A DISTENSIBLE-TUBE WAVE ENERGY CONVERTER WITH A DISTRIBUTED POWER-TAKE-OFF

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1. Summary

A distensible-tube Wave Energy Converter (WEC) operates by converting the wave energy into "bulge waves" in a large distensible (e.g. rubber) water-filled tube floating perpendicular to the wave crests, see [1]. The distensibility is chosen so that the natural bulge-wave speed matches that of the water waves; a resonant interaction then occurs, and large bulge waves are generated, concentrating the wave energy. The best power take-off (PTO) mechanism appears to be one distributed along the whole length of the tube, see [1] fig.13. This paper analyses that option, considering in particular a non-linear PTO in which air in chambers in the tube is pumped though one-way valves between low and high pressure pneumatic accumulators.

The analysis progresses by considering the following cases in turn:

a) A steady sinusoidal bulge wave in a tube floating in still water. This is a very simple case.

b) A sinusoidal bulge wave as in (a) but decaying with time (while remaining the same everywhere along the tube) due to a linear dissipation (or power-take-off) term in the tube material properties.

c) A steady sinusoidal bulge wave, with a linear dissipation term, which is "forced" by a water wave to propagate at the same speed, rather than the natural propagation speed in (a) and (b). The steady state is maintained by the power input from the water wave – this is thus a wave energy converter (WEC). The power is calculated both from the dissipation term in tube material properties, and from the water waves produced by the tube – and the answers are shown to be identical, as they should be.

d) Same case as (c) but with a non-linear dissipation term, with hysteresis loss above a fixed pressure limit, as in a distributed PTO with common pneumatic accumulators. This is approximated by linearising the hysteresis into a linear stiffness and linear dissipation term.

e) Same case as (d), but in irregular water waves. Successive waves are approximated as members of long regular wave trains. This allows the power to be calculated, the pressure limit to be optimised. It is shown that a 600m long device 3.1m diameter with an air spring and a distributed air PTO with common accumulators, should produce 500 kW annual average pneumatic power, in the Benbecula wave climate.

Since all solutions have the same wavelength, we denote the wavenumber by k in all cases. The speed of the water wave is denoted by C, its frequency kC by ω and its period $2\pi/\omega$ by T. The natural speed of the bulge wave is denoted by C^*, its pressure by p^*, its frequency kC^* by ω^* and its period $2\pi/\omega^*$ by T^*. The pressure in the water wave is denoted by p, and the pressure in the tube is (in all cases) denoted by p^*. A special case is the "resonant" one when $C = C^*$. The water wave period needed to achieve this, with no dissipation term, is denoted by T_R, and its wavenumber by $k_R = (2\pi/T_R)^2/g$. Except for g, k, x, t, π, τ and ω (and ω^*), which are a universal notation in the literature, small letters are used for functions of time (and space), and capitals for constants (often complex constants).

2. Bulge wave in a tube floating in still water

Consider a sinusoidal bulge wave of wavenumber *k*, propagating in the positive *x*- direction, as shown at the initial time *t = 0* in Figure 1 below. We ignore for the moment the energy lost by wave radiation (it is calculated in Section 5), so the bulge wave does not decay with time.

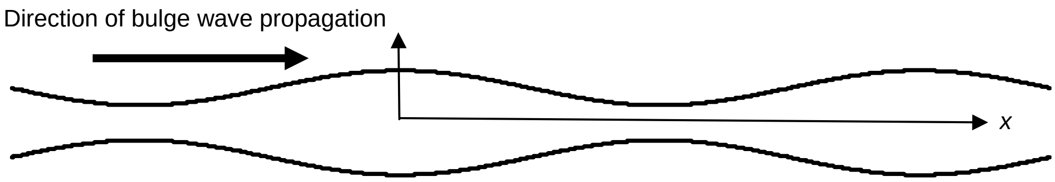

Figure 1: Bulge wave at *t = 0*

The static cross-sectional area of the tube is *A*, and the additional dynamic cross-sectional area is *a(x,t)*, so in complex notation the distension is (the real part of):

$$a/A = Be^{ik(x-C^*t)} \quad (1)$$

where C^* is the natural propagation speed of the bulge wave, and *B* is the bulge-wave amplitude (and is real so that the position of the bulge wave at *t=0* is as shown in Figure 1).

If the longitudinal water velocity in the tube is *v(x,t)* in the direction of *x*, then conservation of fluid volume requires (on linear theory):

$$A\partial v/\partial x + \partial a/\partial t = 0 \quad (2)$$

so that:

$$v = C^*Be^{ik(x-C^*t)} \quad (3)$$

The velocity potential is therefore:

$$(ik)^{-1}C^*Be^{ik(x-C^*t)} \quad (4)$$

so that the pressure $p^*(x,t)$ in the tube is:

$$p^* = \rho C^{*2}Be^{ik(x-C^*t)} \quad (5)$$

From the material properties of the tube the pressure can also be written:

$$p^* = D^{-1}(a/A) = D^{-1}Be^{ik(x-C^*t)} \quad (6)$$

where *D* is the distensibility of the tube. Hence we deduce that the speed of the bulge wave is given by:

$$C^{*2} = (\rho D)^{-1} \quad (7)$$

as is well-known in the literature (e.g. [2], p.94).

3. Bulge wave in still water with power take off (PTO)

Consider now a bulge wave in a tube in still water, as in the previous Section, but with power dissipation – as occurs naturally in a lossy elastic material, or in an electro-active polymer, in which the power is not dissipated but converted into electricity. Our interest in this paper is in an equivalent scheme, in which the distensibility of the tube is provided by the compressibility of internal air compartments, and the power take-off is achieved by using the variable air pressure to pump air from a low-pressure accumulator to a high-pressure one, through one-way valves. A possible arrangement of such a wave energy converter (WEC) is shown in Figure 2 below.

