GEOMETRIC THEORIES OF CONSERVATIVE LIQUID CRYSTAL DYNAMICS

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PLAN OF THE PRESENTATION

Liquid crystal dynamics: two models

Euler-Poincaré reduction

Affine Euler-Poincaré reduction

Euler-Poincaré formulation for liquid crystal dynamics

Eringen implies Ericksen-Leslie

All models are conservative: the terms modeling dissipation have been eliminated. The reason is that we want to understand the geometric nature of these equations. The dissipative terms can be added later. Think: Euler versus Navier-Stokes.

LIQUID CRYSTAL DYNAMICS

Director theory due to Oseen, Frank, Zöcher, Ericksen and Leslie

Micropolar and **microstretch theories**, due to Eringen, which take into account the microinertia of the particles and which is applicable, for example, to *liquid crystal polymers*

Ordered micropolar approach, due to Lhuillier and Rey, which combines the director theory with the micropolar models.

We discuss only nematic liquid crystals ($K_1 = 0$ in free energy; no chirality $\mathbf{n} \cdot \operatorname{curl} \mathbf{n}$). We set all dissipation equal to zero; want to understand the conservative case first.

$\mathcal{D} \subset \mathbb{R}^3$ bounded domain with smooth boundary. All boundary conditions are ignored: in all integration by parts the boundary terms vanish. We fix a volume form μ on $\mathcal{D}$.

ERICKSEN-LESLIE DIRECTOR THEORY
For nematic and cholesteric liquid crystals

Key assumption: only the direction and not the sense of the molecules matter. The preferred orientation of the molecules around a point is described by a unit vector $\mathbf{n} : \mathcal{D} \to S^2$, called the *director*, and $\mathbf{n}$ and $-\mathbf{n}$ are assumed to be equivalent.

Ericksen-Leslie equations (*Ericksen [1966], Leslie [1968]*) in a domain $\mathcal{D}$, constraint $\|\mathbf{n}\| = 1$, are:

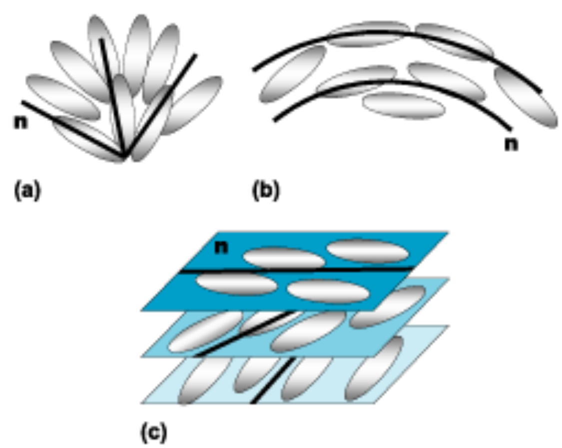
$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad} \dfrac{\partial F}{\partial \rho^{-1}} - \partial_j\left(\rho \dfrac{\partial F}{\partial \mathbf{n}_{,j}} \cdot \nabla \mathbf{n}\right), \\ \rho J \dfrac{D^2}{Dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0, \quad \mathbf{h} = \rho \dfrac{\partial F}{\partial \mathbf{n}} - \partial_i\left(\rho \dfrac{\partial F}{\partial \mathbf{n}_{,i}}\right) \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho \mathbf{u}) = 0, \qquad \dfrac{D}{Dt} := \dfrac{\partial}{\partial t} + \nabla_{\mathbf{u}} = \dfrac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \end{cases}$$

$\mathbf{u}$ *Eulerian velocity*, ρ *mass density*, $\mathbf{n} : \mathcal{D} \to \mathbb{R}^3$ *director* ($\mathbf{n}$ equivalent to -$\mathbf{n}$), J *microinertia constant*, and $F(\mathbf{n}, \mathbf{n}_{,i})$ is the *free energy*:

A standard choice for F is the *Oseen-Zöcher-Frank free energy*:

$$\rho F(\rho^{-1}, \mathbf{n}, \nabla \mathbf{n}) = \frac{1}{2} K_{11} \underbrace{(\operatorname{div} \mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2} K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2}_{\textbf{twist}} + \frac{1}{2} K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl} \mathbf{n}\|^2}_{\textbf{bend}},$$

associated to the basic type of director distorsions nematics:



(a) splay, (b) bend, (c) twist

WHAT IS THE VARIATIONAL/HAMILTONIAN STRUCTURE OF THESE EQUATIONS?

ERINGEN MICROPOLAR THEORY

First key assumption: Replace point particles by *small deformable bodies*: **microfluids**. Examples: *liquid crystals, blood, polymer melts, bubbly fluids, suspensions with deformable particles, biological fluids*. *Eringen [1978], [1979], [1981],...*

A material particle P in a microfluid is characterized by its position X and by a vector Ξ attached to P that denotes the orientation and intrinsic deformation of P. Both X and Ξ have their own motions: $X \mapsto x = \eta(X,t)$ and $\Xi \mapsto \xi = \chi(X,\Xi,t)$ called, respectively, the *macromotion* and *micromotion*.

Second key assumption: Material bodies are very small, so a linear approximation in Ξ is permissible for the micromotion:

$$\xi = \chi(X,t)\Xi,$$

where $\chi(X,t) \in \mathrm{GL}(3)^+ := \{A \in \mathrm{GL}(3) \mid \det(A) > 0\}$.

The classical Eringen theory considers only three possible groups in the description of the micromotion of the particles:

$$\mathsf{GL}(3)^{+}(\textit{micromorphic}) \supset \mathsf{K}(3)(\textit{microstretch}) \supset \mathsf{SO}(3)(\textit{micropolar}),$$

$$\mathsf{K}(3) = \left\{ A \in \mathsf{GL}(3)^{+} \mid \text{there exists } \lambda \in \mathbb{R} \text{ such that } AA^{T} = \lambda I_3 \right\}$$

is a closed subgroup of $\mathsf{GL}(3)^{+}$; associated to rotations and stretch.

The general theory admits other groups describing the micromotion.

We will study only micropolar fluids, i.e., the **order parameter group** is

$$\mathcal{O} := \mathsf{SO}(3)$$

.

