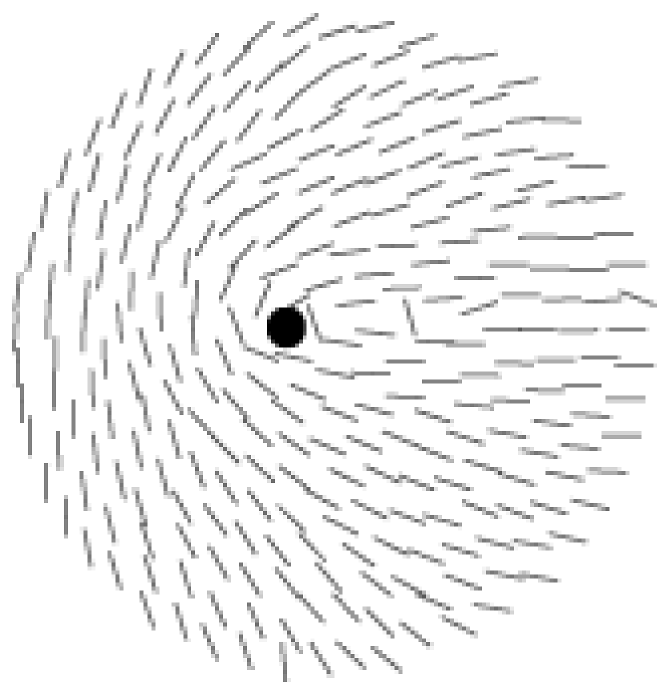
Defects in Nematic Liquid Crystals

Patricia Bauman
Department of Mathematics
Purdue University
West Lafayette, IN 47906

Introduction

Studies of thin films of nematic liquid crystals indicate topological defects of $\pm$ 1/2-degree disclination lines in sufficiently thin films, with point defect-boojoms at the surface in thicker films.



- Chiccoli, Feruli, Lavrentovich, Pasini, Shiyanovskii, and Zannoni: Topological defects in schlieren textures of biaxial and uniaxial nematics (2002).

  Did theoretical and Monte Carlo studies of flat thin 3D films of biaxial and uniaxial nematics with tangential boundary conditions to study structure and evolution of topological defects.

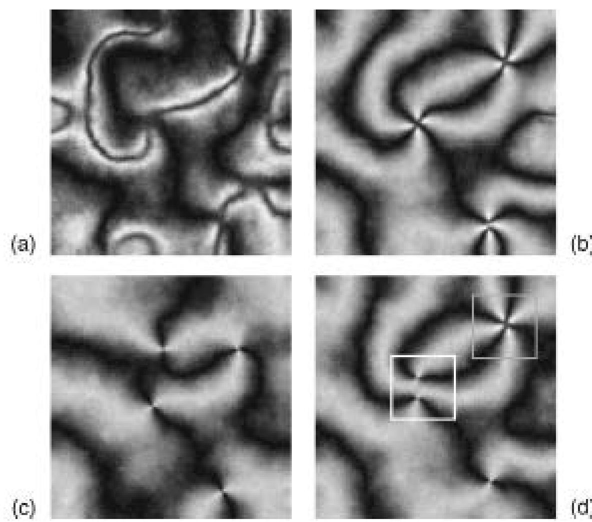


FIG. 2. [(a)-(d)] from top left to bottom right). Evolution of a BN texture ($\lambda=0.2$) for a $120\times120\times8$ lattice at $T^*=0.1$ and $J=1/2$ after 5 (a), 9 (b), 13 (c), and 60 (d) kcycles. Here we employ $n_x=1.54$, $n_y=1.51$, and $n_z=1.61$.

- Vitelli and Nelson(2006). ”Nematic textures in spherical shells”

  Showed formally that tangential fields with $\frac{1}{2}$ degree defects are stable for sufficiently thin films. Found that as the thickness increases, pairs of half-hedgehogs become energetically favorable.

- Ferandez-Nieves, Vitelli, Utada, Link, Marquez, Nelson and Weits (2007). Novel defect structures in nematic liquid crystal shells

  Did experiments on thin shells of nematic liquid crystals surrounding a double-emulsion droplet and observed $\frac{1}{2}$-degree defects for thin shells.

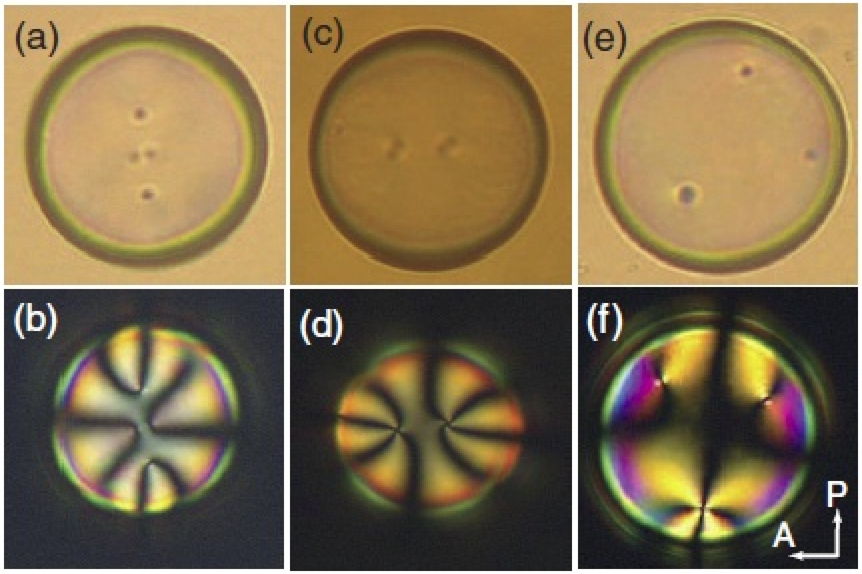


FIG. 3 (color online). Three type of thin shells commonly observed, distinguished by the number and type of defects: (a),(b) Four defects, (c),(d) two defects, and (e),(f) three defects. For all shells $2a = 103$ μm and $2R = 110$ μm.