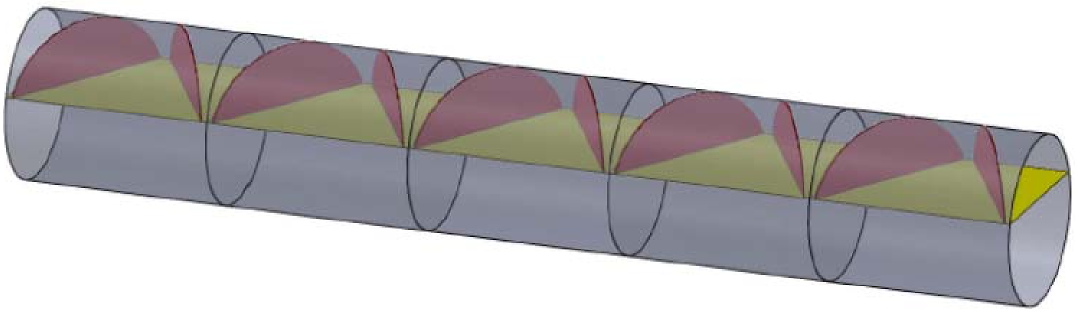

Figure 2. Possible WEC arrangement

The internal air compartments are above the flexible membrane (shown yellow), and between the rigid diagonal bulkheads (shown red). They communicate via one-way valves to two pneumatic accumulators running along the top of the tube (not shown), which smooth the power to a turbine operating between them. The pressure difference between the accumulators can be adjusted by means of the turbine characteristics, so as to optimise the power output – and also to ensure survivability, by means of a pressure-relief valve around the turbine, which will limit the pressure difference between the accumulators.

For our present linear theory, we idealise the additional flow to the accumulators as linearly proportional to the tube pressure p^* - the power is absorbed by this linear flow resistance, between the air in the tube and a single air reservoir at the static tube pressure. We thus write the distension of the tube as:

$$a/A \quad = \quad Dp^* \quad + \quad \Delta\int p^* dt \qquad (8)$$

where the second term is the "PTO distension", giving a distension proportional to the time-integral of the tube pressure p^*, and thus to the time-integral of the flow-rate to the reservoir. The constant of proportionality is the "distensibility rate" Δ.

As a result of this additional power-absorption term, the bulge wave will decay with time, while remaining sinusoidal in shape. Its steady-state shape of $Be^{ik(x-C^*t)}$ without dissipation (see (1)) becomes*:*

$$a/A \quad = \quad Be^{ik(x-C^*t) - t/\tau} \qquad (9)$$

where τ is the (real) decay-time of the bulge waves. From (2) the water velocity from the bulge wave, in the direction of *x,* is:

$$v \quad = \quad (C^*+(ik\tau)^{-1})Be^{ik(x-C^*t) - t/\tau} \qquad (10)$$

from which the pressure follows as in (5) as:

$$\rho(C^*+(ik\tau)^{-1})^2 Be^{ik(x-C^*t) - t/\tau} \qquad (11)$$

which must be equal to the pressure (8) calculated from the tube distension, thus:

$$\rho(C^*+(ik\tau)^{-1})^2 Be^{ik(x-C^*t) - t/\tau} \quad = \quad (D - \Delta/(ikC^* + \tau^{-1}))^{-1} Be^{ik(x-C^*t) - t/\tau}$$

i.e. $$\rho(C^{*2} + 2C^*(ik\tau)^{-1} - (k\tau)^{-2})(D(ikC^* + \tau^{-1}) - \Delta) \quad = \quad ikC^* + \tau^{-1} \qquad (12)$$

Equating the real parts of (12):

$$\rho(C^{*2} - (k\tau)^{-2})(D\tau^{-1} - \Delta) + \rho 2C^{*2}D\tau^{-1} = \tau^{-1}$$

i.e. $$\rho C^{*2}(3D - \Delta\tau) - \rho(k\tau)^{-2}(D - \Delta\tau) = 1$$

i.e. $$C^{*2}(3 - \Delta D^{-1}\tau) - (k\tau)^{-2}(1 - \Delta D^{-1}\tau) = \rho^{-1}D^{-1}$$

i.e. $$C^{*2} \quad = \quad \{\rho^{-1}D^{-1} - (k\tau)^{-2}(\Delta D^{-1}\tau - 1)\}/(3 - \Delta D^{-1}\tau) \qquad (13)$$

Equating the imaginary parts of (12)

$$\rho 2C^*(k\tau)^{-1}(\Delta - D\tau^{-1}) + \rho(C^{*2} - (k\tau)^{-2})DkC^* \quad = \quad kC^*$$

i.e. $$\rho 2(k\tau)^{-2}(\Delta\tau - D) + \rho(C^{*2} - (k\tau)^{-2})D = 1$$

i.e. $$(k\tau)^{-2}(2\Delta D^{-1}\tau - 3) + C^{*2} \quad = \quad \rho^{-1}D^{-1}$$

i.e. $$C^{*2} \quad = \quad \rho^{-1}D^{-1} - (k\tau)^{-2}(2\Delta D^{-1}\tau - 3) \qquad (14)$$

If we set the delay time τ as:

$$\tau = 2D/\Delta \qquad (15)$$

then both (13) and (14) give the bulge wave speed C^* as

$$C^{*2} \quad = \quad \rho^{-1}D^{-1} - (k\tau)^{-2} \quad = \quad C^{*2}_0 - (k\tau)^{-2} \qquad (16)$$

were C^*_0 is the bulge wave speed (7) with no dissipation. Thus the dissipation term slows down the bulge wave slightly, as well as making it decay with time.

4. Steady bulge wave in regular water wave

As well as the "free-running" bulge wave just described, a bulge wave can also be "forced" by a water wave[1], so that it travels at the water wave speed C, rather than the natural bulge-wave speed C^*. The tube is then acting as a wave energy converter, by virtue of the PTO described in the previous Section. We ignore for the moment the change to the water wave produced by the tube - it is clearly small if the tube diameter is small, and is calculated in Section 5.