Eringen's equations for non-dissipative micropolar liquid crystals:

$$\begin{cases} \rho\dfrac{D}{Dt}\mathbf{u}_l = \partial_l \dfrac{\partial\Psi}{\partial\rho^{-1}} - \partial_k\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^a}\boldsymbol{\gamma}_l^a\right), \quad \rho\sigma_l = \partial_k\left(\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}_k^l}\right) - \varepsilon_{lmn}\rho\dfrac{\partial\Psi}{\partial\boldsymbol{\gamma}_m^a}\boldsymbol{\gamma}_n^a, \\ \dfrac{D}{Dt}\rho + \rho\,\mathrm{div}\,\mathbf{u} = 0, \qquad \dfrac{D}{Dt}j_{kl} + (\varepsilon_{kpr}j_{lp} + \varepsilon_{lpr}j_{kp})\boldsymbol{\nu}_r = 0, \\ \dfrac{D}{Dt}\boldsymbol{\gamma}_l^a = \partial_l\boldsymbol{\nu}_a + \nu_{ab}\boldsymbol{\gamma}_l^b - \boldsymbol{\gamma}_r^a\partial_l u_r, \qquad \dfrac{D}{Dt} := \dfrac{\partial}{\partial t} + \mathbf{u}\cdot\nabla \quad \text{mat. deriv.} \end{cases}$$

sum on repeated indices, $\mathbf{u} \in \mathfrak{X}(\mathcal{D})$ Eulerian velocity, $\rho \in \mathcal{F}(\mathcal{D})$ mass density, $\boldsymbol{\nu} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$ microrotation rate, where we use the standard isomorphism between $\mathfrak{so}(3)$ and $\mathbb{R}^3$, $j_{kl} \in \mathcal{F}(\mathcal{D}, \mathrm{Sym}(3))$ microinertia tensor (symmetric), σ_k spin inertia is defined by

$$\sigma_k := j_{kl}\frac{D}{Dt}\boldsymbol{\nu}_l + \varepsilon_{klm}j_{mn}\boldsymbol{\nu}_l\boldsymbol{\nu}_n = \frac{D}{Dt}(j_{kl}\boldsymbol{\nu}_l),$$

$\gamma = (\gamma_i^{ab}) \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ wryness tensor, related to (η, χ) by

$$\gamma = -\eta_*(\nabla\chi)\chi^{-1} =: \widehat{\gamma} = (\widehat{\boldsymbol{\gamma}_i^a}),$$

and $\Psi = \Psi(\rho^{-1}, j, \gamma) : \mathbb{R} \times \mathrm{Sym}(3) \times \mathfrak{gl}(3) \to \mathbb{R}$ is the free energy.

WHAT IS THE VARIATIONAL/HAMILTONIAN STRUCTURE OF THESE EQUATIONS?

WHAT IS THE RELATION BETWEEN ERICKSEN-LESLIE AND ERINGEN THEORY?

Eringen's claim: Eringen theory recovers Ericksen-Leslie theory in the rod-like assumption $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ with the choice $\boldsymbol{\gamma} = \nabla \mathbf{n} \times \mathbf{n}$.

Once we have the Euler-Poincaré formulation, it will be clear that $\gamma = \nabla \mathbf{n} \times \mathbf{n}$ cannot be considered as a definition!

$\Diamond$ Other subsystem: G-strands; ongoing work with Gay-Balmaz and Holm. Constant j.

This statement has been controversial due to mistakes:

e.g. paper by Rymarz [1990]

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MORE ABOUT THE RELATIONS BETWEEN THE ERICKSEN–LESLIE–PARODI AND ERINGEN–LEE THEORIES OF NEMATIC LIQUID CRYSTALS

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Abstract—This paper is a further consideration of the relation between two main phenomenological theories of nematic liquid crystals: Ericksen–Leslie–Parodi (ELP) and Eringen–Lee (EL). The aim of the study is to establish the generality of the conclusion which claims that the ELP theory is a particular case of the EL theory.

According to the analysis presented in the paper this conclusion may be treated as true but only after modification of the constitutive equation for ${}_Em_{ij}$ in the EL theory by one term arising from splay deformation. The results of the study are formulated in four conclusions given at the end of the paper.

This was soon reconsidered in Eringen [1993]

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AN ASSESSMENT OF DIRECTOR AND MICROPOLAR THEORIES OF LIQUID CRYSTALS

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Abstract—It is shown that of the two prominent theories of liquid crystals, the Micropolar theory is more general than the Director theory. Under special assumptions, for liquid crystals having rigid rod-like molecular elements, the Micropolar theory reduces to the Director theory. The relationship between the two theories is established fully. An assessment is made of the limitations of both theories and on their domain of applicability.

more precisely

The present discussion is concerned with the relationship of OFEL and E-theories. Already discussions exist in this regard (e.g. [12] and [13]). Rymarz [12] has shown that OFEL-theory (he calls it ELP-theory) is a special case of E-theory, if the stress potential is modified by a splay term $K(\operatorname{div}\mathbf{n})^2$. Below I shall show that this splay term is already present in the E-theory. Consequently, his statement should be modified to: *The OFEL-theory is a special case of the E-theory*.

The major contributions of this article is not only in this correction, but in the display of relations between the two theories, critical examination of their physical foundations and domain of applicability of each theory in regard to liquid crystals that possess more complicated molecular structures. In particular, this paper should serve the following purposes:

Since then, this remains an open problem.

We solve this problem using techniques of geometric mechanics:

(1) we show under which assumptions, Eringen reduces to Ericksen-Leslie

(2) we establish the correct relation between γ and $\mathbf{n}$ under these assumptions.

$\rightsquigarrow$ Both Eringen and Rymarz are partially right!

EULER-POINCARÉ REDUCTION

Poincaré 1901: Left (right) invariant Lagrangian $L : TG \to \mathbb{R}$, $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$. For $g(t) \in G$, let $\xi(t) = g(t)^{-1}\dot{g}(t)$ $\left(\dot{g}(t)g(t)^{-1} \in \mathfrak{g}\right)$.

Then the following are equivalent:

(i) $g(t)$ satisfies the Euler-Lagrange equations for L on G.

(ii) The variational principle

$$\delta \int_a^b L(g(t), \dot{g}(t))dt = 0$$

holds, for variations with fixed endpoints.

(iii) The ***Euler-Poincaré equations*** hold: $\dfrac{d}{dt}\dfrac{\delta l}{\delta \xi} = \pm \mathrm{ad}^*_\xi \dfrac{\delta l}{\delta \xi}$.

(iv) The ***Euler-Poincaré variational principle***

$$\delta \int_a^b l(\xi(t))dt = 0$$

holds on $\mathfrak{g}$, for variations $\delta\xi = \dot{\eta} \pm [\xi, \eta]$, where $\eta(t)$ is an arbitrary path in $\mathfrak{g}$ that vanishes at the endpoints, i.e $\eta(a) = \eta(b) = 0$.

Geometry has led to analytic questions. I am not aware of any serious analysis results for such constrained variational principles.

Reconstruction

Solve the Euler-Lagrange equations for a left invariant $L : TG \to \mathbb{R}$

- Form $l := L|_{\mathfrak{g}} : \mathfrak{g} \to \mathbb{R}$
- Solve the Euler-Poincaré equations: $\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi}$, $\xi(0) = \xi_0$
- Solve linear equation with time dependent coefficients (quadrature): $\dot{g}(t) = g(t)\xi(t)$, $g(0) = e$
- For any $g_0 \in G$ the solution of the Euler-Lagrange equations is $V(t) = g_0 g(t)\xi(t)$ with initial condition $V(0) = g_0\xi_0$.

