Decoupling the Ericksen-Leslie equations

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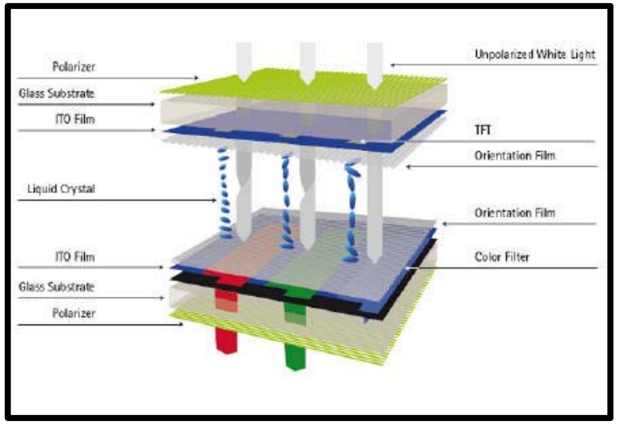
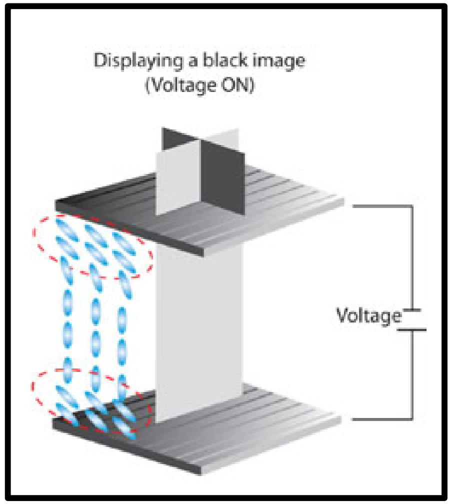
Motivation

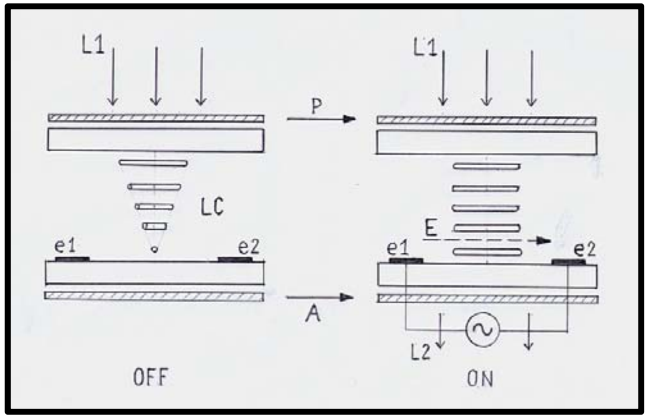
Most liquid crystal display devices are “1-dimensional”