We can write the distension of the tube as:

$$a/A = Be^{ik(x - Ct)} \qquad (17)$$

From which the pressure p^* in the tube follows as before from its distensibility parameters D and Δ, except that in (8) the tube pressure p^* is now replaced by the pressure *difference* between p^* and the pressure p in the water wave outside, thus:

$$a/A = D(p^* - p) + \Delta\int(p^* - p)\,dt \qquad (18)$$

This difference $p^* - p$ is henceforth referred to as the tube "pressurisation". Thus (putting $kC = \omega$):

$$p^* - p = (D + \Delta i/\omega)^{-1}Be^{ik(x - Ct)} \qquad (19)$$

From the distension (17) the fluid mechanical argument of (1) – (5) gives the tube pressure p^* as:

$$p^* = \rho C^2 Be^{ik(x-Ct)} \qquad (20)$$

So if we use (20) to substitute for p^* in (19), and write D_C for the distensibility of a dissipation-free tube with natural bulge-wave speed equal to the water wave speed C (i.e. $C = 1/\surd(\rho D_C)$) then we obtain the water wave pressure p as:

$$p = [D_C^{-1} - (D + \Delta i/\omega)^{-1}]\,Be^{ik(x - Ct)} \qquad (21)$$

Relative to this datum, the tube pressurisation (19) is:

$$(p^* - p)/p = (D + \Delta i/\omega)^{-1}/\{D_C^{-1} - (D + \Delta i/\omega)^{-1}\}$$

$$= \{(D + \Delta i/\omega))/D_C - 1\}^{-1}$$

$$= \{(1 + iT/(\pi\tau))D/D_C - 1\}^{-1}$$

$$= \{(1 + iT/(\pi\tau))T^2/T_R^{\ 2} - 1\}^{-1} \qquad (22)$$

where T_R is the "resonant" wave period at which the water wave speed $C_R = gT_R/(2\pi)$ is the bulge-wave speed without dissipation, viz. $1/\surd(\rho D)$, so that $D/D_C = T^2/T_R^{\ 2}$.

A convenient datum for tube distension is the equivalent distension in the water wave (i.e. the distension seen in an infinitely-distensible tube, whose cross-section will be stretched into a vertical ellipse in wave crests, and squashed into a horizontal ellipse in wave troughs), which is $k\{p/(\rho g)\} = p/(\rho C^2)$. Relative to this datum the tube elastic distension (i.e. the first term in (18)) is:

[1] The vertical motion of the tube produced by the water wave is immaterial to the forcing, since the internal and external pressure have the same vertical hydrostatic pressure gradient. If the water wave causes the tube to follow the water surface, for example, then the external pressure variations will be nil, because the variations in external hydrostatic pressure are cancelling the water wave pressure seen at a fixed point. The bulge waves in the tube will now be forced by the equal variations in the internal hydrostatic pressure.

$$D(p^* - p)/(p/(\rho C^2)) = D\rho C^2(p^* - p)/p = (C^2/C_R^2)(p^* - p)/p = (T^2/T_R^2)(p^* - p)/p$$

$$= (T^2/T_R^2)\{(1 + iT/(\pi\tau))T^2/T_R^{\,2} - 1\}^{-1} \qquad (23)$$

And the tube PTO distension (i.e. the second term in (18)), relative to the same datum, is:

$$(\Delta i/\omega)(p^* - p)/(p/(\rho C^2)) = (\Delta i/(D\omega))D(p^* - p)/(p/(\rho C^2))$$

$$= (iT/(\pi\tau))(T^2/T_R^2)\{(1 + iT/(\pi\tau))T^2/T_R^{\,2} - 1\}^{-1} \qquad (24)$$

Finally the power per unit length is the product of the PTO rate-of-distension (i.e. (24) times $-i\omega A$) and its pressurisation (22), and the product of the two data (p and $p/(\rho C^2)$), which has a mean value of:

$$½\omega A\{p^2/(\rho C^2)\}(T/(\pi\tau))(T^2/T_R^2)\{(T^2/T_R^{\,2} - 1)^2 + (T/(\pi\tau))^2(T^2/T_R^{\,2})^2\}^{-1} \qquad (25)$$

A suitable datum for power is the mean power flux in the water wave, over an area A, which is $½p\omega\{p/(\rho g)\}A = k^{-1}½\omega A\{p^2/(\rho C^2)\} = k_R^{-1}½\omega A\{p^2/(\rho C^2)\}(T^2/T_R^2)$ where k_R is the resonant wavenumber $(2\pi/T_R)^2/g$. Relative to this datum the mean power over a length k_R^{-1} of tube is:

$$(T/(\pi\tau))\{(T^2/T_R^{\,2} - 1)^2 + (T/(\pi\tau))^2(T^2/T_R^{\,2})^2\}^{-1} \qquad (26)$$

Figure 3 below shows the amplitudes of the pressurisation (22), the elastic distension (23), the PTO distension (24) and the power over tube length k_R^{-1} (26) all plotted as a function of T/T_R, for various values of τ/T_R.