EP reduction: free rigid body, ideal fluids, KdV
EP reduction for semidirect products: heavy rigid body, compressible fluids, MHD, GFD. *Holm, Marsden, Ratiu [1998]*
Geometry of complex fluids. *Holm [2002]*

AFFINE EULER-POINCARÉ REDUCTION

Right G-representation on V, $(v,g) \in V \times G \mapsto vg \in V$, induces:

- right G-representation on V^*: $(a,g) \in V^* \times G \mapsto ag \in V^*$
- right $\mathfrak{g}$-representation on V: $(v,\xi) \in V \times \mathfrak{g} \mapsto v\xi \in V$
- right $\mathfrak{g}$-representation on V^*: $(a,\xi) \in V^* \times \mathfrak{g} \mapsto a\xi \in V^*$

Duality pairings: $\langle\, , \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ and $\langle\, , \rangle_V : V^* \times V \to \mathbb{R}$

Affine *right* representation: $\theta_g(a) = ag + c(g)$, where $c \in \mathcal{F}(G, V^*)$ is a right group one-cocycle, i.e., $c(fg) = c(f)g + c(g)$, $\forall f, g \in G$. This implies that $c(e) = 0$ and $c(g^{-1}) = -c(g)g^{-1}$. Note that

$$\left.\frac{d}{dt}\right|_{t=0} \theta_{\exp(t\xi)}(a) = a\xi + \mathbf{d}c(\xi), \quad \xi \in \mathfrak{g}, \quad a \in V^*,$$

where $\mathbf{d}c : \mathfrak{g} \to V^*$ is defined by $\mathbf{d}c(\xi) := T_e c(\xi)$. Useful to introduce:

- $\mathbf{d}c^{\mathsf{T}} : V \to \mathfrak{g}^*$ by $\langle \mathbf{d}c^{\mathsf{T}}(v), \xi \rangle_{\mathfrak{g}} := \langle \mathbf{d}c(\xi), v \rangle_V$, for $\xi \in \mathfrak{g}$, $v \in V$

- $\diamond : V \times V^* \to \mathfrak{g}^*$ by $\langle v \diamond a, \xi \rangle_{\mathfrak{g}} := -\langle a\xi, v \rangle_V$ for $\xi \in \mathfrak{g}$, $v \in V$, $a \in V^*$

- then: $\langle a\xi + \mathbf{d}c(\xi), v \rangle_V = \langle \mathbf{d}c^{\mathsf{T}}(v) - v \diamond a, \xi \rangle_{\mathfrak{g}}$

- the semidirect product $S = G \circledS V$ with group multiplication

$$(g_1, v_1)(g_2, v_2) := (g_1 g_2, v_2 + v_1 g_2), \quad g_i \in G, \quad v_i \in V$$

- its Lie algebra $\mathfrak{s} = \mathfrak{g} \circledS V$ with bracket

$$\mathrm{ad}_{(\xi_1, v_1)}(\xi_2, v_2) := [(\xi_1, v_1), (\xi_2, v_2)] = ([\xi_1, \xi_2], v_1\xi_2 - v_2\xi_1)$$

- then for $(\xi, v) \in \mathfrak{s}$ and $(\mu, a) \in \mathfrak{s}^* = \mathfrak{g}^* \times V^*$ we have

$$\mathrm{ad}^*_{(\xi, v)}(\mu, a) = (\mathrm{ad}^*_\xi \mu + v \diamond a, a\xi)$$

In a physical problem (like liquid crystals) we are given:

• $L : TG \times V^* \to \mathbb{R}$ right G-invariant under the action $(v_h, a) \in T_hG \times V^* \stackrel{g}{\longmapsto} (v_h g, \theta_g(a)) = (v_h g, ag + c(g)) \in T_{hg}G \times V^*$.

• So, if $a_0 \in V^*$, define $L_{a_0} : TG \to \mathbb{R}$ by $L_{a_0}(v_g) := L(v_g, a_0)$. Then L_{a_0} is right invariant under the lift to TG of right translation of $G^c_{a_0}$ on G, where $G^c_{a_0}$ is the θ-isotropy group of a_0.

• Right G-invariance of L permits us to define $l : \mathfrak{g} \times V^* \to \mathbb{R}$ by

$$l(v_g g^{-1}, \theta_{g^{-1}}(a_0)) = L(v_g, a_0).$$

• Curve $g(t) \in G$, let $\xi(t) := \dot{g}(t)g(t)^{-1} \in \mathfrak{g}$, $a(t) = \theta_{g(t)^{-1}}(a_0) \in V^*$ Then $a(t)$ as the unique solution of the following affine differential equation with time dependent coefficients

$$\dot{a}(t) = -a(t)\xi(t) - \mathbf{d}c(\xi(t)),$$

with initial condition $a(0) = a_0 \in V^*$.

The following are equivalent:

(i) With a_0 held fixed, Hamilton's variational principle

$$\delta \int_{t_1}^{t_2} L_{a_0}(g(t), \dot{g}(t))dt = 0,$$

holds, for variations $\delta g(t)$ of $g(t)$ vanishing at the endpoints.

(ii) $g(t)$ satisfies the Euler-Lagrange equations for L_{a_0} on G.

(iii) The constrained variational principle

$$\delta \int_{t_1}^{t_2} l(\xi(t), a(t))dt = 0,$$

holds on $\mathfrak{g} \times V^*$, upon using variations of the form

$$\delta\xi = \frac{\partial \eta}{\partial t} - [\xi, \eta], \quad \delta a = -a\eta - \mathbf{d}c(\eta),$$

where $\eta(t) \in \mathfrak{g}$ vanishes at the endpoints.

(iv) The affine Euler-Poincaré equations hold on $\mathfrak{g} \times V^*$:

$$\frac{\partial}{\partial t}\frac{\delta l}{\delta \xi} = -\,\mathrm{ad}^*_\xi \frac{\delta l}{\delta \xi} + \frac{\delta l}{\delta a} \diamond a - \mathbf{d}c^{\mathsf{T}}\left(\frac{\delta l}{\delta a}\right).$$

Lagrangian Approach to Continuum Theories of Perfect Complex Fluids

To apply the previous theorem to complex fluids one makes two key observations:

1. Complex fluids have internal degrees of freedom encoded by the order parameter Lie group $\mathcal{O}$

2. New kind of advection equation: $\dfrac{D}{Dt}\boldsymbol{\gamma}_l^a = \partial_l \boldsymbol{\nu}_a + \nu_{ab}\boldsymbol{\gamma}_l^b - \boldsymbol{\gamma}_r^a \partial_l u_r$

Geometrically, this means:

1. Enlarge the "particle relabeling group" Diff($\mathcal{D}$) to the semidirect product $G =$ Diff($\mathcal{D}$) Ⓢ $\mathcal{F}(\mathcal{D}, \mathcal{O})$, $\mathcal{F}(\mathcal{D}, \mathcal{O}) := \{\chi : \mathcal{D} \to \mathcal{O}$ smooth$\}$

2. The usual advection equations (for the mass density, the entropy, the magnetic field, etc) need to be augmented by a new advected quantity on which the group G acts by an *affine representation*.

Algebraic structure of the symmetry group of complex fluids:

Diff($\mathcal{D}$) acts on $\mathcal{F}(\mathcal{D}, \mathcal{O})$ via the *right* action

$$(\eta, \chi) \in \text{Diff}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathcal{O}) \mapsto \chi \circ \eta \in \mathcal{F}(\mathcal{D}, \mathcal{O}).$$

Therefore, the group multiplication is given by

$$(\eta, \chi)(\varphi, \psi) = (\eta \circ \varphi, (\chi \circ \varphi)\psi).$$

Fix a volume form μ on $\mathcal{D}$, so identify densities with functions, one-form densitities with one-forms, etc.