Twisted Nematic



Vertically Aligned Nematic

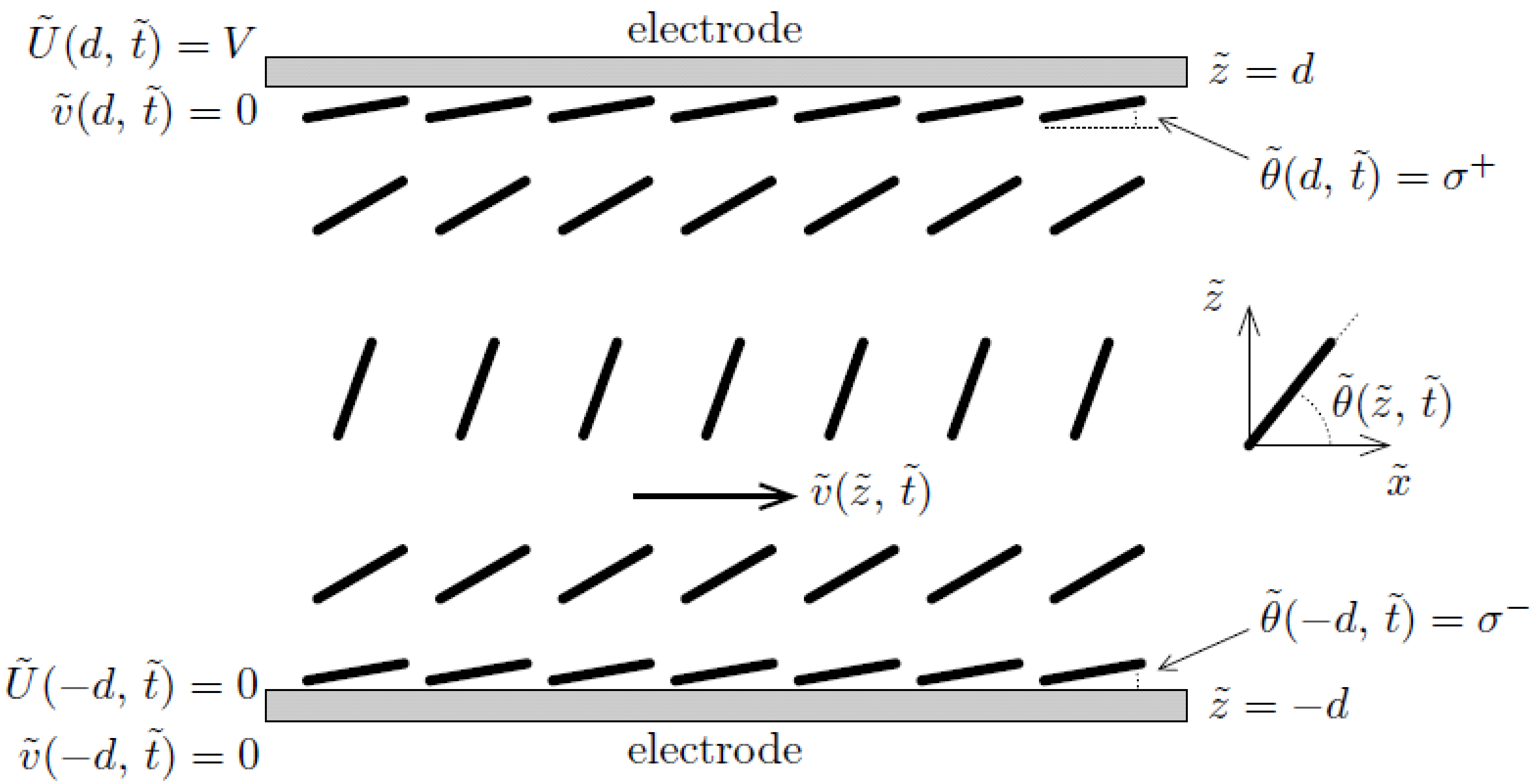


In-Plane Switching

Images:
http://www.connessioni.biz
http://www.merck.com
http://www.wikipedia.org

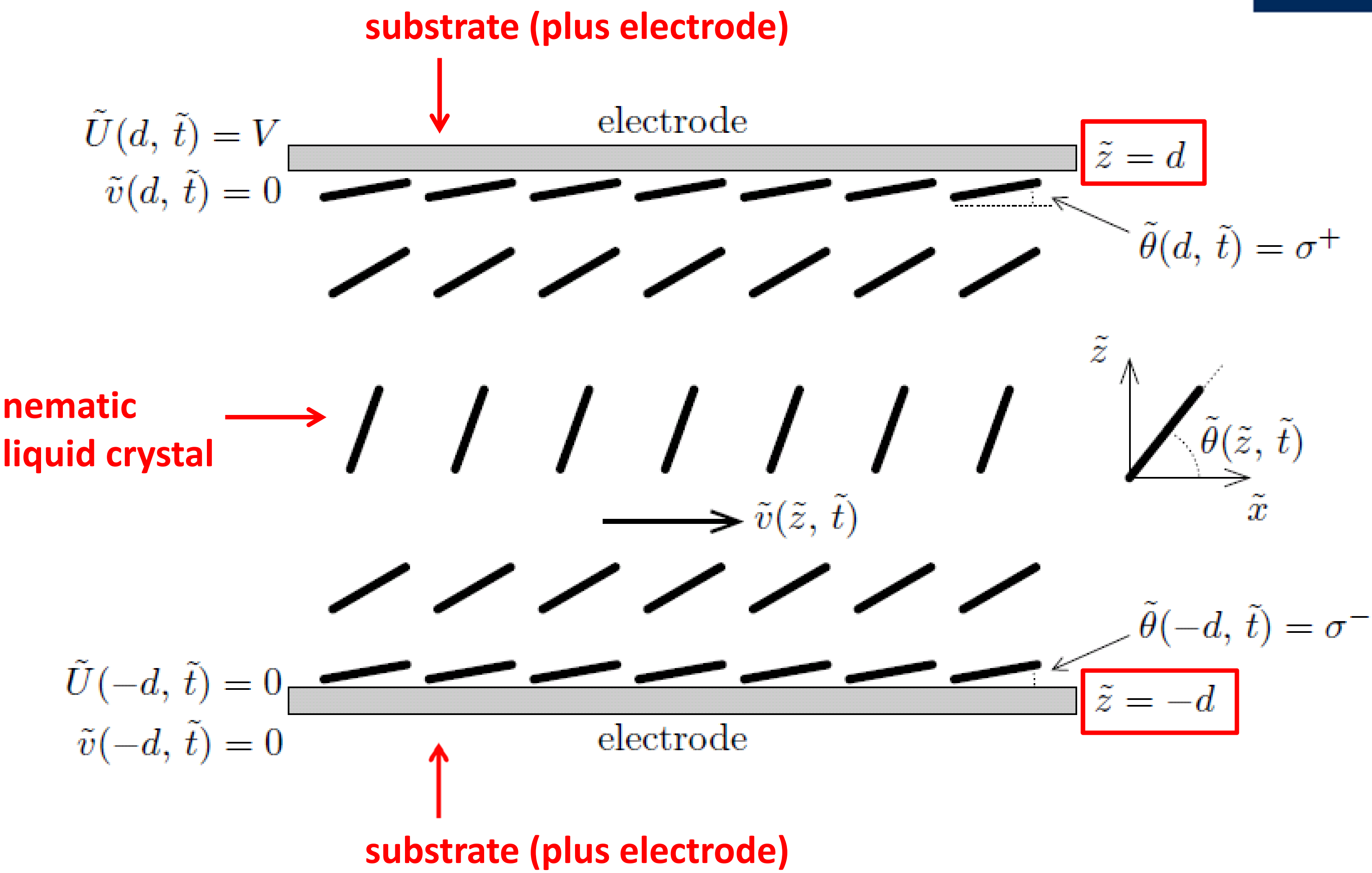
1d nematic layer

- **A standard nematic cell (splay-bend Freedericksz cell)**



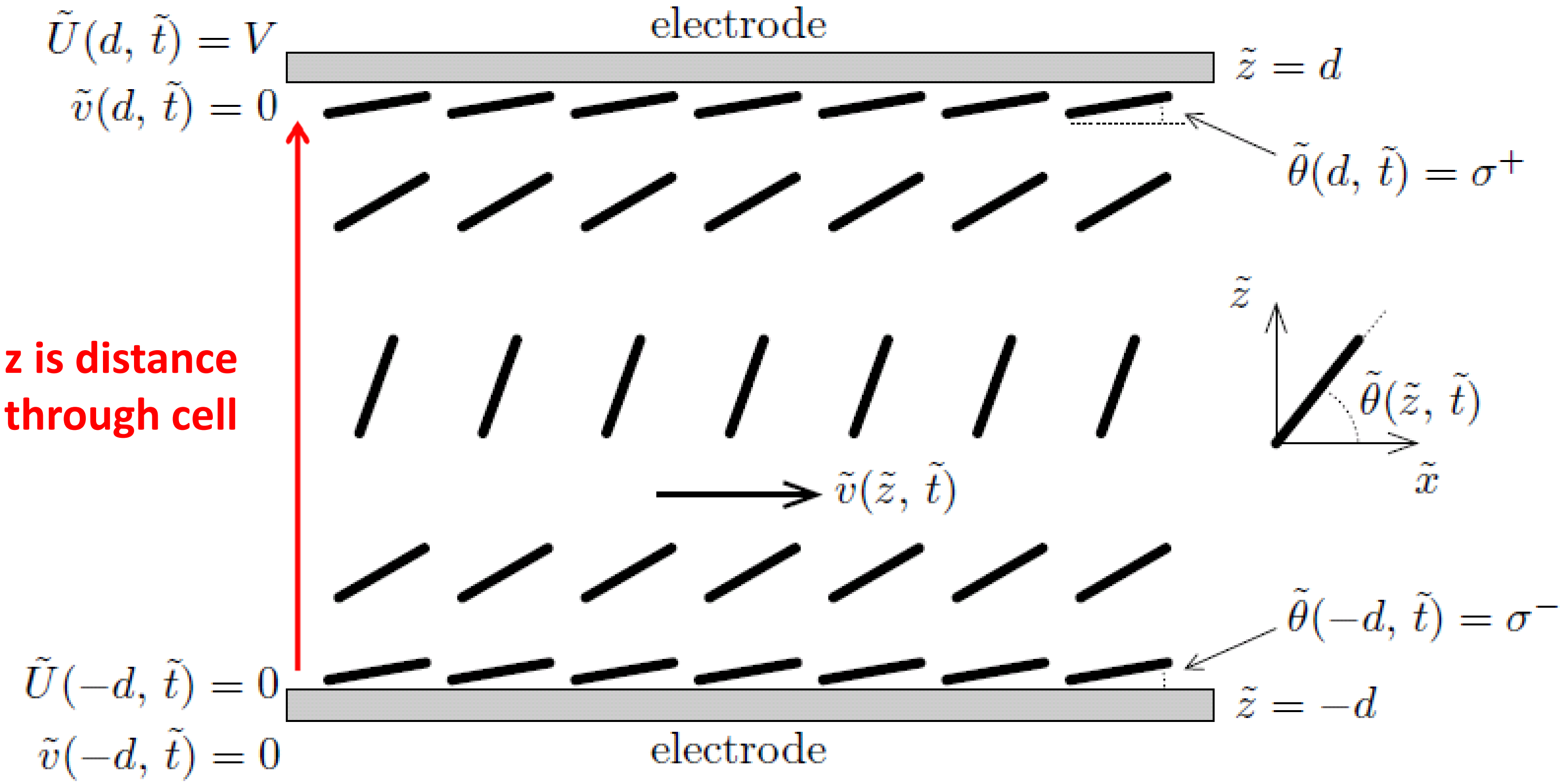
1d nematic layer

- **geometry**



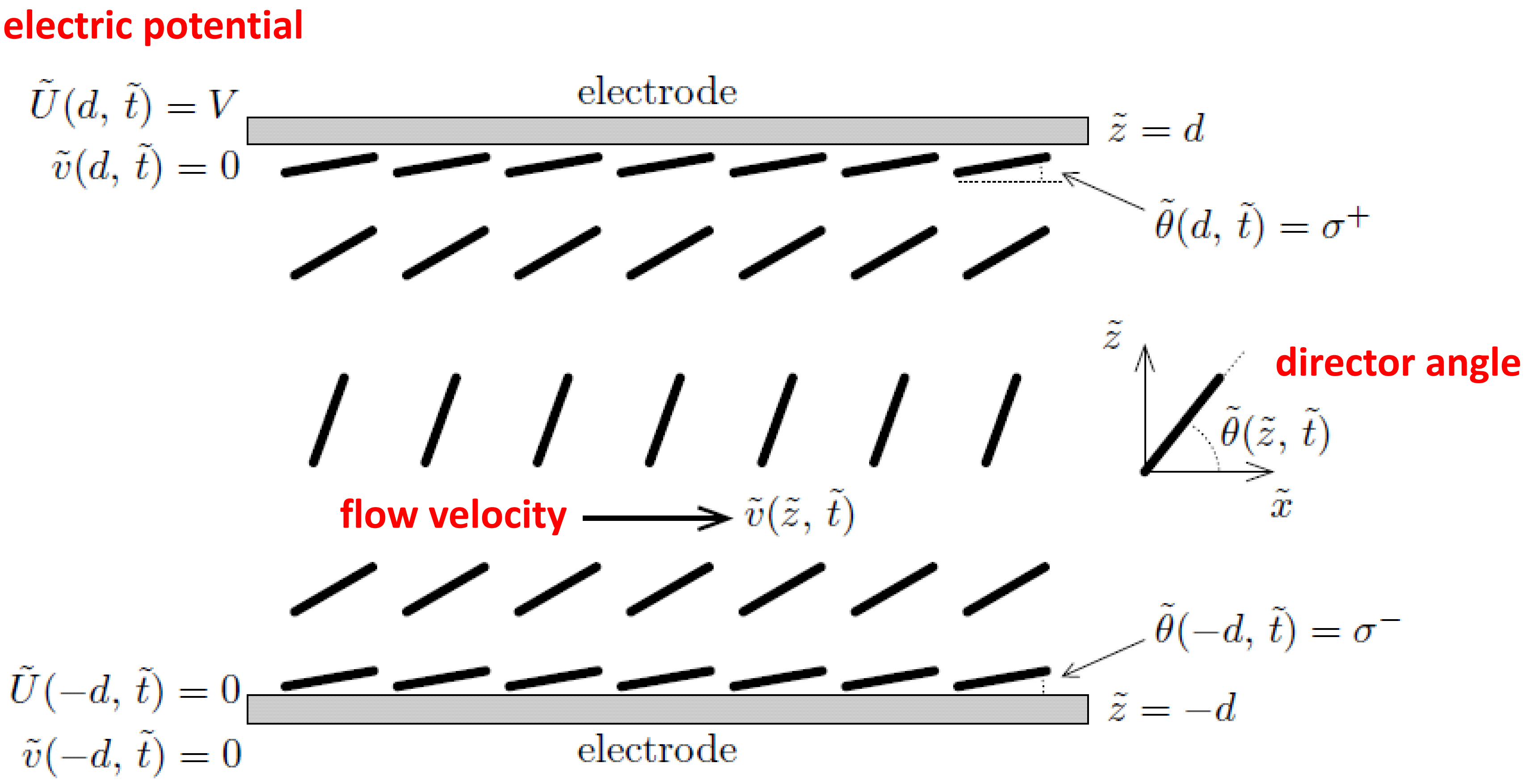
1d nematic layer

- **independent variables: one space + time**



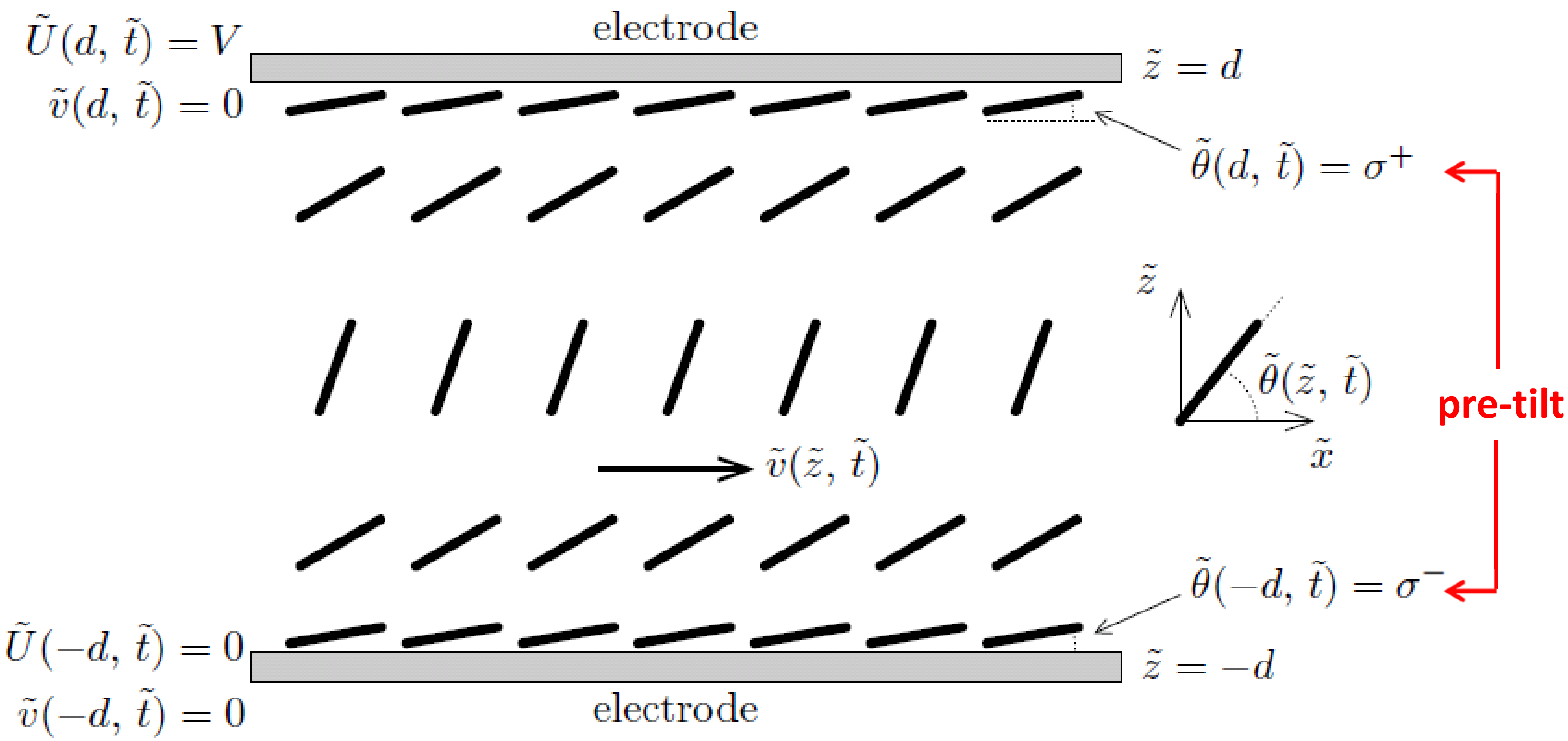
1d nematic layer

- **dependent variables**



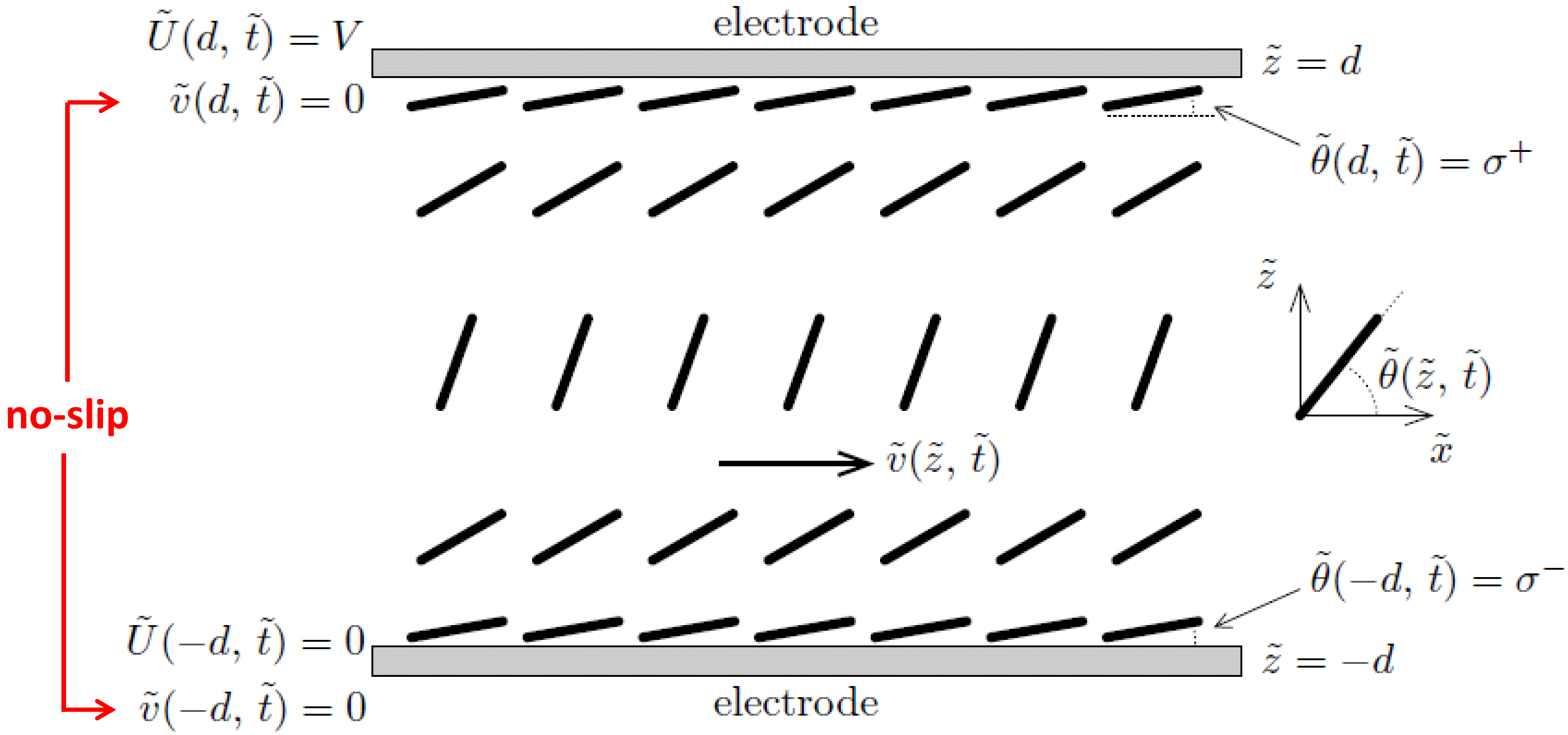
1d nematic layer

- **Boundary conditions**



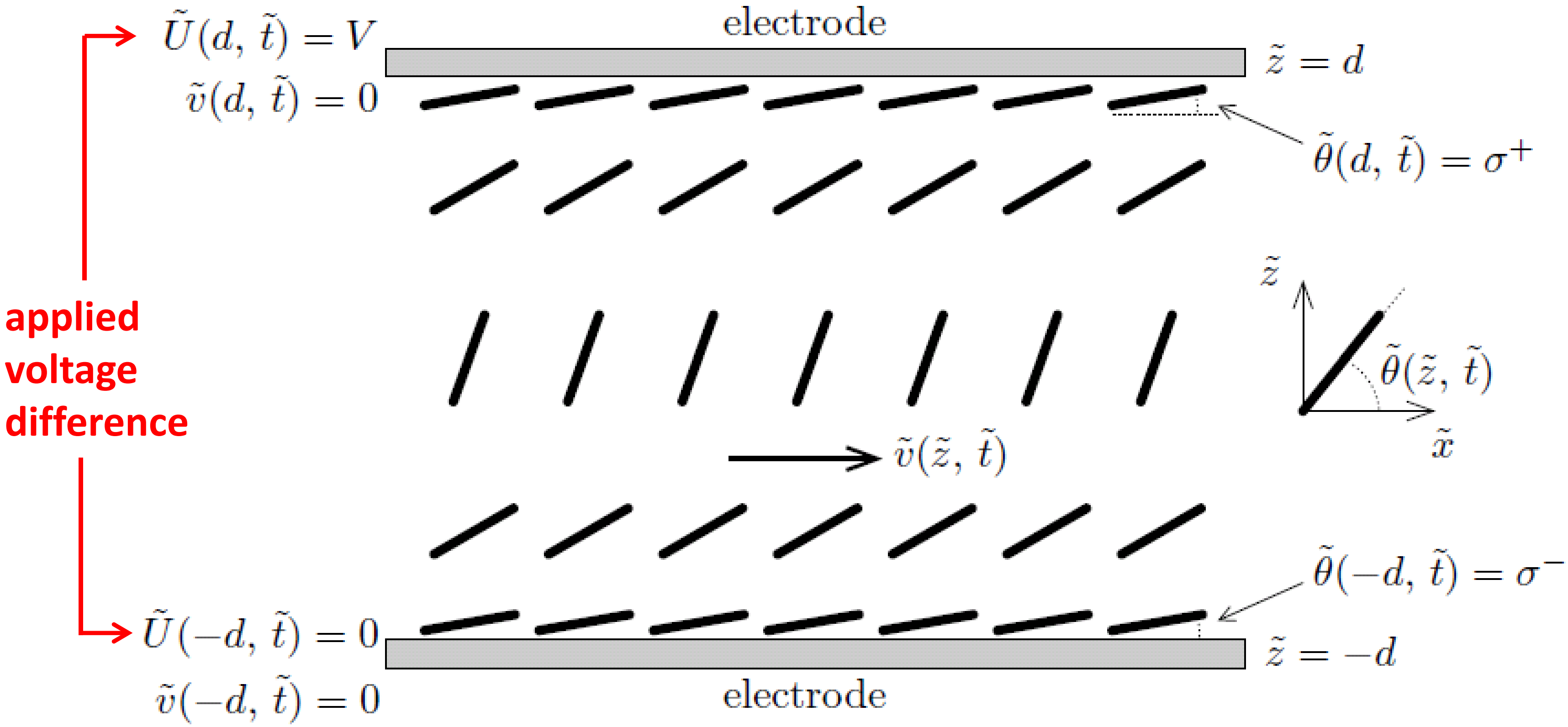
1d nematic layer

- **Boundary conditions**



1d nematic layer

- **Dependent variables**



E-L equations

- Ericksen-Leslie equations: assuming director stays in the xz-plane