- Kim, Shiyanskii, Lavrentovich

  Recent experiments on defects in thin films of lyotropic liquid crystals

QUESTION:

Do Landau-de Gennes models exhibit 1/2-degree disclination line defects as predicted for thin nematic films with appropriate boundary conditions?

Joint work with Jinhae Park and Dan Phillips

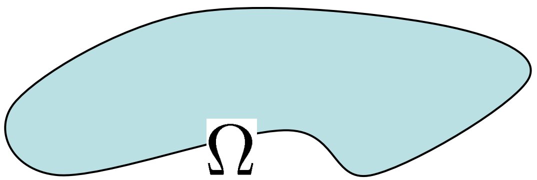
- Consider a class of equilibria used to describe thin nematic liquid crystals with disclination-line defects. The structure of the material is described by a minimizer for the Landau–de Gennes energy

$$F_\varepsilon(Q) = \int_\Omega [f_e(\nabla Q) + \varepsilon^{-2} f_b(Q)].$$

- Here Ω is a bounded simply connected smooth domain in $\mathbb{R}^2$ and $Q = Q(x, y)$ is a matrix-valued function in $W^{1,2}(\Omega; \mathcal{S})$, where

$$\mathcal{S} = \{Q \in \mathbb{R}^{3\times 3} \colon tr\ Q = 0, Q = Q^t\}.$$

- We will consider Dirichlet boundary conditions of nonzero degree on $\partial\Omega$.



Tensor Model for Liquid Crystals

- The Landau-de Gennes model is based on a phenomological theory in which a matrix $Q(\vec{x})$ in

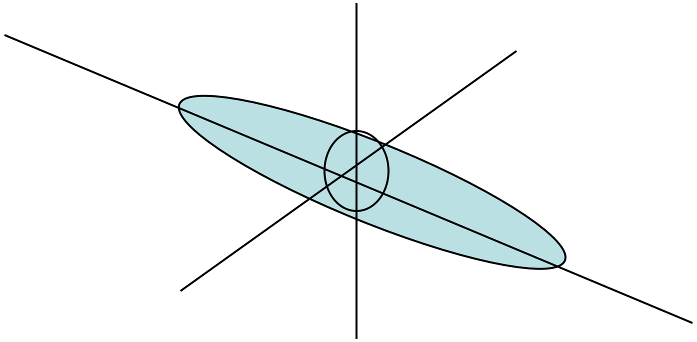
$$\mathcal{S} = \{Q \in \mathbb{R}^{3\times 3} \colon tr\ Q = 0, Q = Q^t\}.$$

  models the second moments of the average orientations of liquid crystal molecules near $\vec{x}$ through its principal axes (eigenvectors) and its eigenvalues.

- Matrices in $\mathcal{S}$ have an orthonormal set of eigenvectors and can be written as

$$Q = s_1 \mathbf{n} \otimes \mathbf{n} + s_2 \mathbf{k} \otimes \mathbf{k} - \frac{1}{3}(s_1 + s_2) I$$

  where $\mathbf{n}$ and $\mathbf{k}$ are orthogonal unit vectors in $\mathbb{R}^3$.



For Q in $\mathcal{S}$ we say that

- Q is isotropic if all its eigenvalues are equal. This occurs when $s_1 = s_2 = 0$.
- Q is uniaxial if exactly two of its eigenvalues are equal. This occurs when either $s_1 = 0$ and $s_2 \neq 0$, or $s_2 = 0$ and $s_1 \neq 0$, or $s_1 = s_2 \neq 0$.
- Q is biaxial when all eigenvalues are distinct. This occurs for all other values of s_1 ands_2.

- Here we consider

$$Q \in \mathcal{S}_0 = \{Q \in \mathcal{S} \colon Q_{31} = Q_{32} = 0\}, \text{i.e.}$$

$$Q = \begin{bmatrix} q_1 & q_2 & 0 \\ q_2 & q_3 & 0 \\ 0 & 0 & -q_1 - q_3 \end{bmatrix}$$

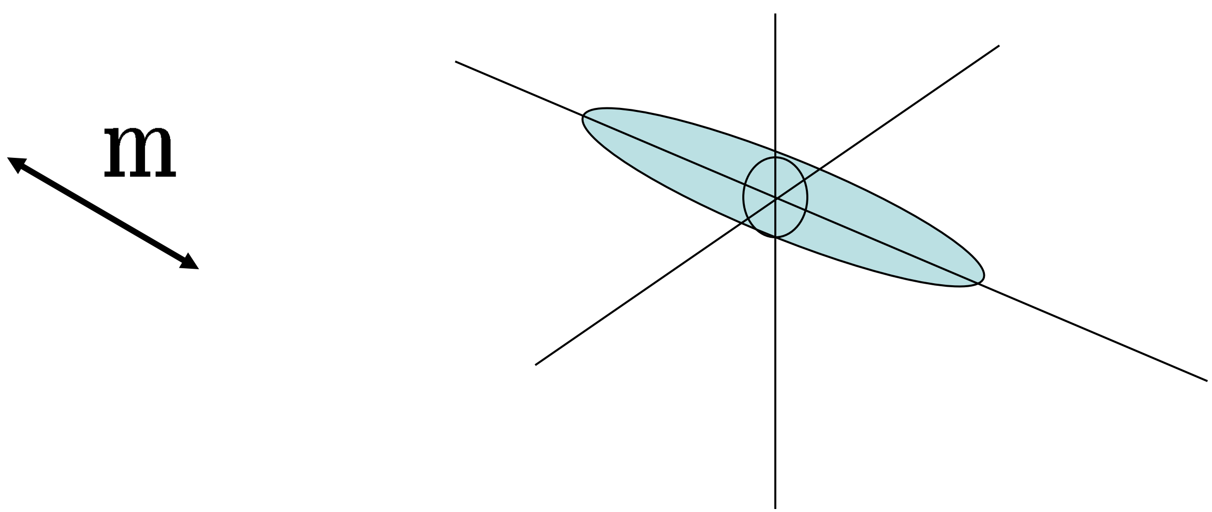
This corresponds to a thin film in which the top and bottom surfaces are treated so as to fix $\mathbf{e}_3$ as a principal axis (eigenvector of Q).