The Lie algebra $\mathfrak{g}$ of the semidirect product group is

$$\mathfrak{g} = \mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}) \ni (\mathbf{u}, \boldsymbol{\nu}),$$

and the Lie bracket is computed to be

$$\text{ad}_{(\mathbf{u}, \nu)}(\mathbf{v}, \zeta) = (\text{ad}_{\mathbf{u}} \mathbf{v}, \text{ad}_{\nu} \zeta + \mathbf{d}\nu \cdot \mathbf{v} - \mathbf{d}\zeta \cdot \mathbf{u}),$$

where $\text{ad}_{\mathbf{u}} \mathbf{v} = -[\mathbf{u}, \mathbf{v}]$, $\text{ad}_{\nu} \zeta \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\text{ad}_{\nu} \zeta(x) := \text{ad}_{\nu(x)} \zeta(x)$, and $\mathbf{d}\nu \cdot \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathfrak{o})$ is given by $\mathbf{d}\nu \cdot \mathbf{v}(x) := \mathbf{d}\nu(x)(\mathbf{v}(x))$.

The dual Lie algebra is identified with

$$\mathfrak{g}^* = \Omega^1(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathfrak{o}^*) \ni (\mathbf{m}, \kappa),$$

through the pairing

$$\langle (\mathbf{m}, \kappa), (\mathbf{u}, \nu) \rangle = \int_{\mathcal{D}} (\mathbf{m} \cdot \mathbf{u} + \kappa \cdot \nu)\, \mu.$$

The dual map to $\mathrm{ad}_{(\mathbf{u},\nu)}$ is

$$\mathrm{ad}^*_{(\mathbf{u},\nu)}(\mathbf{m}, \kappa) = \Big(\pounds_{\mathbf{u}} \mathbf{m} + (\operatorname{div} \mathbf{u})\mathbf{m} + \kappa \cdot \mathbf{d}\nu, \mathrm{ad}^*_\nu \kappa + \operatorname{div}(\mathbf{u}\kappa) \Big).$$

Explanation of the symbols:

- $\kappa \cdot \mathbf{d}\nu \in \Omega^1(\mathcal{D})$ denotes the one-form defined by

$$\Big(\kappa \cdot \mathbf{d}\nu\Big)(v_x) := \kappa(x)(\mathbf{d}\nu(v_x))$$

- $\mathrm{ad}^*_\nu \kappa \in \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ denotes the $\mathfrak{o}^*$-valued mapping defined by

$$\Big(\mathrm{ad}^*_\nu \kappa\Big)(x) := \mathrm{ad}^*_{\nu(x)}(\kappa(x)).$$

- $\mathbf{u}\kappa$ is the 1-contravariant tensor field with values in $\mathfrak{o}^*$ defined by

$$(\mathbf{u}\kappa)(\alpha_x) := \alpha_x(\mathbf{u}(x))\kappa(x) \in \mathfrak{o}^*.$$

So $\mathbf{u}\kappa$ is a generalization of the notion of a vector field. $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ denotes the space of all $\mathfrak{o}^*$-valued 1-contravariant tensor fields.

• div$(\mathbf{u})$ denotes the divergence of the vector field $\mathbf{u}$ with respect to the fixed volume form μ. Recall that it is defined by the condition

$$(\operatorname{div} \mathbf{u})\mu = \boldsymbol{\pounds}_{\mathbf{u}}\mu.$$

This operator can be naturally extended to the space $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ as follows. For $w \in \mathfrak{X}(\mathcal{D}, \mathfrak{o}^*)$ we write $w = w_a \varepsilon^a$ where (ε^a) is a basis of $\mathfrak{o}^*$ and $w_a \in \mathfrak{X}(\mathcal{D})$. We define div : $\mathfrak{X}(\mathcal{D}, \mathfrak{o}^*) \rightarrow \mathcal{F}(\mathcal{D}, \mathfrak{o}^*)$ by

$$\operatorname{div} w := (\operatorname{div} w_a)\varepsilon^a.$$

Note that if $w = \mathbf{u}\kappa$ we have

$$\operatorname{div}(\mathbf{u}\kappa) = \mathbf{d}\kappa \cdot \mathbf{u} + (\operatorname{div} \mathbf{u})\kappa.$$

Split the space of advected quantities in two: usual ones and new ones that involve affine actions and cocycles.

GEOMETRY OF THE ERICKSEN-LESLIE EQUATIONS

• Symmetry group: $G = \text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, SO(3)) \ni (\eta, \chi)$, macromotion and micromotion.

• Advected variables: $V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \mathbb{R}^3) \ni (\rho, \mathbf{n})$, mass density and director field.

• Representation of G on V^*:

$$(\rho, \mathbf{n}) \mapsto \left(J(\eta)(\rho \circ \eta), \chi^{-1}(\mathbf{n} \circ \eta) \right).$$

• Associated infinitesimal actions and diamond operations:

$\mathbf{n}\mathbf{u} = \nabla \mathbf{n} \cdot \mathbf{u}$, $\quad \mathbf{n}\boldsymbol{\nu} = \mathbf{n} \times \boldsymbol{\nu}$, $\quad \mathbf{m} \diamond_1 \mathbf{n} = -\nabla \mathbf{n}^\mathsf{T} \cdot \mathbf{m}$ and $\mathbf{m} \diamond_2 \mathbf{n} = \mathbf{n} \times \mathbf{m}$, where $\boldsymbol{\nu}, \mathbf{m}, \mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3)$.

• No cocycle.

- EP equations for $\Big(\mathrm{Diff}(\mathcal{D})\,\circledS\,\mathcal{F}(\mathcal{D},\mathrm{SO}(3))\Big)\,\circledS\,\Big(\mathcal{F}(\mathcal{D})\times\mathcal{F}(\mathcal{D},\mathbb{R}^3)\Big)$:

$$\begin{cases}\dfrac{\partial}{\partial t}\dfrac{\delta\ell}{\delta\mathbf{u}}=-\pounds_{\mathbf{u}}\dfrac{\delta\ell}{\delta\mathbf{u}}-\operatorname{div}\mathbf{u}\dfrac{\delta\ell}{\delta\mathbf{u}}-\dfrac{\delta l}{\delta\boldsymbol{\nu}}\cdot\mathbf{d}\boldsymbol{\nu}+\rho\,\mathbf{d}\dfrac{\delta\ell}{\delta\rho}-\left(\nabla\mathbf{n}^{\mathsf{T}}\cdot\dfrac{\delta\ell}{\delta\mathbf{n}}\right)^{\flat},\\ \dfrac{\partial}{\partial t}\dfrac{\delta\ell}{\delta\boldsymbol{\nu}}=\boldsymbol{\nu}\times\dfrac{\delta\ell}{\delta\boldsymbol{\nu}}-\operatorname{div}\left(\dfrac{\delta\ell}{\delta\boldsymbol{\nu}}\mathbf{u}\right)+\mathbf{n}\times\dfrac{\delta\ell}{\delta\mathbf{n}},\end{cases}$$