$$\gamma_1 \tilde{\theta}_{\tilde{t}} = (K_1 \cos^2 \tilde{\theta} + K_3 \sin^2 \tilde{\theta})\, \tilde{\theta}_{\tilde{z}\tilde{z}} + (K_3 - K_1) \sin \tilde{\theta} \cos \tilde{\theta}\, (\tilde{\theta}_{\tilde{z}})^2 + \epsilon_0 \Delta\epsilon\, (\tilde{U}_{\tilde{z}})^2 \sin \tilde{\theta} \cos \tilde{\theta} - \tilde{m}(\tilde{\theta})\, \tilde{v}_{\tilde{z}},$$

$$\rho \tilde{v}_{\tilde{t}} = \left(\tilde{g}(\tilde{\theta})\, \tilde{v}_{\tilde{z}} + \tilde{m}(\tilde{\theta})\, \tilde{\theta}_{\tilde{t}}\right)_{\tilde{z}},$$

$$\tilde{m}(\tilde{\theta}) = \alpha_3 \cos^2 \tilde{\theta} - \alpha_2 \sin^2 \tilde{\theta},$$

$$\tilde{g}(\tilde{\theta}) = \frac{1}{2}\left(\alpha_4 + (\alpha_5 - \alpha_2) \sin^2 \tilde{\theta} + (\alpha_3 + \alpha_6) \cos^2 \tilde{\theta}\right) + \alpha_1 \sin^2 \tilde{\theta} \cos^2 \tilde{\theta},$$