Each Q in $\mathcal{S}_0$ has eigenvectors: $\mathbf{m} = \langle m_1, m_2, 0\rangle$, $\mathbf{m}^\perp = \langle -m_2, m_1, 0\rangle$, $\mathbf{e}_3$

$$\text{and } Q = s_1 \mathbf{m} \otimes \mathbf{m} + s_2 \mathbf{m}^\perp \otimes \mathbf{m}^\perp - \frac{1}{3}\,(s_1 + s_2) I.$$

- An example of a uniaxial liquid crystal in this class is:

$$Q = s_1(\mathbf{m} \otimes \mathbf{m} - \tfrac{1}{3} I) \text{ with } s_1 \neq 0$$



Boundary Value Problem:

Recall:

$$F_\varepsilon(Q) = \int_\Omega [f_e(\nabla Q) + \varepsilon^{-2} f_b(Q)].$$

- The elastic energy density f_e is given by

$$\begin{aligned} f_e &= \frac{L_1}{2}\, Q_{ij,k}\, Q_{ij,k} + \frac{L_2}{2}\, Q_{ij,j}\, Q_{ik,k} \\ &+ \frac{L_3}{2}\, Q_{ij,k}\, Q_{ik,j}. \end{aligned}$$

We assume

$$L_1 > 0 \text{ and } L_1 + L_2 + L_3 > 0.$$

- The bulk energy density is such that $f_b = f_b(Q)$ is smooth, $f_b \geq 0$, and $f_b(Q) = 0$ for $Q \in \mathcal{S}$ if and only if

$$Q \in \Lambda_s = \{Q \in \mathcal{S}\colon Q = s(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}I) \text{ for some } \mathbf{m} \in \mathbb{S}^2\}.$$

  Here s is a fixed nonzero number. Thus the energy well is a set of uniaxial states.

- Liquid crystals satisfy the principle of frame indifference and are macroscopically isotropic. Hence:

$$f_b(RQR^t) = f_b(Q) \quad \text{for all } R \in O(3) \text{ and } Q \in \mathcal{S}$$

  and f_b depends only on the invariants of Q.

- We require: $f_b(Q)$ should satisfy the above properties and some additional structural assumptions which include the following classic example from the physics literature:

$$
\begin{aligned}
f_b^0(Q) &= \mathfrak{a}\, tr(Q^2) - \frac{2\mathfrak{b}}{3}\, tr(Q^3) + \frac{\mathfrak{c}}{2}(tr(Q^2))^2 + \mathfrak{d} \\
&= \mathfrak{a}(\sum_{i=1}^{3} \lambda_i^2) - \frac{2\mathfrak{b}}{3}(\sum_{i=1}^{3} \lambda_i^3) + \frac{\mathfrak{c}}{2}(\sum_{i=1}^{3} \lambda_i^2)^2 + \mathfrak{d}.
\end{aligned}
$$

where λ_i are the eigenvalues of Q.

Taking $\mathfrak{b}, \mathfrak{c} > 0$, $\mathfrak{a} < \frac{\mathfrak{b}^2}{27\mathfrak{c}}$, and an appropriate choice of $\mathfrak{d}$, we have $f_b^0 \geq 0$ and $f_b^0(Q) = 0$ if and only if $Q \in \Lambda_s$ where $s = \frac{1}{4\mathfrak{c}}(\mathfrak{b} + \sqrt{\mathfrak{b}^2 - 24\mathfrak{a}\mathfrak{c}})$.

- We consider fixed uniaxial nematic boundary conditions of the form:

$$Q_0(x,y) = s(\mathbf{n}_0(x,y) \otimes \mathbf{n}_0(x,y) - \frac{1}{3}\, I) \quad \text{for } (x,y) \in \partial\Omega,$$

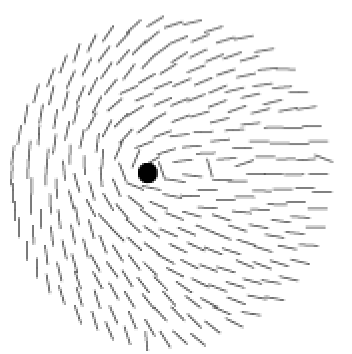
$$\text{with } Q_0 \in C^3(\partial\Omega) \text{ and } \mathbf{n}_0 \perp \mathbf{e}_3.$$

- We define the *degree* of Q_0 by

$$\frac{1}{2\pi} \int_0^1 \mathbf{n}_0(\gamma(t))^\perp \cdot \frac{d\mathbf{n}_0(\gamma(t))}{dt}\, dt \colon = \deg Q_0$$

where $\gamma : [0,1] \to \partial\Omega$ is a smooth parametrization and $\mathbf{n}_0(\gamma(t) \in C[0,1)$.

- Note: $\mathbf{n}_0(\gamma(0)) = \pm\mathbf{n}_0(\gamma(1))$ so $\deg Q_0 = k/2$ for some $k \in \mathbb{N}$. We assume $k > 0$.