- The advection equations are:

$$\begin{cases}\dfrac{\partial}{\partial t}\rho+\operatorname{div}(\rho\mathbf{u})=0,\\ \dfrac{\partial}{\partial t}\mathbf{n}+\nabla\mathbf{n}\cdot\mathbf{u}+\mathbf{n}\times\boldsymbol{\nu}=0.\end{cases}$$

- Reduced Lagrangian for nematic and cholesteric liquid crystals:

$$\ell(\mathbf{u},\boldsymbol{\nu},\rho,\mathbf{n}):=\frac{1}{2}\int_{\mathcal{D}}\rho\|\mathbf{u}\|^2\mu+\frac{1}{2}\int_{\mathcal{D}}\rho J\|\boldsymbol{\nu}\|^2\mu-\int_{\mathcal{D}}\rho F(\rho^{-1},\mathbf{n},\nabla\mathbf{n})\mu.$$

- EP equations for this ℓ: yield

$$(motion) \quad \begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \text{grad}\,\dfrac{\partial F}{\partial \rho^{-1}} - \partial_i\left(\rho\dfrac{\partial F}{\partial \mathbf{n}_{,i}}\cdot\nabla\mathbf{n}\right), \\ \rho J\dfrac{D}{Dt}\boldsymbol{\nu} = \mathbf{h}\times\mathbf{n}, \end{cases}$$

$$(advection) \quad \begin{cases} \dfrac{\partial}{\partial t}\rho + \text{div}(\rho\mathbf{u}) = 0, \\ \dfrac{D}{Dt}\mathbf{n} = \boldsymbol{\nu}\times\mathbf{n}, \end{cases}$$

- Recovering the Ericksen-Leslie equations:

Observation: if $\boldsymbol{\nu}$ and $\mathbf{n}$ are solutions of the EP equations then:

(i) $\|\mathbf{n}_0\| = 1$ *implies* $\|\mathbf{n}\| = 1$ *for all time.*

(ii) $\frac{D}{Dt}(\mathbf{n}\cdot\boldsymbol{\nu}) = 0$. *Therefore,* $\mathbf{n}_0\cdot\boldsymbol{\nu}_0 = 0$ *implies* $\mathbf{n}\cdot\boldsymbol{\nu} = 0$ *for all time.*

(iii) *Suppose that* $\mathbf{n}_0 \cdot \boldsymbol{\nu}_0 = 0$ *and* $\|\mathbf{n}_0\| = 1$*. Then*

$$\frac{D}{Dt}\mathbf{n} = \boldsymbol{\nu} \times \mathbf{n} \quad \textit{becomes} \quad \boldsymbol{\nu} = \mathbf{n} \times \frac{D}{Dt}\mathbf{n}$$

and

$$\rho J \frac{D}{Dt}\boldsymbol{\nu} = \mathbf{h} \times \mathbf{n} \quad \textit{becomes} \quad \rho J \frac{D^2}{Dt^2}\mathbf{n} - 2q\mathbf{n} + \mathbf{h} = 0.$$

Therefore:

If $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ *is a solution of the Euler-Poincaré equations with initial conditions* $\mathbf{n}_0$ *and* $\boldsymbol{\nu}_0$ *satisfying* $\|\mathbf{n}_0\| = 1$ *and* $\mathbf{n}_0 \cdot \boldsymbol{\nu}_0 = 0$*, then* $(\mathbf{u}, \rho, \mathbf{n})$ *is a solution of the Ericksen-Leslie equations.*

Conversely:

if $(\mathbf{u}, \rho, \mathbf{n})$ is a solution of Ericksen-Leslie equations, define

$$\boldsymbol{\nu} := \mathbf{n} \times \frac{D}{Dt}\mathbf{n} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3).$$

Then, $(\mathbf{u}, \boldsymbol{\nu}, \rho, \mathbf{n})$ is a solution of the Euler-Poincaré equations.

GEOMETRY OF THE ERINGEN EQUATIONS

• Symmetry group: same group as before $G = \text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathcal{O})$.

• Advected variables: $V^* = \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3)) \ni (\rho, j, \gamma)$, mass density, microinertia tensor, strain.

• Representation: $(\eta, \chi) \in \text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \text{SO}(3))$ acts *linearly* on the advected quantities $(\rho, j) \in \mathcal{F}(\mathcal{D}) \times \mathcal{F}(\mathcal{D}, \text{Sym}(3))$, by

$$(\rho, j) \mapsto \left(J(\eta)(\rho \circ \eta), \chi^{\mathsf{T}}(j \circ \eta)\chi \right), \quad \chi^{\mathsf{T}} = \chi^{-1}.$$

• Affine representation: $(\eta, \chi) \in \text{Diff}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \text{SO}(3))$ acts on $\gamma \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$ by an *affine* representation

$$\gamma \mapsto \chi^{-1}(\eta^* \gamma)\chi + \chi^{-1} \nabla \chi.$$

Note that γ transforms as a connection.

- The reduced Lagrangian of Eringen's theory:

$$\ell : \left[\mathfrak{X}(\mathcal{D}) \circledS \mathcal{F}(\mathcal{D}, \mathbb{R}^3)\right] \circledS \left[\mathcal{F}(\mathcal{D}) \oplus \mathcal{F}(\mathcal{D}, \mathsf{Sym}(3)) \oplus \Omega^1(\mathcal{D}, \mathfrak{so}(3))\right] \to \mathbb{R}$$

$$\ell(\mathbf{u}, \boldsymbol{\nu}, \rho, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} \rho \|\mathbf{u}\|^2 \mu + \frac{1}{2}\int_{\mathcal{D}} \rho\,(j\boldsymbol{\nu}\cdot\boldsymbol{\nu})\,\mu - \int_{\mathcal{D}} \rho \Psi(\rho^{-1}, j, \gamma)\mu.$$

- The affine Euler-Poincaré equations for ℓ are:

$$\begin{cases} \rho\left(\dfrac{\partial}{\partial t}\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u}\right) = \operatorname{grad}\dfrac{\partial \Psi}{\partial \rho^{-1}} - \partial_k\left(\rho \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a_k}\boldsymbol{\gamma}^a\right), \\ j\dfrac{D}{Dt}\boldsymbol{\nu} - (j\boldsymbol{\nu}) \times \boldsymbol{\nu} = -\dfrac{1}{\rho}\operatorname{div}\left(\rho\dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}}\right) + \boldsymbol{\gamma}^a \times \dfrac{\partial \Psi}{\partial \boldsymbol{\gamma}^a}, \\ \dfrac{\partial}{\partial t}\rho + \operatorname{div}(\rho\mathbf{u}) = 0, \qquad \dfrac{D}{Dt}j + [j, \nu] = 0, \\ \dfrac{\partial}{\partial t}\gamma + \boldsymbol{\pounds}_{\mathbf{u}}\gamma + \mathbf{d}^\gamma \nu = 0, \quad \widehat{\boldsymbol{\nu}} = \nu \in \mathcal{F}(\mathcal{D}, \mathfrak{so}(3)), \end{cases}$$

where $\mathbf{d}^\gamma$ is the covariant γ-derivative defined by $\mathbf{d}^\gamma \nu(\mathbf{v}) := \mathbf{d}\nu(\mathbf{v}) + [\gamma(\mathbf{v}), \nu]$. This system recovers Eringen's equations.