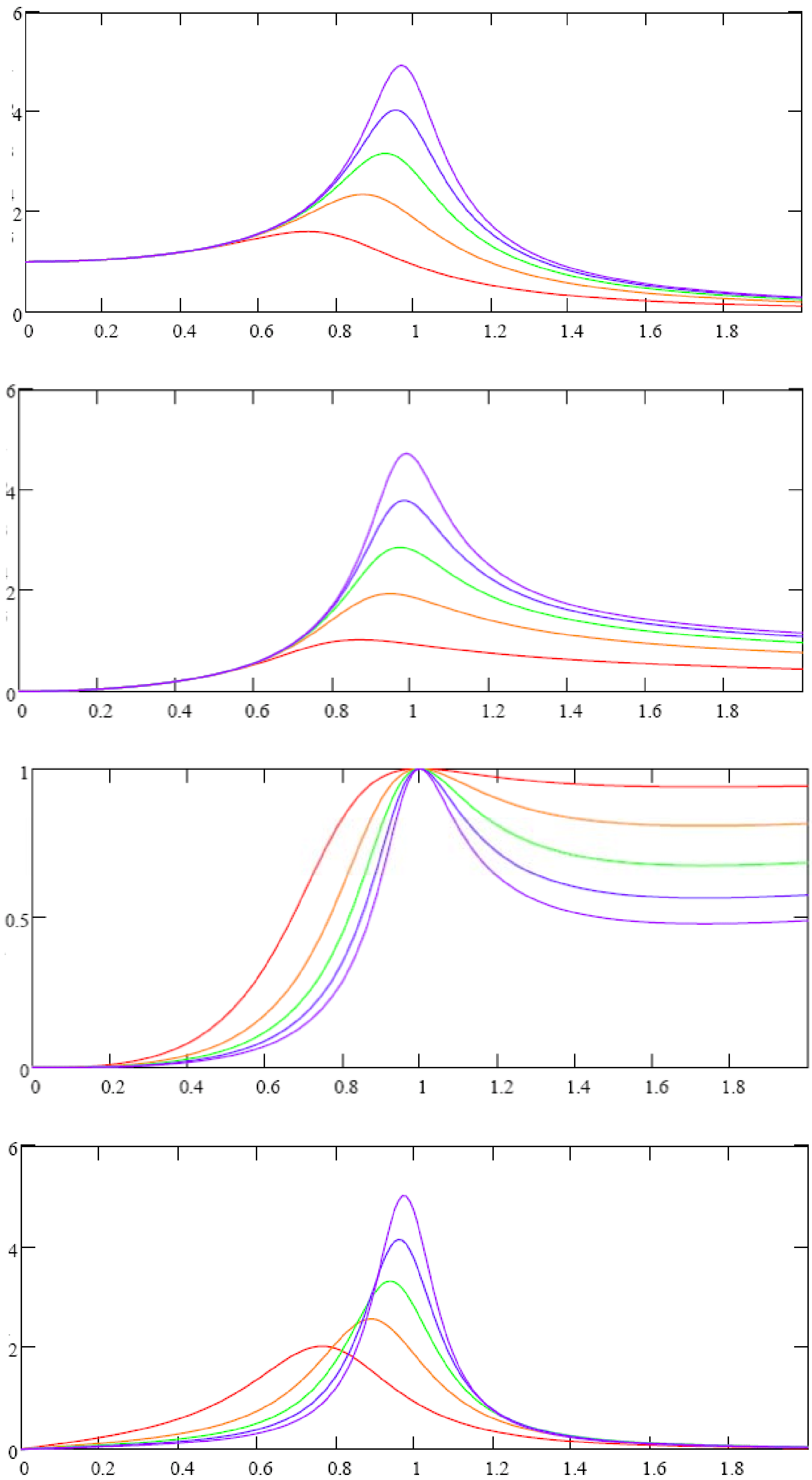

Figure 3. Top to bottom: tube pressurisation (relative to water wave pressure), elastic distension (relative to distension in water wave), PTO distension (ditto), and power over tube length k_R^{-1} (relative to power in water wave over tube cross-section). Horizontal axis is (water wave period T)/(resonant period T_R). Delay times are $0.3T_R$ (red), $0.6T_R$ (orange), $0.9T_R$ (green), $1.2T_R$ (blue), $1.5T_R$ (violet).

At resonance, it can be seen that the top two graphs in Figure 3 both have amplitude $\pi\tau/T_R$, as predicted by (22) and (23). Thus for a large pressure amplification factor at resonance, we require τ to be as large as possible in relation to the resonant wave period T_R.

The phase of (22) and (23) are also significant – if the phase of the water wave is e^{ikx} = cos(kx) at $t = 0$, then at resonance the phase of the tube pressurisation and elastic distension (but not the PTO distension (24), which is 90 degrees out of phase with the elastic distension) is $-ie^{ikx}$ = sin(kx). Since the waves are travelling in the direction of increasing x, see Figure 1, the pressurisation wave (and the elastic distension) is "surfing" on the front of the water wave, exactly 90 degrees out of phase with it. See [1] fig. 2. This is how energy is being extracted from the water wave – the tube is heavier when it is being lifted, and lighted when it is being lowered.

It can be seen from the third graph in Figure 3 that the PTO distension is never greater than the distension in the water wave, for the range of decay-times considered – or indeed for any decay time, from the form of (24). This is a useful safety feature, limiting the distension of the PTO.

The last graph in Figure 3 is the power output from the PTO, which shows greater powers at resonance at large delay-times, but smaller bandwidths.

A parameter of economic importance is the "energy storage time" defined as:

(max energy stored per unit tube length)/(mean power per unit tube length) (27)

This is because the maximum energy stored in the tube sets its capital cost, and of course the power produced sets the revenue from it. The "energy storage time" is readily calculated as:

$$\text{Energy storage time} = \{½D/|p^*-p|^2\}/\{½\Delta/|p^*-p|^2\} = D/\Delta = \tau/2 \qquad (28)$$

Thus for all wave periods the economics of the device are controlled very simply, by the decay-time τ. For the best economics, τ should be as small as possible – which is to say, from (22), that the pressure amplification factor at resonance should be as small as possible. This will obviously increase the cost of the power take-off turbine (low pressures mean high costs), so a compromise is necessary between tube cost and turbine cost.

5. Wave diffraction and radiation

The water waves produced by the tube can be calculated by treating it as a line of point sources, each radiating waves equally in all directions. The pressure in the radiated wave far from a single source is well known and is given in [3] eqns (2.1) and (2.3) as:

$$-i\rho g V e^{-i\omega t} k^2 e^{i(kR-\pi/4)}/\sqrt{(2\pi kR)} \qquad (29)$$

where R is distance from the source and $\rho g V e^{-i\omega t}$ is the varying buoyancy force (written $Fe^{-i\omega t}$ in eqn (2.3) of [3]) felt by the source as a consequence of its varying immersed volume $Ve^{-i\omega t}$. In our case of a bulging tube, there is zero source strength if the tube is completely transparent to the waves, i.e. has infinite distensibiliy so that the water inside behaves exactly as it would were the tube absent. In this case the distension of the tube will not be zero, as noted in the previous section (where this "natural" distension is used as a datum), but is $k\{p/(\rho g)\} = p/(\rho C^2)$. The total distension of the tube is $p^*/(\rho C^2)$ from (17) and (20), so we conclude that the varying immersed volume of the point sources along the tube is $A(p^*-p)/(\rho C^2)$, per unit length.