E-L equations

- Ericksen-Leslie equations: assuming director stays in the xz-plane

angular momentum

$$\gamma_1 \tilde{\theta}_{\tilde{t}} = \left(K_1 \cos^2\tilde{\theta} + K_3 \sin^2\tilde{\theta}\right)\tilde{\theta}_{\tilde{z}\tilde{z}} + (K_3 - K_1)\sin\tilde{\theta}\cos\tilde{\theta}\,(\tilde{\theta}_{\tilde{z}})^2 + \epsilon_0 \Delta\epsilon\,(\tilde{U}_{\tilde{z}})^2 \sin\tilde{\theta}\cos\tilde{\theta} - \tilde{m}(\tilde{\theta})\,\tilde{v}_{\tilde{z}},$$

$$\rho \tilde{v}_{\tilde{t}} = \left(\tilde{g}(\tilde{\theta})\,\tilde{v}_{\tilde{z}} + \tilde{m}(\tilde{\theta})\,\tilde{\theta}_{\tilde{t}}\right)_{\tilde{z}},$$

linear momentum

$$\tilde{m}(\tilde{\theta}) = \alpha_3 \cos^2\tilde{\theta} - \alpha_2 \sin^2\tilde{\theta},$$

$$\tilde{g}(\tilde{\theta}) = \frac{1}{2}\left(\alpha_4 + (\alpha_5 - \alpha_2)\sin^2\tilde{\theta} + (\alpha_3 + \alpha_6)\cos^2\tilde{\theta}\right) + \alpha_1 \sin^2\tilde{\theta}\cos^2\tilde{\theta},$$

E-L equations

- Ericksen-Leslie equations: assuming director stays in the xz-plane

elastic terms

director rotation

$$\gamma_1 \tilde{\theta}_{\tilde{t}} = (K_1 \cos^2 \tilde{\theta} + K_3 \sin^2 \tilde{\theta})\, \tilde{\theta}_{\tilde{z}\tilde{z}} + (K_3 - K_1) \sin \tilde{\theta} \cos \tilde{\theta}\, (\tilde{\theta}_{\tilde{z}})^2$$

$$+ \epsilon_0 \Delta\epsilon\, (\tilde{U}_{\tilde{z}})^2 \sin \tilde{\theta} \cos \tilde{\theta} - \tilde{m}(\tilde{\theta})\, \tilde{v}_{\tilde{z}},$$

director-flow coupling

electric field

$$\rho \tilde{v}_{\tilde{t}} = \left(\tilde{g}(\tilde{\theta})\, \tilde{v}_{\tilde{z}} + \tilde{m}(\tilde{\theta})\, \tilde{\theta}_{\tilde{t}}\right)_{\tilde{z}},$$

$$\tilde{m}(\tilde{\theta}) = \alpha_3 \cos^2 \tilde{\theta} - \alpha_2 \sin^2 \tilde{\theta},$$

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E-L equations

- Ericksen-Leslie equations: assuming director stays in the xz-plane

$$\gamma_1 \tilde{\theta}_{\tilde{t}} = \left(K_1 \cos^2\tilde{\theta} + K_3 \sin^2\tilde{\theta}\right)\tilde{\theta}_{\tilde{z}\tilde{z}} + (K_3 - K_1)\sin\tilde{\theta}\cos\tilde{\theta}\,(\tilde{\theta}_{\tilde{z}})^2 + \epsilon_0 \Delta\epsilon\,(\tilde{U}_{\tilde{z}})^2 \sin\tilde{\theta}\cos\tilde{\theta} - \tilde{m}(\tilde{\theta})\,\tilde{v}_{\tilde{z}},$$

flow inertia

$$\rho \tilde{v}_{\tilde{t}} = \left(\tilde{g}(\tilde{\theta})\,\tilde{v}_{\tilde{z}} + \tilde{m}(\tilde{\theta})\,\tilde{\theta}_{\tilde{t}}\right)_{\tilde{z}},$$

director-flow coupling

fluid viscosity

$$\tilde{m}(\tilde{\theta}) = \alpha_3 \cos^2\tilde{\theta} - \alpha_2 \sin^2\tilde{\theta},$$

$$\tilde{g}(\tilde{\theta}) = \frac{1}{2}\left(\alpha_4 + (\alpha_5 - \alpha_2)\sin^2\tilde{\theta} + (\alpha_3 + \alpha_6)\cos^2\tilde{\theta}\right) + \alpha_1 \sin^2\tilde{\theta}\cos^2\tilde{\theta},$$

effective viscosity

Maxwell’s equation

- Maxwell’s equation for the electric potential

$$\left(\left(\epsilon_\perp + \Delta\epsilon \sin^2 \tilde{\theta}\right) \tilde{U}_{\tilde{z}}\right)_{\tilde{z}} = 0.$$

Maxwell's equation

- Maxwell's equation for the electric potential

$$\left((\epsilon_\perp + \Delta\epsilon \sin^2 \tilde{\theta}) \tilde{U}_{\tilde{z}}\right)_{\tilde{z}} = 0.$$

effective permittivity

Non-dimensionalised equations

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$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z\,,$$

$$\frac{\tau_2}{\tau_1}\zeta\, v_t = \left(-\zeta\, g(\theta)\, v_z - m(\theta)\,\theta_t\right)_z\,,$$

$$0 = \left((e_\perp + \sin^2\theta)\, U_z\right)_z\,,$$

$$\tilde{z} = d\, z, \qquad \tilde{t} = \frac{d^2\gamma_1}{K_3}\, t,$$

Non-dimensionalised equations

University of Strathclyde Glasgow

$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z,$$

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$$\tilde{z} = d\,z, \qquad \tilde{t} = \frac{d^2\gamma_1}{K_3}\,t,$$

elastically driven director relaxation

Non-dimensionalised equations

University of Strathclyde Glasgow

$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z,$$

$$\frac{\tau_2}{\tau_1}\zeta\, v_t = (-\zeta\, g(\theta)\, v_z - m(\theta)\,\theta_t)_z\,,$$

$$0 = ((e_\perp + \sin^2\theta)\, U_z)_z\,,$$

$$k = \frac{K_1}{K_3}, \quad \lambda = -\frac{\alpha_2 d}{K_3}, \quad \zeta = \frac{\gamma_1 d}{K_3}, \quad \tau_1 = \frac{d^2\gamma_1}{K_3}, \quad \tau_2 = -\frac{d^2\rho}{\alpha_2},$$

$$\Delta e = \frac{\epsilon_0\Delta\epsilon}{K_3}, \quad e_\perp = \frac{\epsilon_\perp}{\Delta\epsilon}.$$

Non-dimensionalised equations

$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z\,,$$

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elastic constant ratio

Non-dimensionalised equations

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$$\theta_t = \left(k\cos^2\theta + \sin^2\theta\right)\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z\,,$$

$$\frac{\tau_2}{\tau_1}\zeta\, v_t = \left(-\zeta\, g(\theta)\, v_z - m(\theta)\,\theta_t\right)_z\,,$$

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director rotation and fluid inertia timescales

Non-dimensionalised equations

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$$\theta_t = \left(k\cos^2\theta + \sin^2\theta\right)\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z\,,$$

$$\frac{\tau_2}{\tau_1}\zeta\, v_t = \left(-\zeta\, g(\theta)\, v_z - m(\theta)\,\theta_t\right)_z\,,$$

$$0 = \left((e_\perp + \sin^2\theta)\, U_z\right)_z\,,$$

$$k = \frac{K_1}{K_3}\,, \quad \lambda = -\frac{\alpha_2 d}{K_3}\,, \quad \zeta = \frac{\gamma_1 d}{K_3}\,, \quad \tau_1 = \frac{d^2\gamma_1}{K_3}, \quad \tau_2 = -\frac{d^2\rho}{\alpha_2},$$

$$\Delta e = \frac{\epsilon_0 \Delta\epsilon}{K_3}\,, \quad e_\perp = \frac{\epsilon_\perp}{\Delta\epsilon}.$$

director rotation and fluid inertia timescales

Non-dimensionalised equations

$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z,$$

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$$0 = \left((e_\perp + \sin^2\theta)\, U_z\right)_z,$$

$$k = \frac{K_1}{K_3}, \quad \lambda = -\frac{\alpha_2 d}{K_3}, \quad \zeta = \frac{\gamma_1 d}{K_3}, \quad \tau_1 = \frac{d^2\gamma_1}{K_3}, \quad \tau_2 = -\frac{d^2\rho}{\alpha_2},$$

$$\Delta e = \frac{\epsilon_0\Delta\epsilon}{K_3}, \quad e_\perp = \frac{\epsilon_\perp}{\Delta\epsilon}.$$

normalised permittivities

Non-dimensionalised equations

- We can solve the electric potential equation,

$$0 = \left((e_\perp + \sin^2\theta) U_z \right)_z ,$$

$$U(z,t) = V \int_{-1}^{z} h(\theta)\,\mathrm{d}z \Big/ \int_{-1}^{1} h(\theta)\,\mathrm{d}z$$

where $h(\theta) = (e_\perp + \sin^2\theta)^{-1}$

so the electric potential is "slaved" to the director configuration

Non-dimensionalised equations

- We now have,

$$\theta_t = (k\cos^2\theta + \sin^2\theta)\,\theta_{zz} + (1-k)\sin\theta\cos\theta\,(\theta_z)^2$$

$$+ \Delta e\,(U_z)^2\,\sin\theta\cos\theta + \lambda\, m(\theta)\, v_z,$$

$$0 = \left(\zeta\, g(\theta)\, v_z + m(\theta)\,\theta_t\right)_z,$$

$$U(z,\,t) = V\int_{-1}^{z} h(\theta)\,\mathrm{d}z \Big/ \int_{-1}^{1} h(\theta)\,\mathrm{d}z$$

with boundary conditions

$$\theta(\pm 1,\,t) = \sigma^{\pm},$$

$$v(-1,\,t) = 0,$$

Decoupling

- Integrate the linear momentum equation from z=-1 to z,

$$0 = \left(\zeta\, g(\theta)\, v_z + m(\theta)\, \theta_t\right)_z ,$$

$$\zeta\, g(\theta)\, v_z + m(\theta)\, \theta_t - \zeta\, g(\sigma^-)\, v_z(-1,\, t) - m(\sigma^-)\, \theta_t(-1,\, t) = 0.$$

but strong anchoring of the director at the substrate means that $\theta_t(-1,\, t) = 0$

so

$$v_z = \frac{g(\sigma^-)}{g(\theta)}\, v_z(-1,\, t) - \frac{m(\theta)}{\zeta\, g(\theta)}\, \theta_t.$$