Here $\deg Q_0 = \frac{1}{2}$

Our problem is to investigate minimizers for F_ε in $\mathcal{A}_0$, in the vanishing elastic energy limit, $\varepsilon \to 0$ where

$$\mathcal{A}_0 = \{Q(x,y) \in W^{1,2}(\Omega; \mathcal{S}_0); \quad Q = Q_0 \text{ on } \partial\Omega\}.$$

- Schopohl and Sluckin (1987). Pointed out that equilibria for F_ϵ valued in $\mathcal{S}_0$ are also equilibria for F_ϵ in $\mathcal{S}$.

Reformulation of the Problem:

- For $Q \in \mathcal{S}_0$, define functions $(\mathbf{p}, r) \equiv (p_1, p_2, r)$ in $W^{1.2}(\Omega)$ by:

$$Q = Q(p,r) = \begin{bmatrix} p_1 + \frac{r}{2} & p_2 & 0 \\ p_2 & \frac{r}{2} - p_1 & 0 \\ 0 & 0 & -r \end{bmatrix}$$

- On $\partial\Omega$ we have $Q(\mathbf{p}_0, r_0) = Q_0 \equiv s((\mathbf{n}_0(x,y) \otimes \mathbf{n}_0(x,y) - \frac{1}{3}\ I)$. Thus

$$Q_0(x,y) = s \begin{bmatrix} n_{01}^2 - \frac{1}{3} & n_{01}n_{02} & 0 \\ n_{01}n_{02} & n_{02}^2 - \frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} p_{01} + \frac{r_0}{2} & p_{02} & 0 \\ p_{02} & \frac{r_0}{2} - p_{01} & 0 \\ 0 & 0 & -r_0 \end{bmatrix}$$

If we write $\mathbf{n}_0(\gamma(t)) = \langle \cos\theta(t), \sin\theta(t), 0 \rangle$, we get

$$\mathbf{p}_0 = \frac{s}{2} \langle \cos 2\theta(t), \sin 2\theta(t), 0 \rangle, r_0 = \frac{s}{3}.$$

Thus $\deg\, \mathbf{p}_0 = k$ and $|\mathbf{p}_0| = \frac{|s|}{2}$.

Recall: f_b depends only on the invariants of Q.

Since $trQ = 0$, these are det $Q = (|\mathbf{p}|^2 - \frac{r^2}{4})r$ and $|Q|^2 = 2|\mathbf{p}|^2 + \frac{3}{2}\, r^2$.

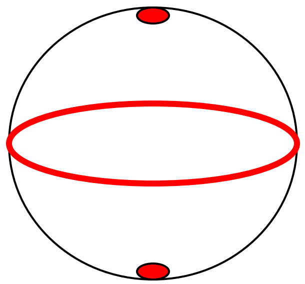
Thus $f_b(Q) = g_b(|\mathbf{p}|^2, r)$, e.g.

$$\begin{aligned}
f_b^0(Q) &= \mathfrak{a}\ tr(Q^2) - \frac{2\mathfrak{b}}{3}\ tr(Q^3) + \frac{\mathfrak{c}}{2}(tr(Q^2))^2 + \mathfrak{d} \\
&= \mathfrak{a}(\sum_{i=1}^{3} \lambda_i^2) - \frac{2\mathfrak{b}}{3}(\sum_{i=1}^{3} \lambda_i^3) + \frac{\mathfrak{c}}{2}(\sum_{i=1}^{3} \lambda_i^2)^2 + \mathfrak{d} \\
&= \mathfrak{a}(2|\mathbf{p}|^2 + \frac{3}{2}r^2) - 2\mathfrak{b}r(|\mathbf{p}|^2 - \frac{r^2}{4}) \\
&+ \frac{\mathfrak{c}}{2}(2|\mathbf{p}|^2 + \frac{3}{2}r^2)^2 + \mathfrak{d} \\
&=: g_b^0(|\mathbf{p}|^2, r).
\end{aligned}$$

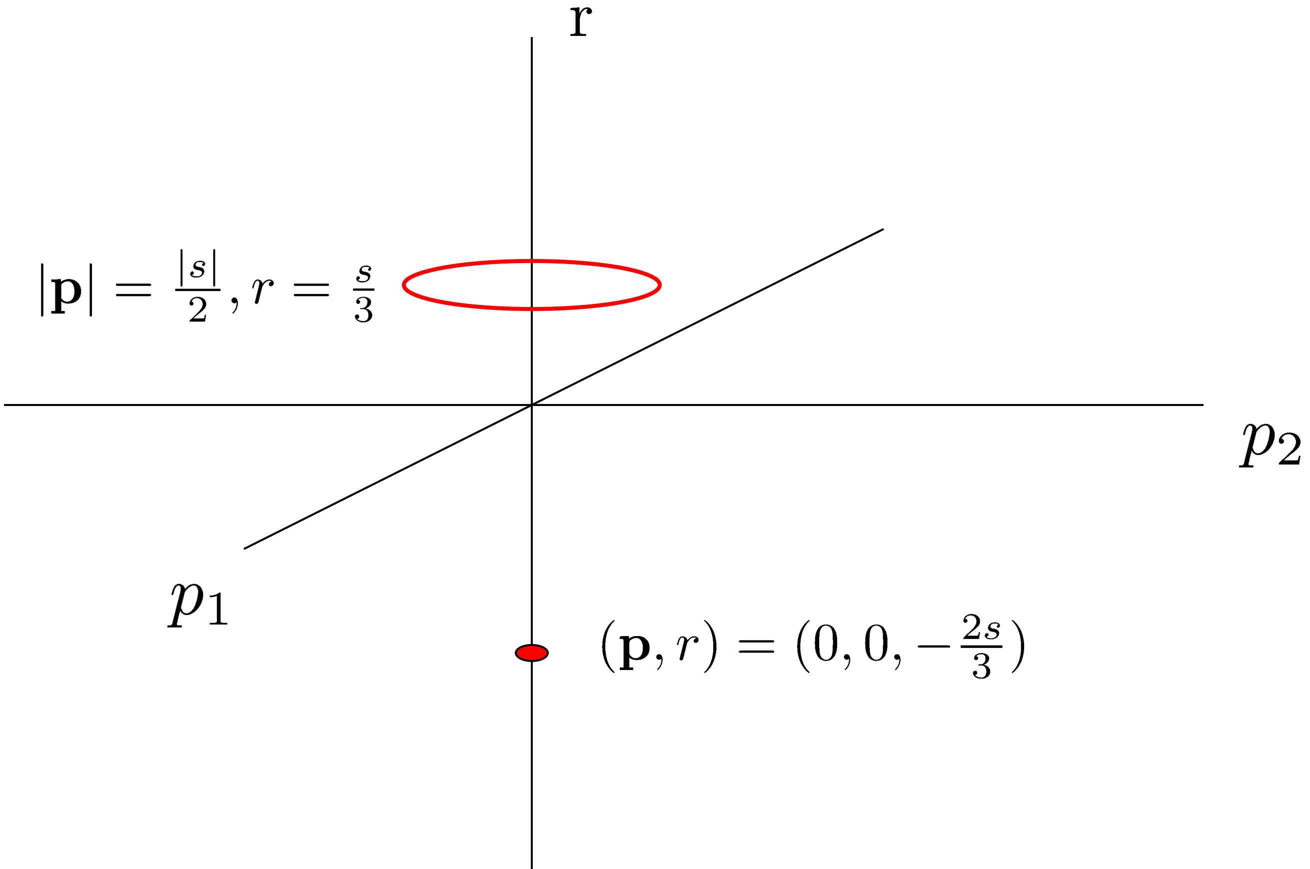
Note that for Q in $\mathcal{S}_0$:

$f_b(Q)$ minimizes at

$$\Lambda_s \cap \mathcal{S}_0 = \{s((\mathbf{m} \otimes \mathbf{m}) - \tfrac{1}{3}I) : \mathbf{m} \in [S^1 \times \{0\}] \cup \{\mathbf{e}_3\}\}$$

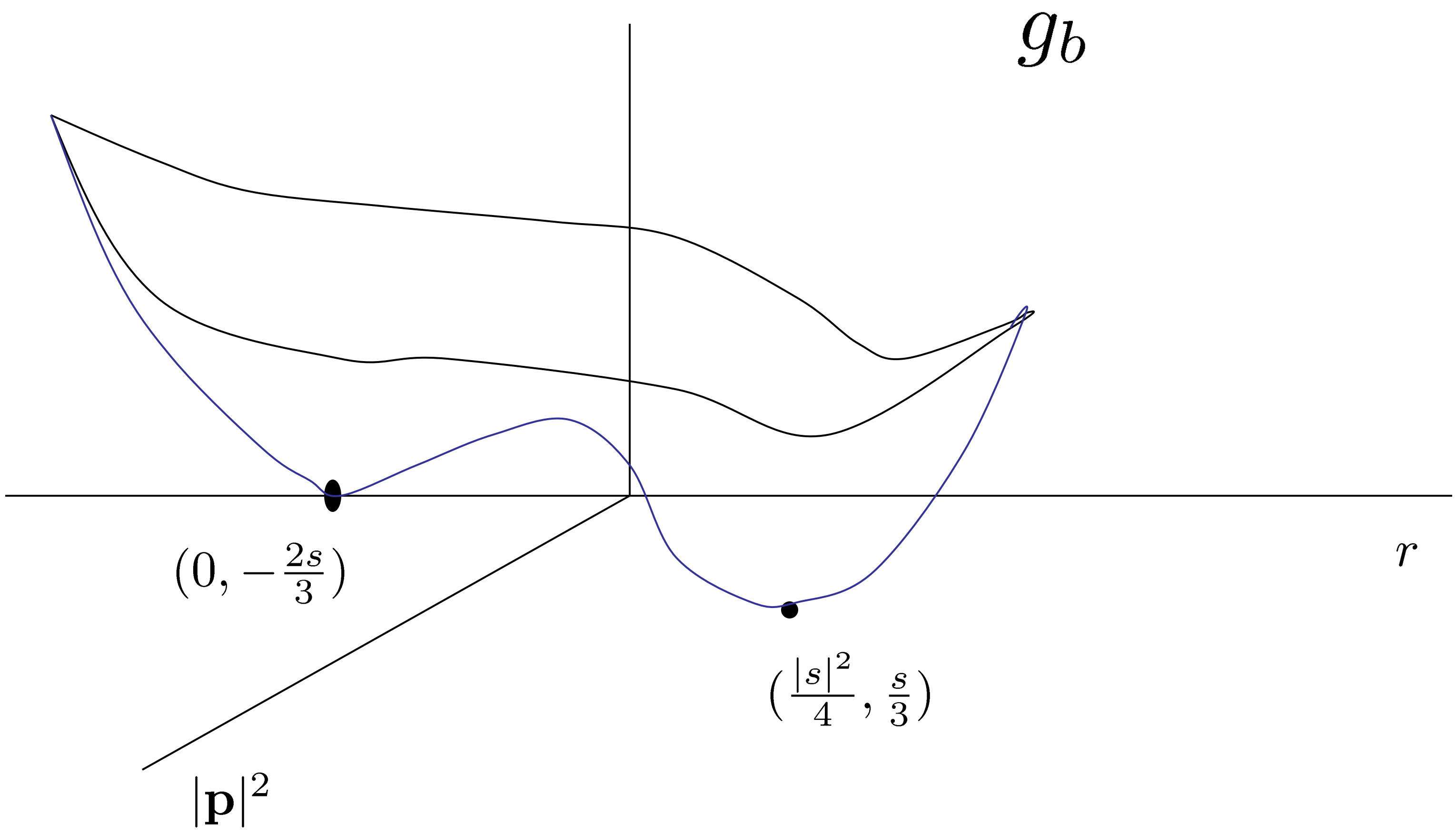


This corresponds to: $g_b(\mathbf{p}, r)$ minimizes at



We assume the following structural conditions for $g_b(\mathfrak{p}, \mathfrak{r}) \equiv g_b(|\mathbf{p}|^2, r)$:

$$\begin{cases} \text{i) } g_b(\mathfrak{p}, \mathfrak{r}) \in C^\infty([0,\infty) \times \mathbb{R}), g_b \geq 0 \text{ and } g_b(\frac{s^2}{4}, \frac{s}{3}) = 0, \\ \\ \text{ii) For some } m_1, m_2, m_3 > 0 \\ \quad |g_{b,\mathfrak{p}}(|\mathbf{p}|^2, r)||\mathbf{p}| + |g_{b,\mathfrak{r}}(|\mathbf{p}|^2, r)| \leq m_1(|\mathbf{p}|^3 + |r|^3) + m_2, \\ \qquad m_3(|\mathbf{p}|^4 + |r|^4) - 1 \leq g_b(|\mathbf{p}|^2, r), \\ \\ \text{iii) For some } \delta, m_4 > 0 \\ \quad m_4((|\mathbf{p}|^2 - \frac{s^2}{4})^2 + |r - \frac{s}{3}|^2) \leq g_b(|\mathbf{p}|^2, r) \\ \qquad\quad \text{for } ||\mathbf{p}| - \frac{|s|}{2}| + |r - \frac{s}{3}| < \delta. \end{cases}$$



g_b
$(0, -\frac{2s}{3})$
r
$(\frac{|s|^2}{4}, \frac{s}{3})$
$|\mathbf{p}|^2$

From the change of variables: $Q \in \mathcal{A}_0 \to (\mathbf{p}, r) \in A_0$, where

$$A_0 = \{(\mathbf{p}, r) \in W^{1,2}(\Omega; \mathbb{R}^2) \times W^{1,2}(\Omega) \colon \mathbf{p} = \mathbf{p}_0,\ r = \frac{s}{3} \text{ on } \partial\Omega\}$$

our problem can be reformulated as:

Minimize

$$G_\varepsilon(\mathbf{p}, r) = \int_\Omega [g_e(\nabla \mathbf{p}, \nabla r) + \varepsilon^{-2} g_b(|\mathbf{p}|^2, r)]$$

over A_0.

- Here g_e is given by:

  If $L_2 + L_3 \geq 0$,

$$\begin{aligned} g_e &= L_1(|\nabla \mathbf{p}|^2 + \frac{3}{4}|\nabla r|^2) \\ &+ \frac{(L_2 + L_3)}{2}((p_{1x} + \frac{r_x}{2} + p_{2y})^2 + (p_{2x} - p_{1y} + \frac{r_y}{2})^2). \end{aligned}$$

  If $0 > L_2 + L_3$,

$$\begin{aligned} g_e &= (L_1 + L_2 + L_3)(|\nabla \mathbf{p}|^2 + \frac{3}{4}|\nabla r|^2) \\ &- \frac{(L_2 + L_3)}{2}((\frac{r_x}{2} - p_{1x} - p_{2y})^2 + (p_{2x} - p_{1y} - \frac{r_y}{2})^2 + |\nabla r|^2). \end{aligned}$$

- It has the property that $g_e(\nabla \mathbf{p}, \nabla r) = f_e(\nabla Q(\mathbf{p}, r)) + f_e'(\nabla Q(\mathbf{p}, r))$ where f_e' is a null Lagrangian.

Main Results:

Theorem A. Let $\{(\mathbf{p}_j, r_j)\}$ be a sequence of minimizers for G_{ε_j} over A_0 such that $\varepsilon_j \downarrow 0$. Then for a subsequence $\{(\mathbf{p}_{j'}, r_{j'})\}$ there exists a harmonic function $h \in C^2(\overline{\Omega})$ and k points $\{a_1, \ldots, a_k\} \subset \Omega$ such that

$$(|\mathbf{p}_{j'(x)}|, r_{j'(x)}) \to (\frac{|s|}{2}, \frac{s}{3}) \text{ in } C_{loc}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\}),$$

$$(\mathbf{p}_{j'(x)}, r_{j'(x)}) \to (\mathbf{p}^*(x), r^*(x)) =: (\frac{|s|}{2} e^{i(h(x)+\sum_{\ell=1}^{k} \theta_\ell(x))}, \frac{s}{3})$$

in $W^{1,2}_{loc}(\overline{\Omega}\backslash\{a_1, \ldots, a_k\})$ and $C^m_{loc}(\Omega\backslash\{a_1, \ldots, a_k\})$ for all $m \geq 0$, where $e^{i\theta_\ell(x)} \simeq (x - a_\ell)/|x - a_\ell|$. Thus $\mathbf{p}_{j'}$ has degree 1 about each a_ℓ and

$$\mathbf{p}_{j'}(\mathbf{x}) = |\mathbf{p}_{j'}(\mathbf{x})| e^{i(h_{j'}(\mathbf{x})+\sum_{\ell=1}^{k} \theta_\ell(\mathbf{x}))} \text{ in } \overline{\Omega}\backslash \bigcup_{\ell=1}^{k} B_\rho(a_\ell) \ \equiv \overline{\Omega}_\rho$$

for all small $\rho > 0$, and j' sufficiently large where $h_{j'}(\mathbf{x}) \in C^2(\overline{\Omega}_\rho)$.