To integrate along the tube we need to introduce the notation:

$$p = Pe^{ik(x-Ct)} \quad \text{and} \quad p^* = P^*e^{ik(x-Ct)} \qquad (30)$$

where P, P^* are (complex) constants as distinct from $p(x,t)$, $p^*(x,t)$ which are (complex) functions of time and space (the physical variable being the real part, as usual). If we take the tube of length L from $x = -L$ to $x = 0$ in Figure 1, we can integrate (29) to obtain the total radiated wave pressure seen at the tail of the tube ($x = 0$) as:

$$\int_{-L}^{0} -i\rho gA\{(P^*-P)/(\rho C^2)\}e^{ik(x-Ct)}k^2e^{i(k(-x)-\pi/4)}dx/\surd(2\pi k(-x))$$

$$= ik^3A(P^*-P)e^{-i(kCt+\pi/4)}\int_{-L}^{0} -dx/\surd(2\pi k(-x))$$

$$= -2ik^3A(P^*-P)e^{-i(kCt+\pi/4)}\surd(L/(2\pi k))$$

$$= -\pi^{-1}\, ik^2A(P^*-P)e^{-i(kCt+\pi/4)}\surd(2\pi kL) \qquad (31)$$

Compared with the undisturbed water wave pressure Pe^{-ikCt} at the tail of the tube, the ratio is:

$$-\pi^{--1}i(k^2A)\{(P^*-P)/P\}\, e^{-i\pi/4}\surd(2\pi kL) \qquad (32)$$

With the dimensions in Section 8 of $A = 5$ m^2 and $L = 600$m, and for a typical wavenumber k of 0.05 and pressure ratio $|(P^*-P)/P\}|$ = 3, the amplitude is:

$$\pi^{-1}(0.05^2\times 5)3\surd(2\pi 0.05\times 600) = 0.15 \qquad (33)$$

Bearing in mind that the pressure ratio $|(P^*-P)/P|$ is often much less than 3, see Figure 7 below, wave diffraction is just starting to degrade the performance, so we conclude that these values of the cross-section A and length L are as large as desirable. A similar conclusion is reached in [1], fig.13, but that analysis slightly over-states the effect, since it includes the "natural" distension of the tube when calculating the source strength.

We can also integrate (29) in a different way, to obtain the waves produced by the tube, in the far field. This enables the power to be calculated, which gives a searching cross-check on (26), and a figure for the power losses by radiation, ignored hitherto. We consider the waves produced by the tube as seen at some large distance R^* from the tail of the tube at $x = 0$, and at an angle ϑ to the downwave direction. The integral (30) for the wave pressure then comes to:

$$\int_{-L}^{0} -i\rho gA\{(P^*-P)/(\rho C^2)\}e^{ik(x-Ct)}k^2e^{i(k(R^*-x\cos\vartheta)-\pi/4)}dx/\surd(2\pi kR^*)$$

$$= ik^3A(P^*-P)e^{i(k(R^*-Ct)-\pi/4)}\int_{-L}^{0} -e^{ikx(1-\cos\vartheta)}dx/\surd(2\pi kR^*) \qquad (34)$$

In the downwave direction $\vartheta = 0$ this is easily evaluated as:

$$-ik^3A(P^*-P)e^{ik(R^*-Ct-\pi/4)}L/\surd(2\pi kR^*) \qquad (35)$$

from which we can deduce the power (neglecting radiation losses) from the "Fundamental Theorem of Wave Power" ([3] eqn. 2.2) as:

$$\{\rho g\omega/(2k^2)\}R[-(P/(\rho g))\{ik^3A(\overline{P^*}-\bar{P})e^{i\pi/4}L/(\rho g)\}e^{-i\pi/4}]$$

$$= (k\omega AL/(2\rho g))R[-iP(\overline{P^*} - \bar{P})] \quad (36)$$

As in the previous Section, it is convenient to express this power per unit length $k_R^{-1} = (T_R/T)^2k^{-1}$ of tube, and relative to the mean power flux in the water wave, over an area A, which is $|P|^2\omega A/(2\rho g)$. Expressed in this way the power is:

$$(T_R/T)^2R[-i(\overline{P^*} - \bar{P})/\bar{P}] \quad (37)$$

Since $(P^*-P)/P = (p^*-p)/p$, we can use (22) to express this as:

$$(T_R/T)^2R\,[-i\{(1 - iT/(\pi\tau))T^2/T_R^{\,2} - 1\}^{-1}]$$

$$= (T_R/T)^2[(T/(\pi\tau))T^2/T_R^{\,2}\{(T^2/T_R^{\,2} - 1)^2 + (T/(\pi\tau))^2(T^2/T_R^{\,2})^2\}^{-1}]$$

$$= (T/(\pi\tau))\{(T^2/T_R^{\,2} - 1)^2 + (T/(\pi\tau))^2(T^2/T_R^{\,2})^2\}^{-1} \quad (38)$$

which is exactly the same result as (26), but obtained by a completely different argument, based on the far field.

Finally, we can calculate the power lost by wave radiation, which has been ignored hitherto. From (34), the "Fundamental Theorem of Wave Power" ([3] eqn. 2.2) gives this as:

$$\underline{\{\rho g\omega/(4k^2)\}[\,k^3A|P^*-P|\,|\int_{-L}^{0} -e^{ikx(1-\cos\vartheta)}dx\,|/(\rho g)]^2}$$

$$= \{(k^2A)^2\omega L^2/(4\rho g)\}\,|P^*-P|^2\,\underline{|L^{-2}\int_{-L}^{0} -e^{ikx(1-\cos\vartheta)}dx\,|^2} \quad (39)$$

where the underbar denotes the average value over all radiation angles ϑ. The function of ϑ in question (i.e. the "spreading function" above the underbar in (39)) gives the angular dependence of radiated wave power, and is readily evaluated for our case of a 600m long tube, for typical wavenumbers k = 0.025, 0.05, and 0.1 corresponding to wave periods of 12.7s, 9.0s and 6.3s. See Figure 4 below.

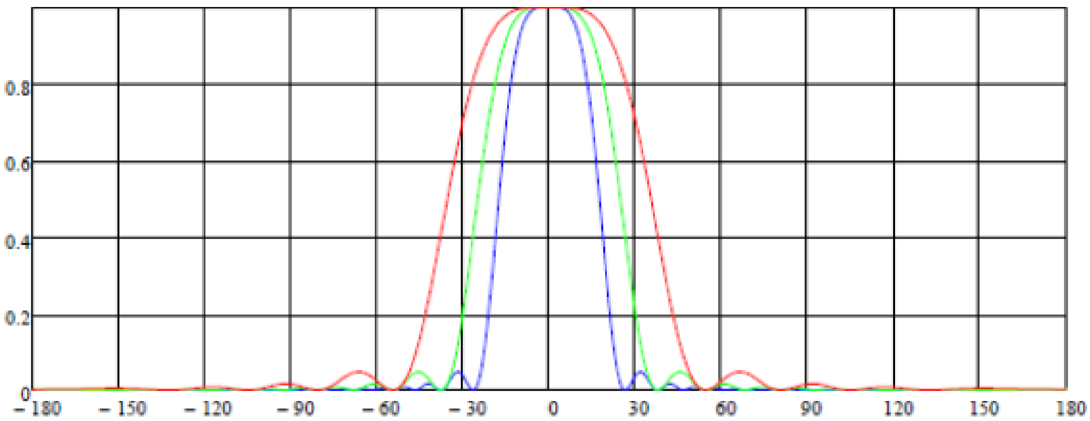

Figure 4: Directional spread of radiated wave power, from downwave. 600m long tube, with k = 0.025 (red), 0.05(green), 0.1 (blue)