Decoupling

- Integrating this equation from z=-1 to 1, and using the non-slip boundary conditions

$$v_z = \frac{g(\sigma^-)}{g(\theta)}\, v_z(-1,\, t) - \frac{m(\theta)}{\zeta\, g(\theta)}\, \theta_t.$$

$$\Downarrow$$

$$v_z(-1,\, t) = \frac{\mathcal{K}}{\zeta\, g(\sigma^-)},$$

where

$$\mathcal{K} = \int_{-1}^{1} \frac{m(\theta)}{g(\theta)}\, \theta_t\, \mathrm{d}z \Big/ \int_{-1}^{1} \frac{1}{g(\theta)}\, \mathrm{d}z.$$

(a function of time only)

Decoupling

- We finally arrive at the expression for the velocity gradient and thus, by integrating, the velocity

$$v_z = \frac{1}{\zeta\, g(\theta)} \left[\mathcal{K} - m(\theta)\, \theta_t \right].$$

$$\Downarrow$$

$$v(z,\,t) = \frac{1}{\zeta}\left[\mathcal{K}\int_{-1}^{z} \frac{1}{g(\theta)}\,\mathrm{d}z - \int_{-1}^{z} \frac{m(\theta)}{g(\theta)}\theta_t\,\mathrm{d}z\right]$$

where

$$\mathcal{K} = \int_{-1}^{1} \frac{m(\theta)}{g(\theta)}\,\theta_t\,\mathrm{d}z \Big/ \int_{-1}^{1} \frac{1}{g(\theta)}\,\mathrm{d}z.$$

so, as with the electric field, the velocity is “slaved” to the director motion

Decoupling

- The decoupled director equation is therefore

$$\left(1+\frac{\lambda\, m^2(\theta)}{\zeta\, g(\theta)}\right)\theta_t = (k\cos^2\theta + \sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{K}\,\lambda\, m(\theta)}{\zeta\, g(\theta)}.$$

where

$$\mathcal{K} = \int_{-1}^{1}\frac{m(\theta)}{g(\theta)}\,\theta_t\,\mathrm{d}z \Big/ \int_{-1}^{1}\frac{1}{g(\theta)}\,\mathrm{d}z.$$

$$U(z,t) = V\int_{-1}^{z} h(\theta)\,\mathrm{d}z \Big/ \int_{-1}^{1} h(\theta)\,\mathrm{d}z$$

Decoupling

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$$\left(1+\frac{\lambda m^2(\theta)}{\zeta g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

$$+\Delta e(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{K}\lambda m(\theta)}{\zeta g(\theta)}.$$

where

$$\mathcal{K} = \int_{-1}^{1}\frac{m(\theta)}{g(\theta)}\theta_t \, \mathrm{d}z \Big/ \int_{-1}^{1}\frac{1}{g(\theta)}\,\mathrm{d}z.$$

$$U(z,t) = V\int_{-1}^{z} h(\theta)\,\mathrm{d}z \Big/ \int_{-1}^{1} h(\theta)\,\mathrm{d}z$$

Decoupling

- Rearrange and integrate between -1 and 1 to get

$$\left(1+\frac{\lambda\,m^2(\theta)}{\zeta\,g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz}+(1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta+\frac{\mathcal{A}\,\lambda\,m(\theta)}{\mathcal{B}\,\zeta\,g(\theta)},$$

where

$$\mathcal{A}=\int_{-1}^{1}\frac{\zeta\,m(\theta)}{\zeta\,g(\theta)+\lambda\,m^2(\theta)}\big[(k\cos^2\theta+\sin^2\theta)\theta_{zz}+(1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e(U_z)^2\sin\theta\cos\theta\big]\,\mathrm{d}z,$$

$$\mathcal{B}=\int_{-1}^{1}\frac{\zeta}{\zeta\,g(\theta)+\lambda\,m^2(\theta)}\,\mathrm{d}z.$$

Flow effects

- The effect of flow enters in two terms

$$\left(1+\frac{\lambda m^2(\theta)}{\zeta g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\lambda m(\theta)}{\mathcal{B}\zeta g(\theta)},$$

rescaled rotational viscosity

non-local forcing term

where

$$\mathcal{A} = \int_{-1}^{1}\frac{\zeta m(\theta)}{\zeta g(\theta)+\lambda m^2(\theta)}\left[(k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e(U_z)^2\sin\theta\cos\theta\right]\mathrm{d}z,$$

$$\mathcal{B} = \int_{-1}^{1}\frac{\zeta}{\zeta g(\theta)+\lambda m^2(\theta)}\,\mathrm{d}z.$$

Flow effects

$$\left(1+\frac{\lambda m^2(\theta)}{\zeta g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

rescaled rotational viscosity

$$+\Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\,\lambda\,m(\theta)}{\mathcal{B}\,\zeta\,g(\theta)},$$

- The extra term in the viscosity is always negative (but greater than -1) since γ_1 and $\tilde{g}(\tilde{\theta})$ are effective viscosities

$$\frac{\lambda m^2(\theta)}{\zeta g(\theta)} = -\frac{\tilde{m}^2(\tilde{\theta})}{\gamma_1\,\tilde{g}(\tilde{\theta})},$$

- So, in this term, flow always speeds director rotation

Flow effects

$$\left(1+\frac{\lambda\, m^2(\theta)}{\zeta\, g(\theta)}\right)\theta_t = (k\cos^2\theta + \sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\,\lambda\, m(\theta)}{\mathcal{B}\,\zeta\, g(\theta)},$$

non-local forcing term

where

$$\mathcal{A} = \int_{-1}^{1} \frac{\zeta\, m(\theta)}{\zeta\, g(\theta) + \lambda\, m^2(\theta)}\left[(k\cos^2\theta + \sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e(U_z)^2\sin\theta\cos\theta\right]\mathrm{d}z,$$

$$\mathcal{B} = \int_{-1}^{1} \frac{\zeta}{\zeta\, g(\theta) + \lambda\, m^2(\theta)}\,\mathrm{d}z.$$

Flow effects

$$\left(1+\frac{\lambda\, m^2(\theta)}{\zeta\, g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

$$+\,\Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\,\lambda\, m(\theta)}{\mathcal{B}\,\zeta\, g(\theta)},$$

non-local forcing term

$g(\theta)$, $\mathcal{B}$, ζ are positive but $m(\theta)$, $\mathcal{A}$, λ can be negative or positive

$$\lambda = -\frac{\alpha_2 d}{K_3}, \quad \zeta = \frac{\gamma_1 d}{K_3}$$

Boundary conditions

- Recall the boundary conditions

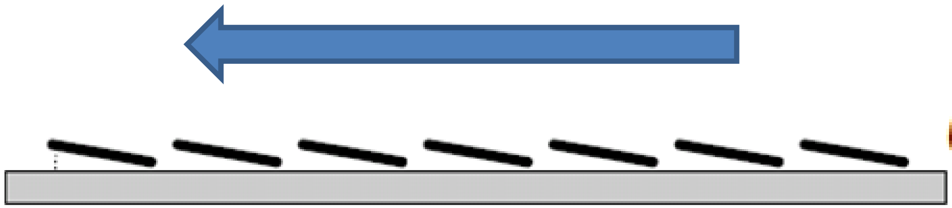
$$\theta(\pm 1, t) = \sigma^{\pm}$$

- We have two standard setups
  - Parallel rubbing $\sigma^{+} = -\sigma^{-} = \sigma$
  - Antiparallel rubbing $\sigma^{+} = \sigma^{-} = \sigma$

Boundary conditions

- Alignment at a substrate is induced through physical rubbing, which can induce a “pre-tilt”.