Theorem B. Let $\{(\mathbf{p}_j, r_j)\}$ be a sequence of minimizers for G_{ε_j} for which $(a_1, \ldots, a_k)$ is a limiting configuration of defects as $\varepsilon_j \downarrow 0$. Then

$$F_{\varepsilon_j}(Q_j) = G_{\varepsilon_j}(\mathbf{p}_j, r_j) - (L_3 - L_2 + |L_3 + L_2|)\frac{s^2 \pi k}{4}.$$

The points $(a_1, \ldots, a_k) = \mathbf{a}$ minimize the reduced energy $W(\mathbf{b})$ formulated by Bethuel-Brezis-Helein in Ginzburg-Landau theory, and we have

$$\begin{aligned}
\lim_{j\to\infty} & \quad [G_{\varepsilon_j}(\mathbf{p}_j, r_j) - \frac{(2L_1 + L_2 + L_3)s^2\pi k}{4} \ln \ \frac{1}{\varepsilon_j}] \\
= & \quad (2L_1 + L_2 + L_3)\frac{s^2}{4} W(\mathbf{a}) + k\gamma.
\end{aligned}$$

Here γ is a fixed constant associated to the energy of each defect core.

Here the reduced energy $W(\mathbf{b})$, defined for $\mathbf{b} = (b_1, \ldots, b_k)$ such that b_i are distinct points in Ω,is defined by:

$$W(\mathbf{b}) = -\pi \sum_{\ell \neq j} \ln |a_\ell - a_j| + \frac{1}{2} \int_{\partial\Omega} \Phi(g \times g_\tau) - \pi \sum_{i=1}^{k} R(a_i).$$

where $g(x) = \mathbf{p}_0(x)/|\mathbf{p}_0(x)|$ in $\partial\Omega$, Φ is the solution of:

$$\Delta\Phi = 2\pi\Sigma_{i=1}^{k}\delta_{a_i} \text{ in } \Omega,$$

$$\frac{\partial\Phi}{\partial\nu} = g \times g_\tau \text{ on } \partial\Omega,$$

$$\int_{\Omega} \Phi = 0,$$

and R is the harmonic function: $R(x) = \Phi(x) - \sum_{i=1}^{k} \ln |x - a_i|$.

Corollary A'. Let $\{Q_j \equiv Q(\mathbf{p}_j, r_j)\}$ be a sequence of minimizers for F_{ε_j} over $\mathcal{A}_0$ such that $\varepsilon_j \downarrow 0$. Then for a subsequence $\{Q_{j'}\}$ and the k points $\{a_1, \ldots, a_k\} \subset \Omega$, the functions $h_{j'}$ and the harmonic function $h(x)$ as above, we have

$$\begin{aligned} Q_{j'} &= s_{j'1}\mathbf{m}_{j'} \otimes \mathbf{m}_{j'} + s_{j'2}\mathbf{m}_{j'}^{\perp} \otimes \mathbf{m}_{j'}^{\perp} \\ &- \frac{1}{3}(s_{j'1} + s_{j'2})I \qquad \text{in } \Omega_\rho, \end{aligned}$$

where

$$\begin{aligned} \mathbf{m}_{j'} &= \langle \cos(\frac{1}{2}(h_{j'} + \sum_{\ell=1}^{k} \theta_\ell)), \sin(\frac{1}{2}(h_{j'} + \sum_{\ell=1}^{k} \theta_\ell)), 0 \rangle, \\ s_{j'1} &= |\mathbf{p}_{j'}| + \frac{3}{2} r_{j'}, \quad s_{j'2} = \frac{3r_{j'}}{2} - |\mathbf{p}_{j'}|. \end{aligned}$$

Thus $Q_{j'}$ converges to a uniaxial field away from $\{a_1, \ldots, a_k\}$, such that if $s > 0$ then

$$Q_{j'} \to s(\mathbf{m} \otimes \mathbf{m} - \frac{1}{3}\ I) \quad \text{in } \Omega \backslash \{a_1, \ldots, a_k\} \text{ as } j' \to \infty,$$

and if $s < 0$ then

$$Q_{j'} \to s(\mathbf{m}^\perp \otimes \mathbf{m}^\perp - \frac{1}{3}\ I) \quad \text{in } \Omega \backslash \{a_1, \ldots, a_k\} \text{ as } j' \to \infty.$$

Moreover

$$\mathbf{m} = \langle \cos(\frac{1}{2}\ (h(x) + \sum_{\ell=1}^{k} \theta_\ell)), \sin(\frac{1}{2}(h(x) + \sum_{i=1}^{k} \theta_\ell)), 0 \rangle$$

and the corresponding Q has degree $\frac{1}{2}$ about each a_ℓ.

Sketch of the Analysis

Difficulties:

A) $g_e(\nabla \mathbf{p}, \nabla r)$ is a coupled quadratic in three unknowns.

B) $g_b(|\mathbf{p}|^2, r)$ has a disconnected energy well.

C) No maximum principle.

Main Ideas

1.) Choose $g_e(\nabla \mathbf{p}, \nabla r)$ that differs from $f_e(\nabla Q(\mathbf{p}, r))$ by a null Lagrangian and satisfies our hypotheses:

Recall $L_1 > 0$, $L_1 + L_2 + L_3 > 0$ and

$$\begin{aligned} f_e(\nabla Q) &= \frac{L_1}{2}\, Q_{ij,k}\, Q_{ij,k} + \frac{L_2}{2}\, Q_{ij,j}\, Q_{ik,k} \\ &+ \frac{L_3}{2}\, Q_{ij,k}\, Q_{ik,j}. \end{aligned}$$

Thus if $L_2 + L_3 > 0$:

$$\begin{aligned} f_e(\nabla Q) - \frac{L_3}{2}(Q_{ij,k}Q_{ik,j} - Q_{ij,j}Q_{ik,k}) &= \frac{L_1}{2}|\nabla Q|^2 + \frac{(L_2 + L_3)}{2}|\mathrm{div}\ Q|^2 \\ &\equiv g_e(\nabla \mathbf{p}, \nabla r) \end{aligned}$$

for $Q = Q(p, r)$.

