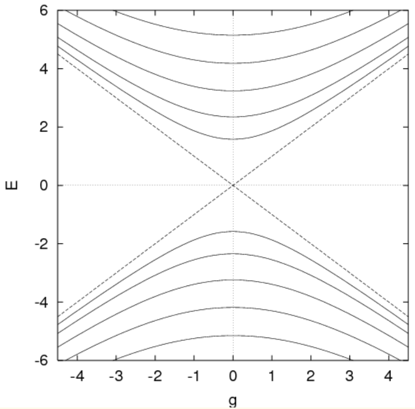
$$\psi = \begin{pmatrix} \sqrt{1-u}\sqrt{1-u^2}^{\,G} G_2(u)\,|GM\rangle_b + \sqrt{1+u}\sqrt{1-u^2}^{\,G-1} G_3(u)\,|GM\rangle_c \\ i\sqrt{1+u}\sqrt{1-u^2}^{\,G} G_4(u)\,|GM\rangle_d + i\sqrt{1-u}\sqrt{1-u^2}^{\,G-1} G_1(u)\,|GM\rangle_a \end{pmatrix},$$

- Inserting the ψ into the Schrödinger equation $E\psi = H\psi$ results in a system of four coupled differential equations:

$$\begin{aligned}
(1-u)\frac{dG_1}{du} &= \left(G + \tfrac{1}{2} - \frac{g(1-u)}{2G+1}\right) G_1 + (E - gu)G_3 - \frac{2g\sqrt{G(G+1)}(1-u^2)}{2G+1} G_4, \\
(1-u)\frac{dG_2}{du} &= \left(G + \tfrac{3}{2} - \frac{g(1-u)}{2G+1}\right) G_2 - (E + gu)G_4 + \frac{2g\sqrt{G(G+1)}}{2G+1} G_3, \\
(1+u)\frac{dG_3}{du} &= -\left(G + \tfrac{1}{2} - \frac{g(1+u)}{2G+1}\right) G_3 - (E + gu)G_1 + \frac{2g\sqrt{G(G+1)}(1-u^2)}{2G+1} G_2, \\
(1+u)\frac{dG_4}{du} &= -\left(G + \tfrac{3}{2} - \frac{g(1+u)}{2G+1}\right) G_4 + (E - gu)G_2 - \frac{2g\sqrt{G(G+1)}}{2G+1} G_1.
\end{aligned}$$

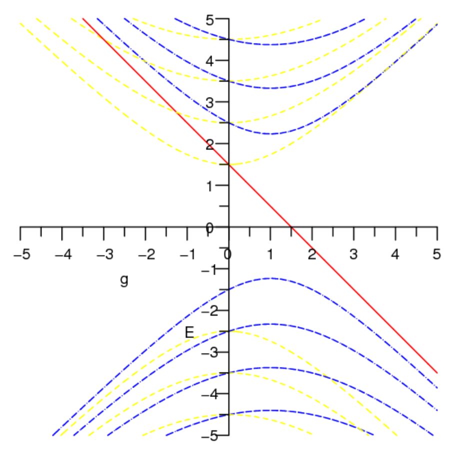
- Equations with opposite parity gives rise to the same system of differential equations but with $g \mapsto -g$.

Spectrum for the vacuum $f(\mu) = 0$



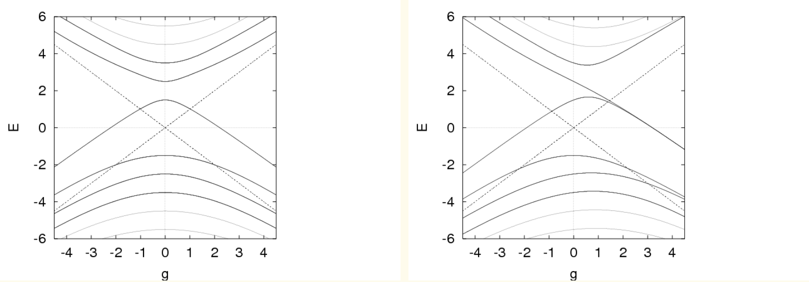
- For $f(\chi) = 0$ the equations decouple into two pairs
- Energy $E = \pm\sqrt{g^2 + \left(N + \frac{3}{2}\right)^2} \quad \text{for} \quad N = 0, 1, 2, \ldots$
- Solutions $G_i(u)$ can be written explicitly in terms of Jacobi polynomials (agrees with Camporesi and Higuchi (1996) for $g = 0$)

Spectrum for $B = 1$ Skyrme field with $f(\mu) = \mu$



- Spectrum is known explicictly and eigenfunctions can be written as polynomials in u (Goatham and SK (2010))
- Most interesting mode: $E_0 = \frac{3}{2} - g$. This mode crosses zero and is related with the non-trivial topology of the Skyrme field (SK (2002))

$f(\mu) = 2\mu$ and $f(\mu) = 3\mu$



- Spectra calculated numerically
- B modes cross zero, where B is the topological charge of the Skyrme field

Hopf solitons

Questions

Can the spectrum be found explicitly for Hopf solitons?
Does the spectrum have a similar dependence on topology?

- Recall: Interaction terms for Hopf solitons:

$$\mathcal{L}_{int} = -ig\bar{\psi}\gamma_5\boldsymbol{\tau}\cdot\boldsymbol{\phi}\psi$$

where $\phi = (0, 0, 1)$ for the vacuum and the Hopf map for $Q = 1$.

- $\hat{\mathbf{G}}$ does not commute with interaction term

Equations for Hopf soliton vacuum

- For the vacuum $\phi = (0,0,1)$ the interaction term is

$$\mathcal{L}_{int} = -ig\bar{\psi}\gamma_5 \boldsymbol{\tau} \cdot \boldsymbol{\phi}\psi$$

- This commutes with G_3 and hence leaves M fixed.
- The interaction term mixes states with G and $G+1$.
- Setting

$$\psi = \begin{pmatrix} G_2(u)\left|GM\right\rangle_b + G_1(u)\left|G+1M\right\rangle_c \\ iG_4(u)\left|GM\right\rangle_d + iG_3(u)\left|G+1M\right\rangle_a \end{pmatrix},$$

gives rise to four coupled first order ODEs.

Hopf map

- The Hopf map in spherical polar coordinates gives

$$\begin{aligned}\phi\cdot\boldsymbol{\tau} &= 2\sin^2\chi\sin\theta\cos\theta(\cos\phi\tau_1+\sin\phi\tau_2)\\ &+ 2\sin\chi\cos\chi\sin\theta(\sin\phi\tau_1-\cos\phi\tau_2)\\ &+ (2\sin^2\chi\sin^2\theta-1)\tau_3.\end{aligned}$$

- $\mathbf{G}$ and $\hat{P}$ do not commute with interaction term, but $\hat{G}_3$ still commutes. Hence M is conserved.
- Interaction term couples many different $|GM\rangle$ states.
- Needs numerical solution.

Conclusion

- Discussed how to couple Skyrmions and Hopf solitons to fermions on S^3
- Showed spectra for Skyrmions (analytical solutions for $B = 0$ and $B = 1$, numerically for $B = 2$ and $B = 3$.)
- Hopf solitons more complicated because generalized angular momentum operator no longer commutes.
- It is possible to calculate spectra in the vacuum state $\phi = (0, 0, 1)$.
- Lowest state for Hopf map can only be calculated numerically. Work in progress...