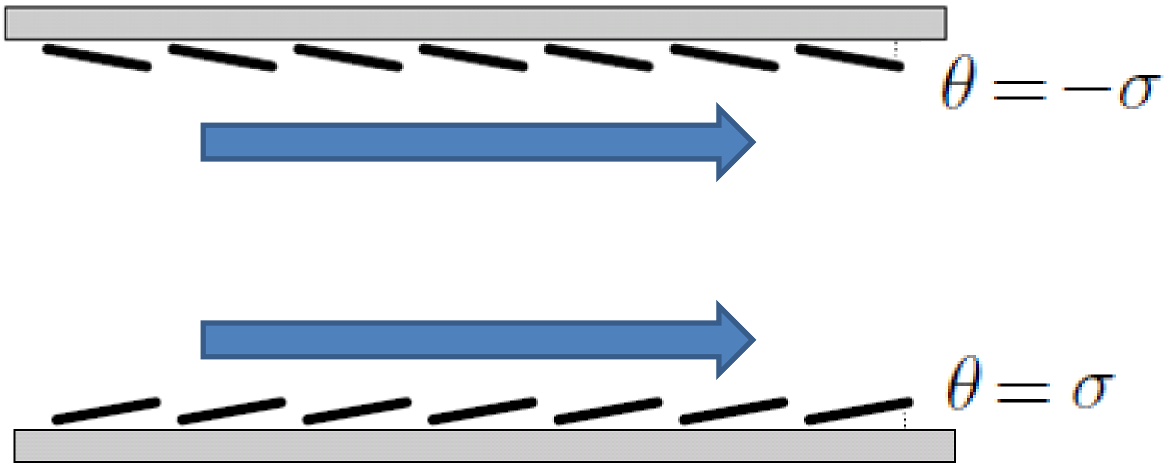
$$\theta = \sigma$$



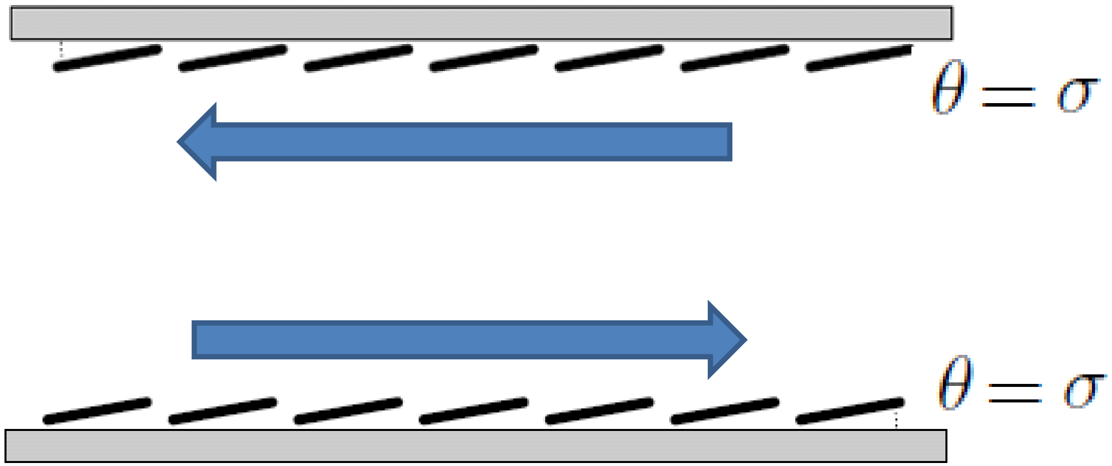
$$\theta = -\sigma$$

Boundary conditions

- Parallel rubbing

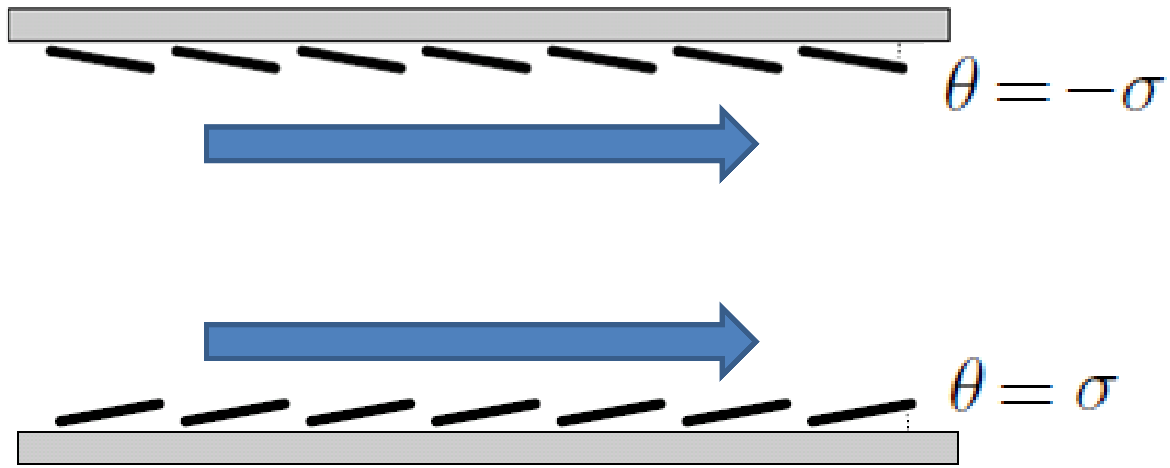


- Antiparallel rubbing

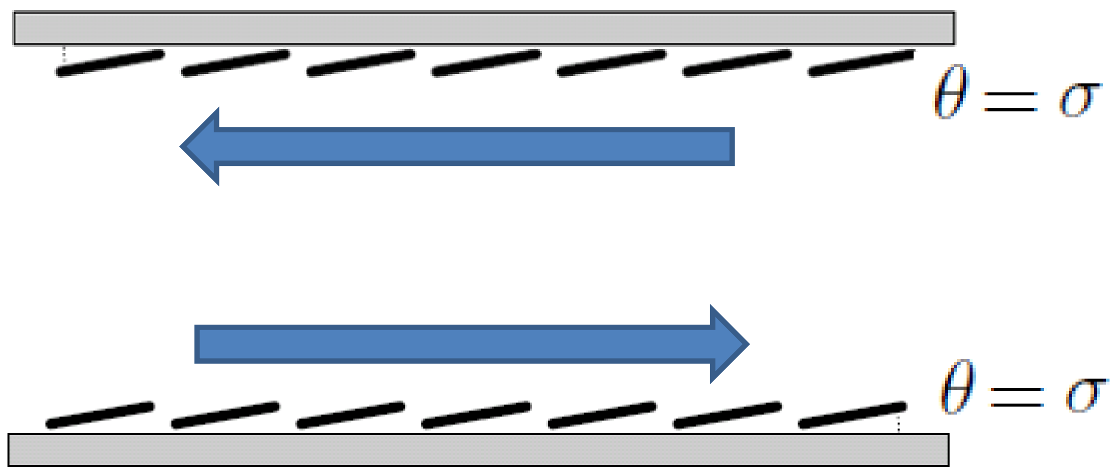


Boundary conditions

- **Parallel rubbing**

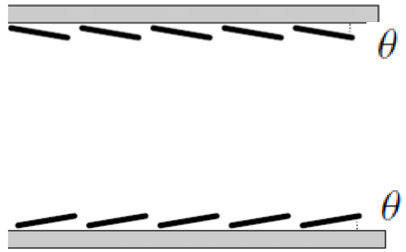


- Antiparallel rubbing



Parallel rubbed anchoring

- With these boundary conditions there is an odd solution (in z) for θ



$$\left(1+\frac{\lambda\, m^2(\theta)}{\zeta\, g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

$$+\,\Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\,\lambda\, m(\theta)}{\mathcal{B}\,\zeta\, g(\theta)},$$

$$\mathcal{A} = \int_{-1}^{1}\frac{\zeta\, m(\theta)}{\zeta\, g(\theta)+\lambda\, m^2(\theta)}\Big[(k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

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Parallel rubbed anchoring

University of Strathclyde Glasgow

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$\theta = -\sigma$

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$\theta = \sigma$

even in θ

$$\mathcal{A} = \int_{-1}^{1} \frac{\zeta\, m(\theta)}{\zeta\, g(\theta) + \lambda\, m^2(\theta)}\Big[(k\cos^2\theta + \sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2$$

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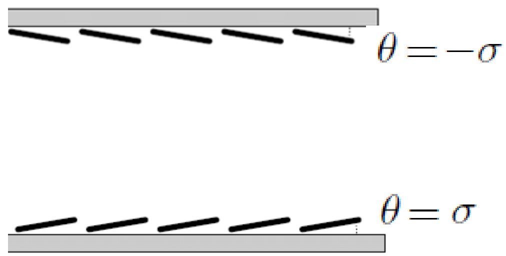

even in θ

odd in θ

$$\mathcal{A} = \int_{-1}^{1} \frac{\zeta\, m(\theta)}{\zeta\, g(\theta)+\lambda\, m^2(\theta)}\Big[(k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e(U_z)^2\sin\theta\cos\theta\Big]\,\mathrm{d}z,$$

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Parallel rubbed anchoring

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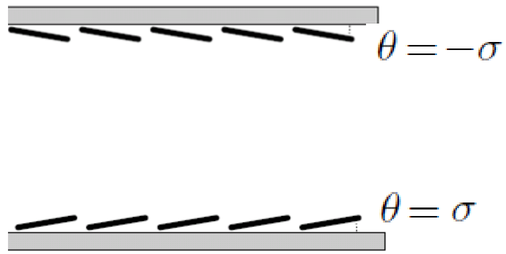

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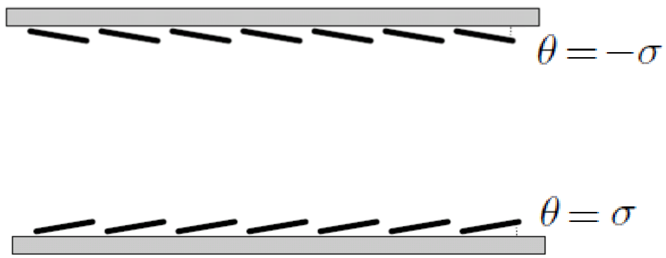
$$\mathcal{B} = \int_{-1}^{1}\frac{\zeta}{\zeta\, g(\theta)+\lambda\, m^2(\theta)}\,\mathrm{d}z.$$

- So for an odd solution (in z) then $\mathcal{A} = 0$ and the non-local term disappears

Parallel rubbed anchoring

- …and the governing equation is

$$\left(1+\frac{\lambda\, m^2(\theta)}{\zeta\, g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta$$



Parallel rubbed anchoring

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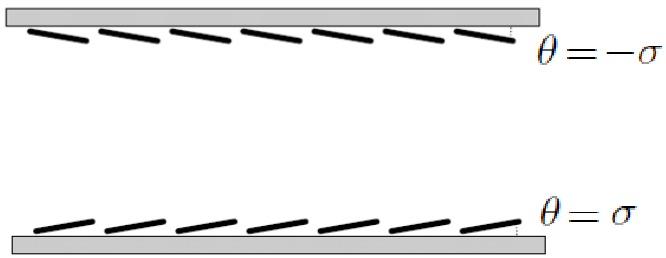
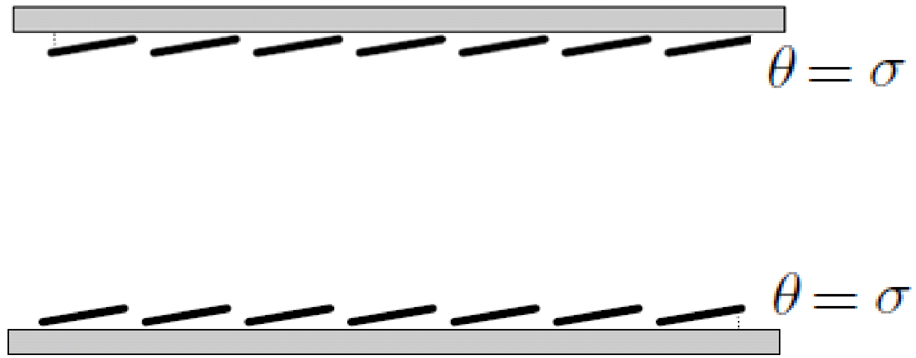
reduced rotational viscosity