The general affine Euler-Poincaré theory applied to many other complex fluids: spin chain, Yang-Mills MHD (classical and superfluid), Hall MHD, multivelocity superfluids (classical and superfluid), HBVK dynamics for superfluid ^{4}He, Volovik-Dotsenko spin glasses, microfluids, Lhuillier-Rey equations (see Gay-Balmaz & Ratiu [2009]).

Kelvin-Noether circulation theorem for micropolar liquid crystals

$$\frac{d}{dt}\oint_{C_t}\mathbf{u}^\flat = \oint_{C_t}\frac{\partial\Psi}{\partial j}\mathbf{d}j + \frac{\partial\Psi}{\partial\gamma}\mathbf{i}\_\mathbf{d}\gamma - \frac{1}{\rho}\operatorname{div}\left(\rho\frac{\partial\Psi}{\partial\gamma}\right)\gamma.$$

The γ-circulation formulated in $\mathbb{R}^3$

$$\frac{d}{dt}\oint_{C_t}\boldsymbol{\gamma} = \oint_{C_t}\boldsymbol{\nu}\times\boldsymbol{\gamma}$$

ERINGEN IMPLIES ERICKSEN-LESLIE

Physically, the Eringen equations should imply the Ericksen-Leslie equations. *Eringen [1993]* proposes

$$j := J(I_3 - \mathbf{n} \otimes \mathbf{n}), \qquad \gamma := \nabla \mathbf{n} \times \mathbf{n}$$

to pass from his equations to the Ericksen-Leslie equations. This is FALSE! Two arguments: brute force computation and symmetry considerations. So, one needs to do something else.

However, not all is wrong:

1. it is true that there is $\Psi(j, \gamma)$ such that

$$\Psi\left(J(I_3 - \mathbf{n} \otimes \mathbf{n}), \nabla \mathbf{n} \times \mathbf{n}\right) = F(\mathbf{n}, \nabla \mathbf{n}).$$

2. the definition $j := J(I_3 - \mathbf{n} \otimes \mathbf{n})$ is geometrically consistent.

WE SHALL USE THE TOOLS OF GEOMETRIC MECHANICS TO GIVE A DEFINITIVE ANSWER.

Note: For simplicity, we consider motionless nematics. The present approach easily generalizes to the flowing case.

STEP I: γ-formulation of Ericksen-Leslie

The material Lagrangian for nematic motionless liquid crystals $\mathcal{L} : T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3)) \to \mathbb{R}$, $\mathcal{D} \subset \mathbb{R}^3$, is thus given by

$$\mathcal{L}(\chi, \dot{\chi}) = \frac{1}{2} J \int_{\mathcal{D}} \|\dot{\chi}\mathbf{n}_0\|^2 \mu - \int_{\mathcal{D}} F(\chi\mathbf{n}_0, \nabla(\chi\mathbf{n}_0))\mu,$$

where, usually $\mathbf{n}_0 = \hat{\mathbf{z}}$, J is the microinertia constant, and F is the Oseen-Frank free energy:

$$F(\mathbf{n}, \nabla\mathbf{n}) = K_2 \underbrace{(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})}_{\textbf{chirality}} + \frac{1}{2} K_{11} \underbrace{(\operatorname{div}\mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2} K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl}\mathbf{n})^2}_{\textbf{twist}} + \frac{1}{2} K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl}\mathbf{n}\|^2}_{\textbf{bend}}.$$

IDEA: Apply two different EP reductions to this Lagrangian.

FIRST EULER-POINCARÉ REDUCTION FOR NEMATICS

Write $\mathcal{L}(\chi, \dot{\chi}) = L_{\mathbf{n}_0}(\chi, \dot{\chi})$, where the Lagrangian

$$L_{\mathbf{n}_0} : T\mathcal{F}(\mathcal{D}, SO(3)) \to \mathbb{R}$$

is invariant under the right action

$$(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$$

of $\psi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3))_{\mathbf{n}_0}$ (the G_{a_0} of the general theory). So get the reduced Euler-Poincaré Lagrangian

$$\ell_1(\boldsymbol{\nu}, \mathbf{n}) = \frac{1}{2} J \int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla \mathbf{n}) \mu,$$

$\widehat{\boldsymbol{\nu}} = \dot{\chi}\chi^{-1}$, $\mathbf{n} = \chi \mathbf{n}_0$. The Euler-Poinaré equations are

$$\begin{cases} \dfrac{d}{dt} \dfrac{\delta \ell_1}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \dfrac{\delta \ell_1}{\delta \boldsymbol{\nu}} + \mathbf{n} \times \dfrac{\delta \ell_1}{\delta \mathbf{n}} \\ \partial_t \mathbf{n} + \mathbf{n} \times \boldsymbol{\nu} = 0 \end{cases}$$

More explicitly, upon denoting $\mathbf{h} = -\delta\ell_1/\delta\mathbf{n}$, one has

$$\begin{cases} J\partial_t\boldsymbol{\nu} = \mathbf{h}\times\mathbf{n} \\ \partial_t\mathbf{n} + \mathbf{n}\times\boldsymbol{\nu} = 0, \end{cases}$$

which are the Ericksen-Leslie equations of nematodynamics if $\|\mathbf{n}_0\| = 1$ and $\boldsymbol{\nu}_0\cdot\mathbf{n}_0 = 0$:

$$J\frac{d^2\mathbf{n}}{dt^2} - 2\underbrace{\left(\mathbf{n}\cdot\mathbf{h} + J\,\mathbf{n}\cdot\frac{d^2\mathbf{n}}{dt^2}\right)}_{=q}\mathbf{n} + \mathbf{h} = 0.$$

SECOND EULER-POINCARÉ REDUCTION FOR NEMATICS

Start with the same Lagrangian. If $\mathbf{n}_0$ is constant, we can write

$$\begin{aligned}\mathcal{L}(\chi,\dot{\chi}) &= \frac{1}{2}J\int_{\mathcal{D}}\|\dot{\chi}\mathbf{n}_0\|^2\mu - \int_{\mathcal{D}}F(\chi\mathbf{n}_0,\nabla(\chi\mathbf{n}_0))\mu \\ &= \frac{1}{2}J\int_{\mathcal{D}}\|\dot{\chi}\mathbf{n}_0\|^2\mu - \int_{\mathcal{D}}F(\chi\mathbf{n}_0,(\nabla\chi)\,\chi^{-1}\cdot\chi\mathbf{n}_0))\mu,\end{aligned}$$

and we view $\mathcal{L}$ as

$$\mathcal{L}(\chi,\dot{\chi}) = L_{(\mathbf{n}_0,\gamma_0=0)}(\chi,\dot{\chi}).$$

This Lagrangian is invariant under the right action

$$(\chi,\mathbf{n}_0,\gamma_0)\mapsto\left(\chi\psi,\psi^{-1}\mathbf{n}_0,\psi^{-1}\gamma_0\psi+\psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D},SO(3))_{(\mathbf{n}_0,0)} = \mathcal{F}(\mathcal{D},S^1)\cap SO(3) = S^1$ (the $G^c_{a_0}$ of the general theory).