Similarly if $L_2 + L_3 < 0$ and Q is in $\mathcal{A}_0$:

$$\begin{aligned} f_e(\nabla Q) &+ \frac{(L_2+L_3)}{2}(Q_{ij,k}Q_{ik,j} - Q_{ij,j}Q_{ik,k}) \\ &= \frac{(L_1+L_2+L_3)}{2}|\nabla Q|^2 - \frac{(L_2+L_3)}{2}|\text{curl } Q|^2 \\ &= \frac{(L_1+L_2+L_3)}{2}|\nabla Q|^2 + \frac{|L_2+L_3|}{2}|\text{curl } Q|^2 \\ &\equiv g_e(\nabla \mathbf{p}, \nabla r) \end{aligned}$$

for $Q = Q(p, r)$.

2.) Asymptotic Estimates of minimizers

- $g_e(\nabla \mathbf{p}, \nabla r)$ is a positive definite quadratic operator with constant coefficients in $\nabla(\mathbf{p}_1, \mathbf{p}_2, r)$ and $g_e(\nabla \mathbf{p}, \nabla r) \geq c(|\nabla \mathbf{p}|^2 + |\nabla r|^2)$

- Minimizers $(\mathbf{p}_\varepsilon, r_\varepsilon)$ for G_ε in A_0 are classical solutions to the boundary value problem

$$\begin{cases} \mathcal{L}_1(\mathbf{p}, r) := -2L_1 \Delta p_1 - (L_2 + L_3)[\Delta p_1 + \frac{1}{2}(r_{xx} - r_{yy})] = -\frac{2p_1}{\varepsilon^2} g_{b,\mathfrak{p}} \\ \mathcal{L}_2(\mathbf{p}, r) := -2L_1 \Delta p_2 - (L_2 + L_3)[\Delta p_2 + r_{xy}] = -\frac{2p_2}{\varepsilon^2} g_{b,\mathfrak{p}} \\ \mathcal{L}_3(\mathbf{p}, r) := -\frac{3}{2} L_1 \Delta r - \frac{(L_2+L_3)}{2}[p_{1xx} - p_{1yy} + 2p_{2xy} + \frac{1}{2}\Delta r] = -\frac{1}{\varepsilon^2} g_{b,\mathfrak{r}} \end{cases} \quad \text{in } \Omega,$$

$$\text{and } r = \frac{s}{3}, \qquad \mathbf{p} = \mathbf{p}_0 \quad \text{on } \partial\Omega,$$

with $|\mathbf{p}_0| = \frac{|s|}{2}$ on $\partial\Omega$ and $\deg\left(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega\right) = k > 0$.

- Using the system of pde's, boundary conditions, and the structural assumptions of g_b, we prove for all $\varepsilon > 0$ sufficiently small:

  i)
  $$\varepsilon^{-2} \int_\Omega g_b |(\mathbf{p}_\varepsilon|^2, r_\varepsilon) \leq M$$

  where M depends on $s, L_1, L_2, L_3, \Omega, k, \|\mathbf{p}_0\|_{W^{1,2}(\partial\Omega)}$ and the constants in the structural assumptions on g_b.

  ii)
  $$|\mathbf{p}_\varepsilon|, |r_\varepsilon|, \varepsilon|\nabla \mathbf{p}_\varepsilon|, \varepsilon|\nabla r_\varepsilon| \leq C(M).$$

iii) $$\int_\Omega ((r_\varepsilon(x) - \frac{s}{3})^2 + (|\mathbf{p}_\varepsilon(x)|^2 - \frac{s^2}{4})^2) \leq C(M)\varepsilon^2.$$

iv) $$\int_\Omega |\nabla r_\varepsilon|^2 \leq C(M)$$

v) $$\frac{1}{2}\int_\Omega |\nabla \mathbf{p}_\varepsilon|^2 \leq \frac{s^2}{4}\ \pi\ k \ln\ \frac{1}{\varepsilon} + C$$

- Setting

$$J_\varepsilon(\mathbf{v}) = \int_\Omega \frac{1}{2}[|\nabla \mathbf{v}|^2 + \frac{1}{2\varepsilon^2}\ (\frac{s^2}{4} - |\mathbf{v}|^2)^2]$$

by iii) and v) we get

$$(*) \qquad J_\varepsilon(\mathbf{p}_\varepsilon) \leq \frac{s^2}{4}\ \pi\ k \ln\ \frac{1}{\varepsilon} + C$$

The fact that $s \neq 0$ and

$$\mathbf{p}_\varepsilon \in \{\mathbf{v} \in W^{1,2}(\Omega;\mathbb{R}^2)\colon \mathbf{v} = \mathbf{p}_0 \text{ on } \partial\Omega\} \text{ such that}$$
$$\mathbf{p}_0 \in C^3(\partial\Omega),\ |\mathbf{p}_0| = \frac{s}{2},\ \deg\,(\frac{\mathbf{p}_0}{|\mathbf{p}_0|}, \partial\Omega) = k > 0,$$
$$(*) \quad J_\varepsilon(\mathbf{p}_\varepsilon) \Box \pi\ \frac{s^2}{4}\ k \ln\ \frac{1}{\varepsilon} + K$$

for $0 < \varepsilon < \eta$ imply from techniques in Ginzburg-Landau theory (cf. Fangua Lin's structure theorem) that a subsequence of $\mathbf{p}_\varepsilon$ and r_ε converges to $\mathbf{p}*$ and s/3, and there exist points $\{a_1^\varepsilon, \ldots, a_k^\varepsilon\}$ converging to $\{a_1, \ldots, a_k\}$ with $|\mathbf{p}_\varepsilon| \geq \frac{|s|}{4}$ on ∂B_m and $\deg\,(\frac{\mathbf{p}_\varepsilon}{|\mathbf{p}_\varepsilon|}, \partial B_m) = 1$ where $B_m := B_{\varepsilon^{\alpha_m}}(a_m^\varepsilon)$.