It may be seen that the radiated waves are increasingly focused on the downwave direction as the wavenumber increases: typically the average value of the spreading function, over all wave angles (i.e. the underbar term in (39)) is 0.15. Using this value, we can calculate the radiated power (39) as a fraction of the produced power (36), thus:

$$\{(k^2A)^2\omega L^2/(4\rho g)\}\ |P^*\text{-}P|^2 0.15\ /\{(k\omega AL/(2\rho g))R[-iP(\overline{P^*}-\bar{P})]\}$$

$$= 0.075(k^2A)(kL)|P^*\text{-}P|^2/R[-iP(\overline{P^*}-\bar{P})] \qquad (40)$$

Taking a typical wavenumber k of 0.05 and the pressure ratio $|(P^*\text{-}P)/P\}|$ = 3 as above, and assuming the tube is operating at optimum phase (so that $-iP(\overline{P^*}-\bar{P})$is real), this comes to:

$$0.075(0.05^2{\times}5)(0.05{\times}600)3\ =\ 8\% \qquad (41)$$

Bearing in mind that the pressure ratio $|(P^*\text{-}P)/P|$ is often much less than 3, see Figure 7 below, we conclude that radiation losses are small. It may also be deduced that the tube is operating far below its maximum theoretical capture width, which is when radiation losses are 50%, see [3]. This is because it is designed to a very different objective, which is to "sweat the assets" as much as possible (see [3]), by operating the tube at its pressure limits for much of the time, as described in the next Section.

6. Tube with pressure-limit PTO, in regular waves

We saw at the end of Section 4 that the economically-important parameter is the delay-time τ. The parameter at our disposal in the WEC in Figure 2, however, is not the delay time τ directly, but the tube pressurisation limits $\pm P_L$ set by the accumulators. To deduce the effective value of τ at a given value of P_L, we need to linearise the action of the accumulators. In regular waves we can define this value of τ as that which gives a sinusoidal bulge wave of pressurisation P_L. It will be a function of the amplitude of the water wave pressure P in the regular wave. From (22) we have:

$$/P/P_L/^2\ =\ (T^2/T_R{}^2\ -1)^2 + (T/(\pi\tau))^2(T^2/T_R{}^2)^2$$

so that: $$\tau/T\ =\ \pi^{-1}[\{/P/P_L/^2 - (T^2/T_R{}^2 - 1)^2\}(T_R{}^2/T^2)^2]^{-0.5} \qquad (42)$$

which is plotted in Figure 5 below. The wave periods T have been chosen in equal steps over a wave period range of 2:1, covering the peaks of the power graphs at the bottom of Figure 3.

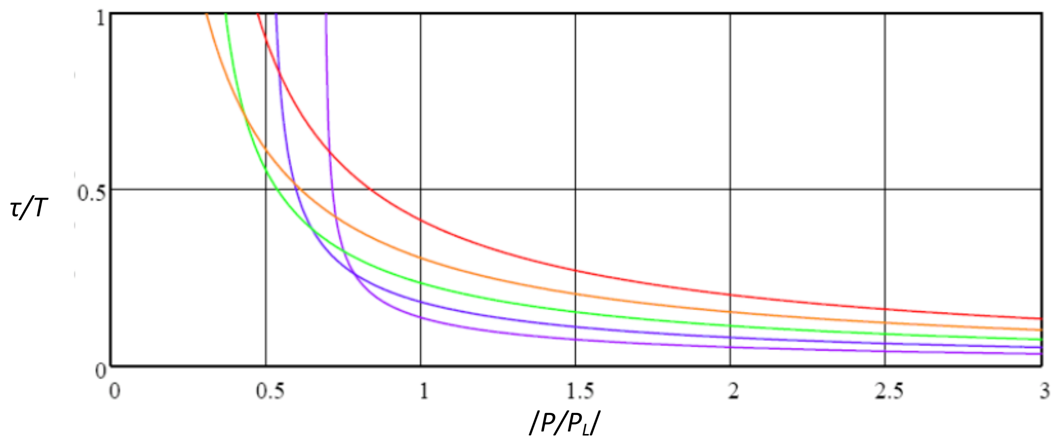

Figure 5. (Delay-time)/(water wave period) as a function of (water wave pressure amplitude P)/(tube pressurisation amplitude P_L). Water wave period $T = 1.12T_R$ (red), $0.98T_R$ (orange), $0.84T_R$ (green), $0.7T_R$ (blue), $0.56T_R$ (purple) .

As can be seen, the delay times τ decrease steadily with $/P/P_L/$, i.e. with the amplitude of the water wave, relative to the pressure head between the accumulators. This is to be expected from the top graph in Figure 3 – as the amplitude of the water wave increases, the required tube dynamic pressure amplification reduces, which implies shorter delay times.

The next two graphs in Figure 3 show that as the delay times decrease (and thus the distensibility-rate Δ rises) more and more of the tube's distension is from the PTO, rather than elastic distension. Eventually, at very large $/P/P_L/$, the water in the tube has nearly the same pressure as that in the water wave, and distends in the same way too, via the PTO.

7. Tube with pressure-limit PTO, in irregular waves

To analyse irregular waves, we can treat each individual wave (one period in duration) in an irregular sequence in a quasi-steady-state manner, i.e. treat it as a member of a regular wave of the same amplitude and period, in which the bulge wave has reached a steady state. This approximation will become progressively more accurate as the duration of the wave groups increases, i.e. as they become increasingly narrow-banded.