So we get the reduced affine Euler-Poincaré Lagrangian

$$\ell_2(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) = \frac{1}{2} J \int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, -\boldsymbol{\gamma} \times \mathbf{n}) \mu.$$

$\widehat{\boldsymbol{\nu}} = \dot{\chi}\chi^{-1}$, $\mathbf{n} = \chi \mathbf{n}_0$, $\gamma = -(\nabla\chi)\,\chi^{-1} \in \Omega^1(\mathcal{D}, \mathfrak{so}(3))$.

$\rightsquigarrow$ **The correct relation bewteen γ and $\mathbf{n}$ is $\nabla \mathbf{n} = \mathbf{n} \times \gamma$ and not $\boldsymbol{\gamma} := \nabla \mathbf{n} \times \mathbf{n}$.**

Notations:
$\gamma = \widehat{\boldsymbol{\gamma}}$. For $\boldsymbol{\gamma} = \boldsymbol{\gamma}_i dx^i \in \Omega^1(\mathcal{D}; \mathbb{R}^3)$, define $\boldsymbol{\gamma} \times \mathbf{n} \in \Omega^1(\mathcal{D}, \mathbb{R}^3)$ by $\boldsymbol{\gamma} \times \mathbf{n} = (\boldsymbol{\gamma}_i \times \mathbf{n})\, dx^i$, or

$$(\boldsymbol{\gamma} \times \mathbf{n})(v_x) = \boldsymbol{\gamma}(v_x) \times \mathbf{n}, \quad v_x \in T_x\mathcal{D}.$$

Important: $L(\chi, \dot{\chi}, \mathbf{n}_0, \gamma_0)$ may not be defined when $\gamma_0 \neq 0$. ℓ_2 is only defined on the orbit of $\gamma_0 = 0$, i.e., if $\gamma = -(\nabla\chi)\chi^{-1}$. However, this does not affect reduction, as long as the expression $L(\chi, \dot{\chi}, \mathbf{n}_0, 0)$ is invariant under the isotropy group of $\gamma_0 = 0$. This occurs in the reduction for molecular strand dynamics with nonlocal interactions (*Ellis, Gay-Balmaz, Holm, Putkaradze, Ratiu [2010]*).

The affine Euler-Poincaré equations are

$$\begin{cases} \dfrac{d}{dt}\dfrac{\delta \ell_2}{\delta \boldsymbol{\nu}} = \boldsymbol{\nu} \times \dfrac{\delta \ell_2}{\delta \boldsymbol{\nu}} + \operatorname{div} \dfrac{\delta \ell_2}{\delta \boldsymbol{\gamma}} + \operatorname{Tr}\left(\boldsymbol{\gamma} \times \dfrac{\delta \ell_2}{\delta \boldsymbol{\gamma}}\right) + \mathbf{n} \times \dfrac{\delta \ell_2}{\delta \mathbf{n}} \\ \partial_t \mathbf{n} + \mathbf{n} \times \boldsymbol{\nu} = 0 \\ \partial_t \boldsymbol{\gamma} + \boldsymbol{\gamma} \times \boldsymbol{\nu} + \nabla \boldsymbol{\nu} = 0, \quad \boldsymbol{\gamma}_0 = 0. \end{cases}$$

If $\gamma_0 \neq 0$, these reduced equations still make sense, and they are an extension of EL dynamics to account for disclination dynamics. Note that these equations consistently preserve the relation $\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$, since

$$\left(\frac{\partial}{\partial t} - \boldsymbol{\nu} \times\right)(\nabla \mathbf{n} - \mathbf{n} \times \boldsymbol{\gamma}) = 0\,.$$

EP equations for ℓ_1 and AEP equations ℓ_2 are equivalent since they are induced by the SAME Euler-Lagrange equations for $\mathcal{L}(\chi, \dot{\chi})$ on $T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$.
Moreover, the AEP equations allow for a generalization of Ericksen-Leslie to the case with disclinations.

STEP II: Eringen micropolar theory contains Ericksen-Leslie director theory as a particular case

Recall:

1. Eringen's Lagrangian (motionless case)

$$\mathcal{L}(\chi,\dot{\chi}) = \frac{1}{2}\int_{\mathcal{D}} \operatorname{Tr}\left((i_0\chi^{-1}\dot{\chi})^T\chi^{-1}\dot{\chi}\right)\mu - \int_{\mathcal{D}} \Psi(\chi j_0\chi^{-1}, \chi\nabla\chi^{-1} + \chi\gamma_0\chi^{-1})\mu,$$

was interpreted as $\mathcal{L} = L_{(j_0,\gamma_0)}$, where $i_0 := \frac{1}{2}\operatorname{Tr}(j_0)I_3 - j_0$. This Lagrangian is invariant under the right affine action

$$(\chi, j_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}j_0\psi, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))_{(j_0,\gamma_0)}$.

2. Reduced Lagrangian

$$\ell_2(\boldsymbol{\nu}, j, \gamma) = \frac{1}{2}\int_{\mathcal{D}} (j\boldsymbol{\nu})\cdot\boldsymbol{\nu}\mu - \int_{\mathcal{D}} \Psi(j,\gamma)\mu.$$

3. Eringen's equation are the affine Euler-Poincaré equations for:
$G = \mathcal{F}(\mathcal{D}, SO(3))$
$V^* = \mathcal{F}(\mathcal{D}, \operatorname{Sym}(3)) \times \Omega^1(\mathcal{D}, \mathfrak{so}(3))$.

II.1 Rod-like assumption

Take as initial condition $j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0)$.
This definition is $\mathcal{F}(\mathcal{D}, SO(3))$-equivariant, so that $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ for all time.

Consider $\mathcal{L}(\chi, \dot{\chi}) = L_{(\mathbf{n}_0, \gamma_0)}(\chi, \dot{\chi}) := L_{(j_0 = J(\mathbf{I} - \mathbf{n}_0 \otimes \mathbf{n}_0), \gamma_0)}(\chi, \dot{\chi})$. This Lagrangian is invariant under the right action

$$(\chi, \mathbf{n}_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$$

of the isotropy subgroup $\mathcal{F}(\mathcal{D}, SO(3))_{(\mathbf{n}_0, \gamma_0)}$.
Reduced Lagrangian

$$\begin{aligned}\ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) : &= \ell_2(\boldsymbol{\nu}, J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma}) \\ &= \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} \Psi(J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \boldsymbol{\gamma})\mu,\end{aligned}$$

Affine Euler-Poincaré equations for ℓ_2' are equivalent to Eringen's equations in which the rod-like assumption has been assumed.

It remains to show that these equations contain as particular case, the Ericksen-Leslie equations.

II.2 No disclination assumption $\gamma_0 = 0$

Same step as earlier: suppose that $\mathbf{n}_0$ is constant and take $\gamma_0 = 0$. So the evolution of γ is given by

$$\gamma = \theta_{\chi^{-1}}(0) = -(\nabla\chi)\chi^{-1}.$$

Since $\mathbf{n} = \chi\mathbf{n}_0$, we get $\nabla\mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$.