- and flow always acts to speed up switching through a reduced viscosity

Antiparallel rubbed anchoring

- In this situation we find a symmetric solution (about z=0)



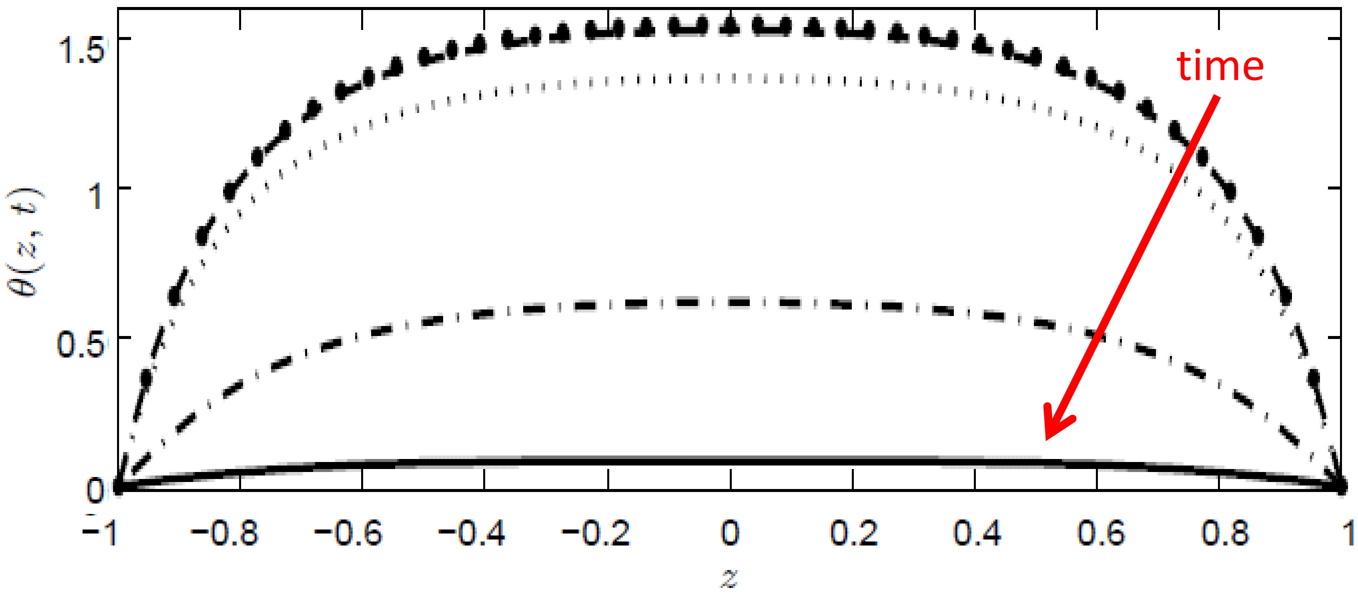
- …but we can’t infer much about the non-local term
- However, the single decoupled equation does give us information about the initial switching

Antiparallel rubbed anchoring

- The are two possible types of relaxation from an initial state

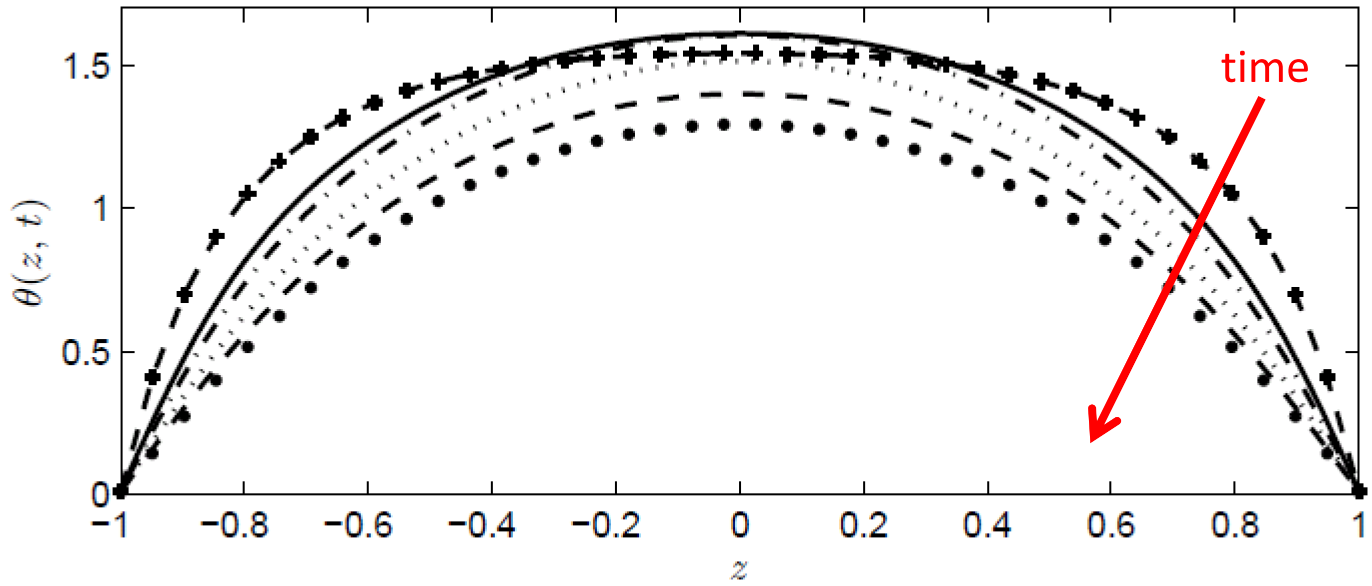
Antiparallel rubbed anchoring

- The are two possible types of relaxation from an initial state
- A simple decay back to the equilibrium state



Antiparallel rubbed anchoring

- The are two possible types of relaxation from an initial state
- Decay back to the equilibrium state after an initial increase in director angle



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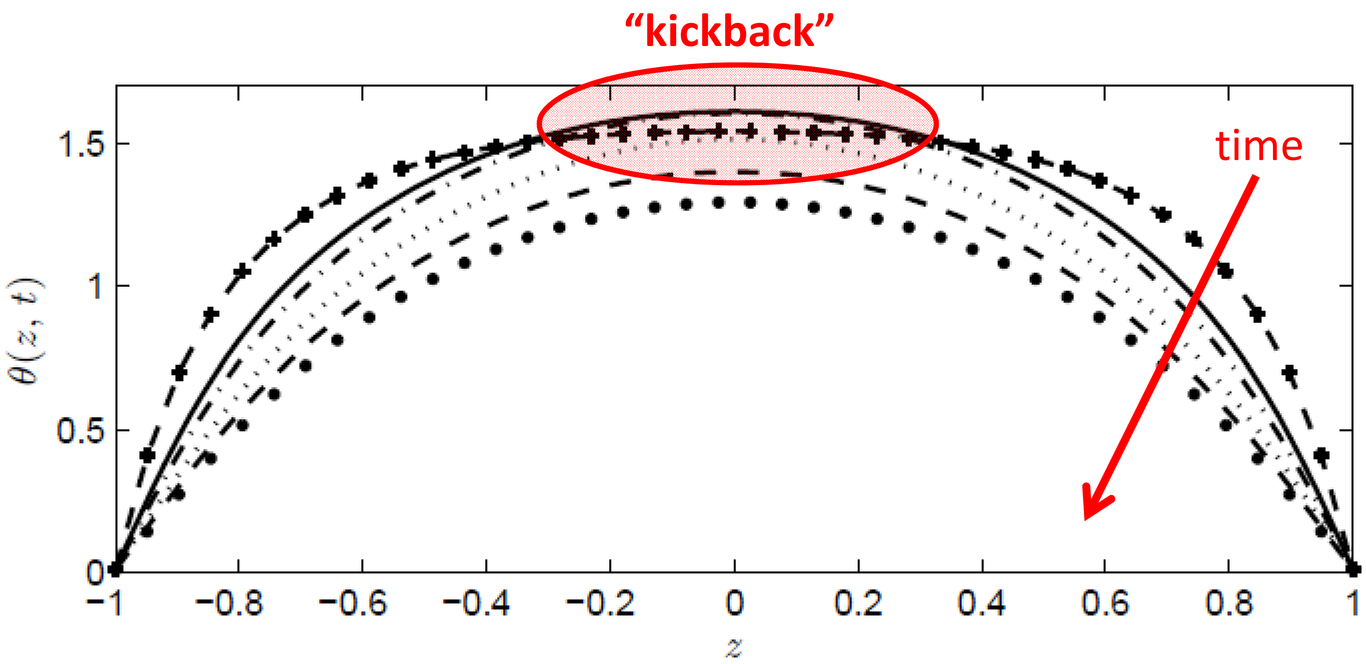