In this way we can obtain, from the known statistics of individual wave amplitudes (which are well-known to have a Rayleigh distribution, if narrow-banded) the performance of the PTO in irregular waves, and thus optimise the PTO pressure limit P_L. We can also optimise the economically-important energy storage time (27), and compare it with that for a linear PTO (over which a pressure-limit PTO has the advantage that its elastic energy storage is limited by P_L).

We first require the power output (26) (which is power relative to the power in the water wave, over the same cross-section, for a tube of length k_R^{-1}) with the delay-time τ set by (29), as in Section 4, so as to give a tube pressurisation P_L . This comes to:

$$W(/P/P_L/) = \frac{[\{/P/P_L/^2 - (T^2/T_R^{\,2} - 1)^2\}(T_R^{\,2}/T^2)^2]^{0.5}}{(T^2/T_R^{\,2} - 1)^2 + [\{/P/P_L/^2 - (T^2/T_R^{\,2} - 1)^2\}(T_R^{\,2}/T^2)^2](T^2/T_R^{\,2})^2} \qquad (43)$$

which is plotted in Figure 6 below

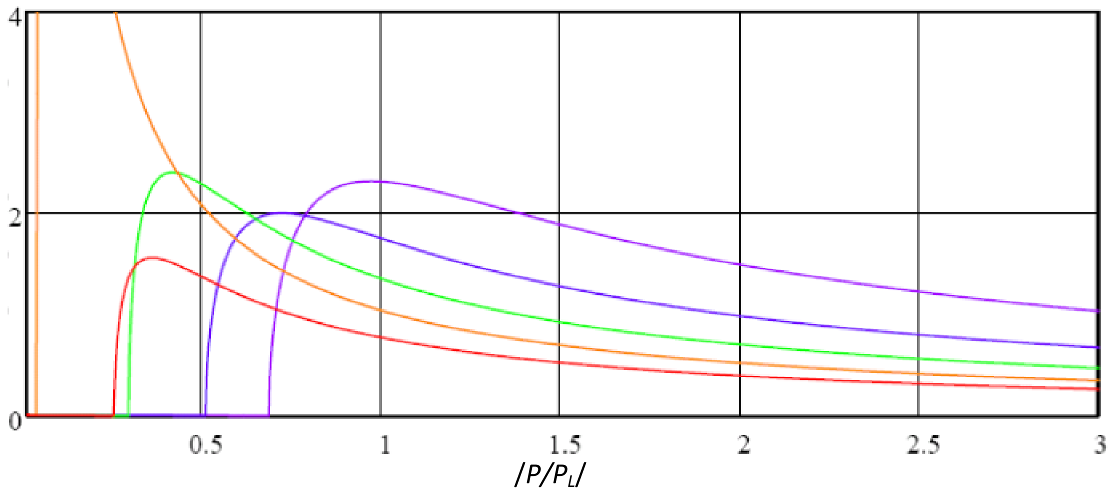

Figure 6. Power output (relative to power in water wave over tube cross-section) in a tube of length k_R^{-1}, as a function of (wave pressure *p*)/(PTO pressure p_L). Water wave period $T = 1.12T_R$ (red), $0.98T_R$ (orange), $0.84T_R$ (green), $0.7T_R$ (blue), $0.56T_R$ (purple), i.e. same colour code as Figure 5.

Except at resonance ($T = T_R$, orange case), it may be seen that there is a threshold value of $/P/P_L/$ below which no power is obtained, because there is insufficient dynamic pressure amplification available to reach the PTO pressure P_L, however small the PTO distension-rate (i.e. however long the delay-time). As the water wave pressure P increases above this threshold, the efficiency of the device reaches a peak, typically when $/P/$ is about half P_L.

In narrow-band irregular waves of root-mean-square pressure P_{RMS}, the amplitudes of the individual pressure cycles P_i have a Rayleigh probability distribution (see e.g. [4], p.…) whose probability density is:

$$(P_i /P_{RMS})exp[-(P_i/P_{RMS})^2/2] \qquad (44)$$

If we define a new datum as the time-averaged water wave power over the tube cross-sectional area, then the power in individual waves (assuming they reach a steady-state so that (43) applies, as discussed above), relative to this datum, will be:

$$½(P_i /P_{RMS})^2W(P_i /P_L) = ½(P_i/P_{RMS})^2W\{(P_i/P_{RMS})(P_{RMS}/P_L)\}. \qquad (45)$$

the factor of ½ arising because for a regular wave $P_{RMS} = P_i/\surd 2$. Since the individual waves all have the same period on our narrow-banded assumption, the time-averaged power relative to the new datum is:

$$\int ½(P_i/P_{RMS})^2W\{(P_i/P_{RMS})(P_{RMS}/P_L)\}(P_i/P_{RMS})exp[-(P_i/P_{RMS})^2/2]d(P_i/P_{RMS}) \qquad (46)$$

This is a function of (P_{RMS}/P_L) only, and is plotted in Figure 7 below.

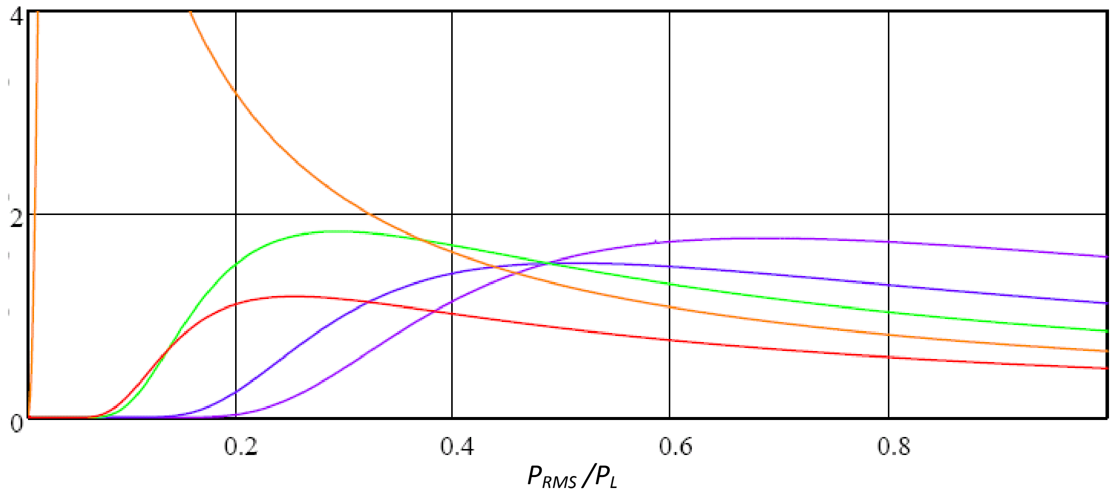

Figure 7. Power output (relative to average power in water waves over tube cross-section) in a tube of length k_R^{-1}, as a function of (RMS wave pressure P_{RMS})/(PTO pressure P_L). Water waves are narrow-banded irregular, of mean period $T = 1.12T_R$ (red), $0.98T_R$ (orange), $0.84T_R$ (green), $0.7T_R$ (blue), $0.56T_R$ (purple), i.e. same colour code as Figure 5.