II.3 Recovering the Oseen-Frank free energy

Recall that $\Psi = \Psi(j, \gamma)$, rod-like assumption $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$, and

$$F(\mathbf{n}, \nabla\mathbf{n}) = K_2 \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})}_{\textbf{chirality}} + \frac{1}{2}K_{11} \underbrace{(\operatorname{div} \mathbf{n})^2}_{\textbf{splay}} + \frac{1}{2}K_{22} \underbrace{(\mathbf{n} \cdot \operatorname{curl} \mathbf{n})^2}_{\textbf{twist}}$$
$$+ \frac{1}{2}K_{33} \underbrace{\|\mathbf{n} \times \operatorname{curl} \mathbf{n}\|^2}_{\textbf{bend}};$$

$K_2 \neq 0$ for cholesterics, $K_2 = 0$ for nematics.
So we need to show that there exists $\Psi = \Psi(j, \gamma)$ such that

$$\Psi(j, \boldsymbol{\gamma}) = \Psi(J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n}), \gamma) = F(\mathbf{n}, \mathbf{n} \times \boldsymbol{\gamma}) = F(\mathbf{n}, \nabla\mathbf{n}).$$

Lemma The Oseen-Frank free energy can be expressed in terms of $\Psi = \Psi(j, \gamma)$ as

$$\Psi(j,\gamma) = \frac{K_2}{J}\,\mathrm{Tr}(j\boldsymbol{\gamma}) + \frac{K_{11}}{J}\Big(\mathrm{Tr}\big((\boldsymbol{\gamma}^A)^2\big)\,(\mathrm{Tr}(j) - J) - 2\,\mathrm{Tr}\big(j(\boldsymbol{\gamma}^A)^2\big)\Big) + \frac{1}{2}\frac{K_{22}}{J^2}\,\mathrm{Tr}^2(j\boldsymbol{\gamma}) - \frac{K_{33}}{J}\,\mathrm{Tr}\Big(\big((\boldsymbol{\gamma} j)^A - J\boldsymbol{\gamma}^A\big)^2\Big).$$

So we can rewrite the reduced Eringen Lagrangian in the rod-like assumption

$$\begin{aligned}\ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) &= \ell_2(\boldsymbol{\nu}, J(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}), \boldsymbol{\gamma}) \\ &= \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu}\times\mathbf{n}\|^2\mu - \int_{\mathcal{D}} \Psi(J(\mathbf{I} - \mathbf{n}\otimes\mathbf{n}), \boldsymbol{\gamma})\mu\end{aligned}$$

as

$$\ell_2'(\boldsymbol{\nu}, \mathbf{n}, \boldsymbol{\gamma}) = \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu}\times\mathbf{n}\|^2\mu - \int_{\mathcal{D}} F(\mathbf{n}, \mathbf{n}\times\boldsymbol{\gamma})\mu,$$

Same substitution in the unreduced Eringen Lagrangian in the rod-like assumption yields $\mathcal{L}(\chi, \dot{\chi}) = L_{(\mathbf{n}_0, \gamma_0 = 0)}(\chi, \dot{\chi})$.

II.4 Recovering Ericksen-Leslie theory

1. Interpret now this $\mathcal{L}(\chi, \dot{\chi})$ as $L_{\mathbf{n}_0}(\chi, \dot{\chi})$ instead of $L_{(\mathbf{n}_0, \gamma=0)}(\chi, \dot{\chi})$.

2. Check that this Lagrangian is $\mathcal{F}(\mathcal{D}, SO(3))_{\mathbf{n}_0}$-invariant under the action $(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$.

3. Implement Euler-Poincaré reduction associated to the action $(\chi, \mathbf{n}_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0\right)$ and obtain the reduced Lagrangian

$$\ell_1'(\boldsymbol{\nu}, \mathbf{n}) = \frac{J}{2}\int_{\mathcal{D}} \|\boldsymbol{\nu} \times \mathbf{n}\|^2 \mu - \int_{\mathcal{D}} F(\mathbf{n}, \nabla \mathbf{n})\mu$$

(Previously we considered affine Euler-Poincaré reduction associated to the action $(\chi, \mathbf{n}_0, \gamma_0) \mapsto \left(\chi\psi, \psi^{-1}\mathbf{n}_0, \psi^{-1}\gamma_0\psi + \psi^{-1}\nabla\psi\right)$, with reduced Lagrangian ℓ_2').

By general reduction theory: EP equations for ℓ_1' and AEP equations ℓ_2' are equivalent since they are induced by the SAME Euler-Lagrange equations for $\mathcal{L}(\chi, \dot{\chi})$ on $T\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$.

It remains to show that the EP equations for ℓ_1' are the Ericksen-Leslie equations. True, by direct verification.

We have thus proved:

THEOREM: The Eringen micropolar theory of liquid crystals contains as a particular case the Ericksen-Leslie director theory. More precisely, the Ericksen-Leslie theory is recovered by assuming rod-like molecules: $j = J(\mathbf{I} - \mathbf{n} \otimes \mathbf{n})$ and absence of disclinations $\gamma_0 = 0$.

Summary of method:

- This is shown by considering two distinct Euler-Poincaré reductions associated with distinct advected quantities.

- This allows us to replace the non-consistent definition $\boldsymbol{\gamma} := \nabla \mathbf{n} \times \mathbf{n}$ by the relation $\nabla \mathbf{n} = \mathbf{n} \times \boldsymbol{\gamma}$ and to solve the inconsistencies in Eringen's approach.

Final remarks: 1.) All the discussion here can be easily extended to moving liquid crystals. One applies EP, respectively affine EP, theory, as discussed earlier. Then the same considerations as above show that Eringen micropolar theory contains Ericksen-Leslie nematodynamics.

2.) Other inconsistencies in the micropolar description: Eringen defines a smectic liquid crystal by $\mathrm{Tr}(\gamma) = \gamma_1^1 + \gamma_2^2 + \gamma_3^3 = 0$. This is *not preserved by the evolution* $\gamma = \eta_* \left(\chi \gamma_0 \chi^{-1} + \chi \nabla \chi^{-1} \right)$.

Consistent with the statement: the equation

$$\frac{\partial \gamma}{\partial t} + \pounds_{\mathbf{u}} \gamma + \mathbf{d} \boldsymbol{\nu} + \gamma \times \boldsymbol{\nu} = 0$$

does not imply that if $\mathrm{Tr}(\gamma_0) = 0$ then $\mathrm{Tr}(\gamma) = 0$ for all time.

Is Eringen's definition of smectic incorrect? Instead of the trace need an $\mathcal{F}(\mathcal{D}, \mathrm{SO}(3))$-invariant function (of γ) under the action

$$\mathbf{v} \mapsto \chi^{-1} \mathbf{v} + \chi^{-1} \nabla \chi, \quad \mathbf{v} \in \mathcal{F}(\mathcal{D}, \mathbb{R}^3), \quad \chi \in \mathcal{F}(\mathcal{D}, \mathrm{SO}(3)).$$

We do not know how to choose a physically reasonable function of this type.

3.) Other difficulties in liquid crystals dynamics may be solved by using the tools of geometric mechanics (disclinations, defects,...)