It may be seen that the maximum power outputs have dropped compared with the regular-wave case in Figure 6, which is to be expected because we are now effectively averaging over many values of $/P/P_L/$. On the other hand the performance is remarkably consistent over the relevant range of

RMS wave pressures and periods. The Benbecula wave climate, for example, has an annual-average power of 4.57 kW/m^2 (see WERATLAS data [5]), with 91% of it between mean wave periods *Te* of 7 and 14 seconds (i.e. a 2:1 period ratio, like that plotted in the above Figures), and 89% of it between significant wave heights *Hs* of 2m and 10m. If we set the pressure limits $\pm p_L$ at 2.5m head of water for example (which give a practical working head of 5m for the turbine), then Figure 7 covers waves up to 2.5m RMS head, i.e. 10m *Hs*, and shows that the performance is sustained down to about 2m *Hs*. It can be sustained below this by reducing the pressure limits $\pm P_L$.

8. Tube dimensions

Overall, Figure 7 shows that the relative power output is approximately 1.5, i.e. 1.5×4.57kW annual average per square metre of tube cross-sectional area, for a tube of length k_R^{-1}. In the above figures $1.12T_R$ corresponds to 14s, so T_R = 14/1.12 = 12.5s, so $k_R^{-1} = g/(2\pi/12.5)^2$ = 38.8m. Thus a 600m long device of 5 m^2 water cross-section will have an annual-average power of 1.5×4.57×5×600/38.8 = 500 kW. The highest power (in 1,000 seastates) given in the WERATLAS data for Benbecula is 32.2 kW/m^2, but it has a long wave period, 14s. At this long period Figure 7 shows that the relative power output has dropped to 0.5, so the power from the device will be 350×(32.2/4.57)/2 = 1200 kW. The device thus has a remarkably small max-to-mean power ratio.

The resonant wave period T_R of 12.5s can conveniently be achieved by making the air compartment in Figure 2 one third of the tube cross-section, with a static pressurisation of 0.5 bar = 50 kPa (so absolute air pressure of 150 kPa), so as to give a comfortable margin over the ±2.5m head = ±0.25 bar pressure limits set by the accumulators. This means that a 1% increase in water cross-section requires a 2% reduction in air cross-section and thus a pressure rise of 150×0.02×1.4 bar = 4.2 kPa assuming adiabatic behaviour with $\gamma = 1.4$. This corresponds to a distensibility of 0.01/4.2 = 0.0024 kPa^{-1} and so from (7) a natural bulge wave speed $C^* = (\rho 0.0024)^{-0.5}$ = 20 m/s and thus a resonant wave period $T_R = 20\times 2\pi/g$ = 12.8s

Since the total tube cross-section including the air is 5×1.5 = 7.5 m^2, the tube diameter is $2\sqrt{(7.5/\pi)}$ = 3.1m. The depth of the tube axis below the still water line comes to 0.4m, and the width of the separating membrane comes to 3.0m. At the working pressure limits of ±0.25 bar, the volumetric change in the air is (0.25/1.5)/1.4 = 12%, which is 4% of the whole tube cross-section. Assuming a parabolic shape, the associated deflection of the separating membrane is thus 2×0.04×7.5/3.0 = 0.2m at its centre. The additional deflection when air passes to/from the accumulators is given by the PTO distension, i.e. the distension in the water part of the tube caused by PTO action. From Figure 3 this is limited to the distension in the water wave, which is $p/(\rho C^2) = \pi\times$height/wavelength, which is limited to about 0.3 by wave breaking. This is 0.3×5/7.5 = 20% of the area of the whole tube, and thus corresponds to an additional deflection of the separating membrane of 0.2×20/4 = 1m at its centre. All these figures appear to be acceptable.

As for flow losses, with the accumulator pressure difference of 50 kPa, the maximum power of 1200 kW corresponds to a flow rate of 1200/50 = 24 m^3/s. If we take the accumulators as 1m diameter, say, and place the turbine at the centre of the tube, then this is a flow speed of $24/(2\times\pi 1^2/4)$ = 15.3 m/s at the turbine, or more typically 10 m/s some distance from it. Given the air density of 1.5×0.001226 = 0.00184 $tonne/m^3$, this corresponds to a pressure drop of $½\times 0.00184\times 10^2$ = 0.1 kPa every 30 diameters or 30m. Since there are 600m of air tube in the circuit, this is a total pressure drop of 0.1×600/30 = 2 kPa, which is an acceptable 4% of the turbine pressure of 50 kPa. The stored energy in the accumulators would be the volumetric change in them produced by their pressurisation of ±0.25bar, which is $2\times 600\times\pi 1^2/4\times(0.25/1.5)/1.4$ = 112 m^2, times half their pressurisation of ±25 kPa, so 112×½×25 = 1400 KJ, or 3 seconds power output at the annual-average power of 500 kW. Given that the device is about 6 wavelengths long at the typical wavelength of

100m, which will give a considerable degree of spatial averaging, this appears to be acceptable

For a scale model, the fraction of water in the tube cross-section can be reduced from 2/3 to 1/3. This gives 4 times the distensibility and thus a scale factor of 4. In fact the static pressurisation can be reduced from 0.5 bar to 0.1 bar, which further increases the distensibility by a factor 1.5/1.1 and thus the scale factor to 5.5. This is a tube of diameter 3.1/5.5 = 0.56m, which is convenient experimentally. However, since the water cross-section in relation to the tube cross-section would be halved, the distortion of the separating membrane would also halve, so such a model would under-estimate this problem.

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