Antiparallel rubbed anchoring

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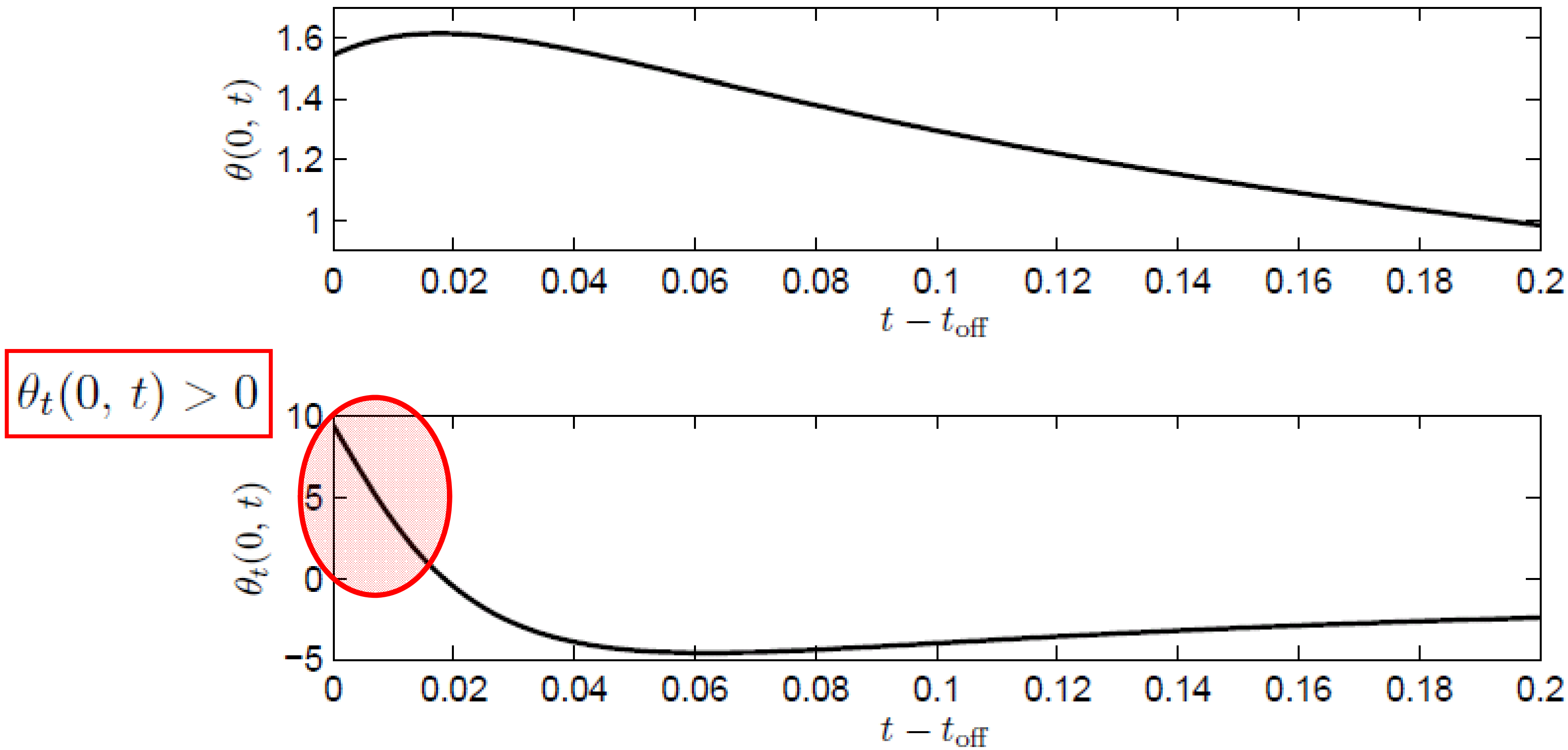
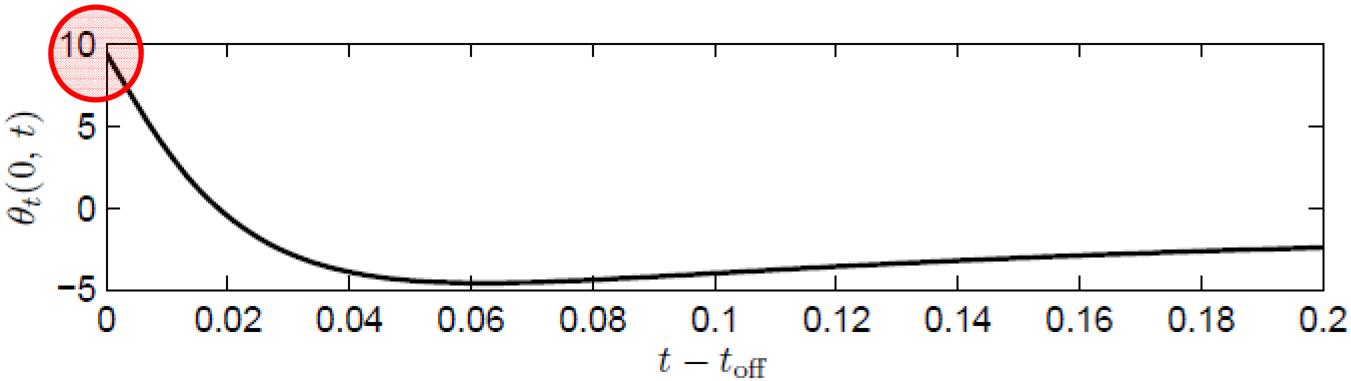


Antiparallel rubbed anchoring

- We can calculate the initial value of θ_t using the initial state $\theta(z,0)$ and the decoupled equation

$$\left(1+\frac{\lambda m^2(\theta)}{\zeta g(\theta)}\right)\theta_t = (k\cos^2\theta+\sin^2\theta)\theta_{zz} + (1-k)\sin\theta\cos\theta(\theta_z)^2 + \Delta e\,(U_z)^2\sin\theta\cos\theta + \frac{\mathcal{A}\lambda m(\theta)}{\mathcal{B}\zeta g(\theta)},$$

$\theta(z,0)$



Antiparallel rubbed anchoring

- For instance, if we apply a voltage V, create a director distortion, and then remove the voltage, we can calculate the initial director response: an initial decrease or increase in the director angle.

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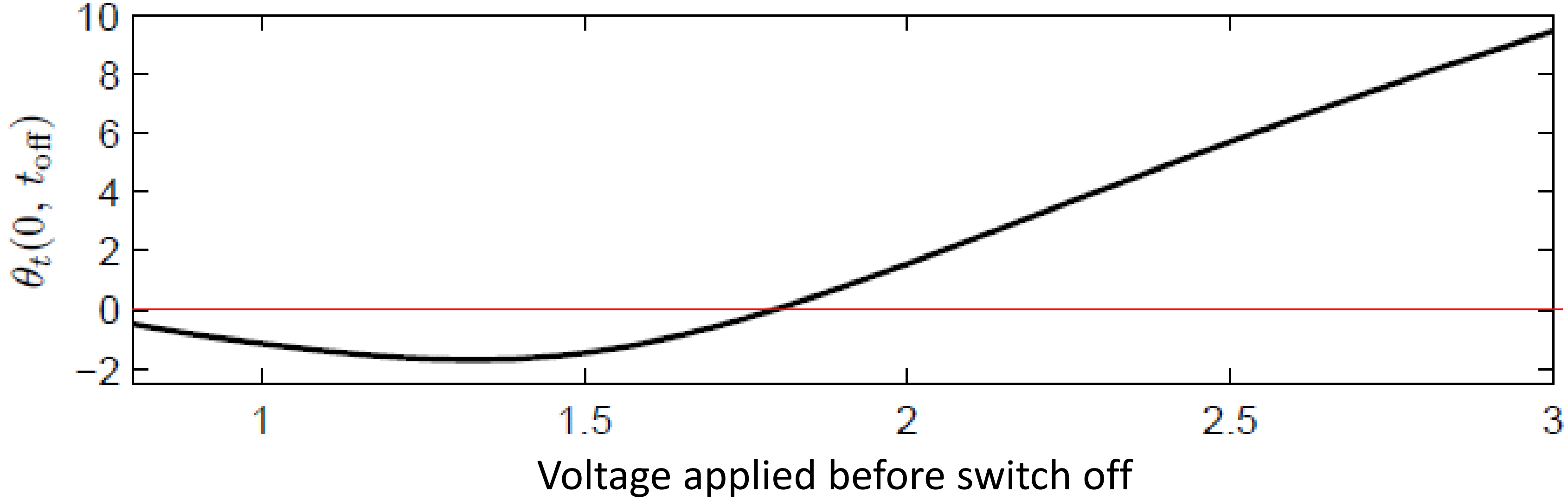

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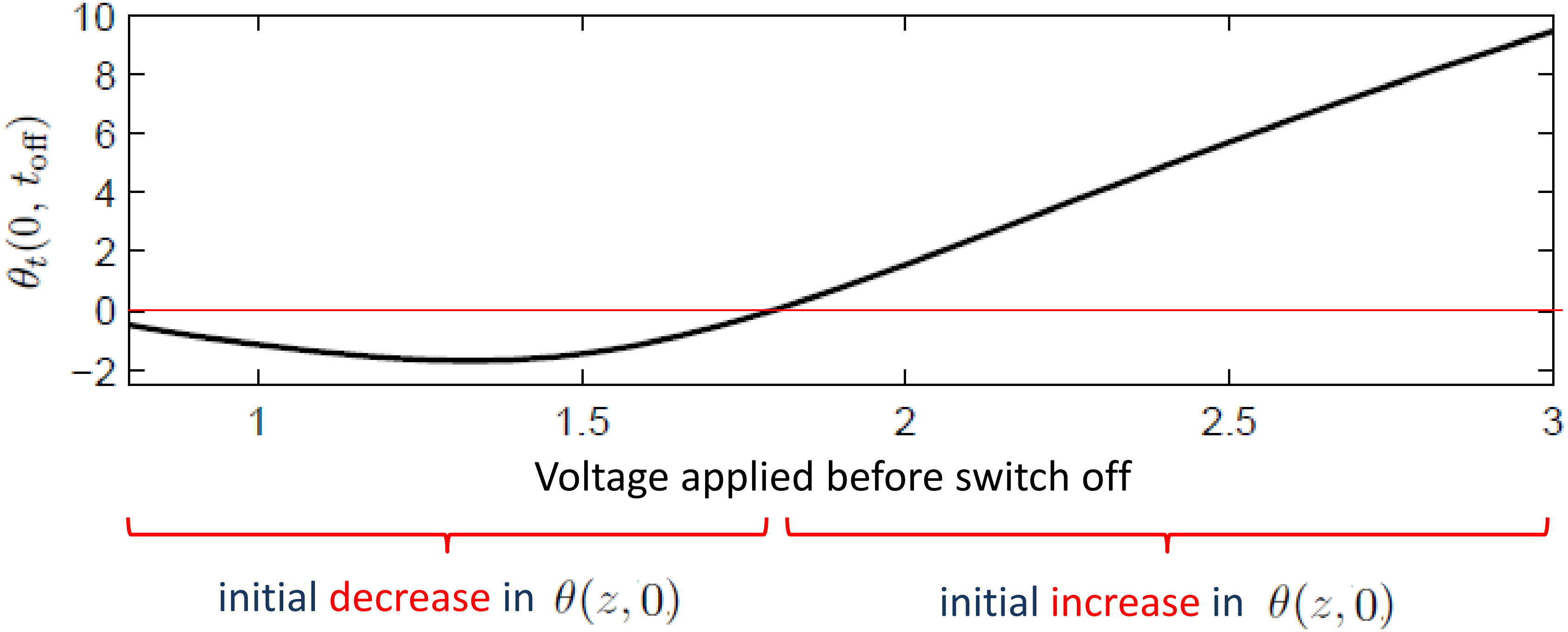
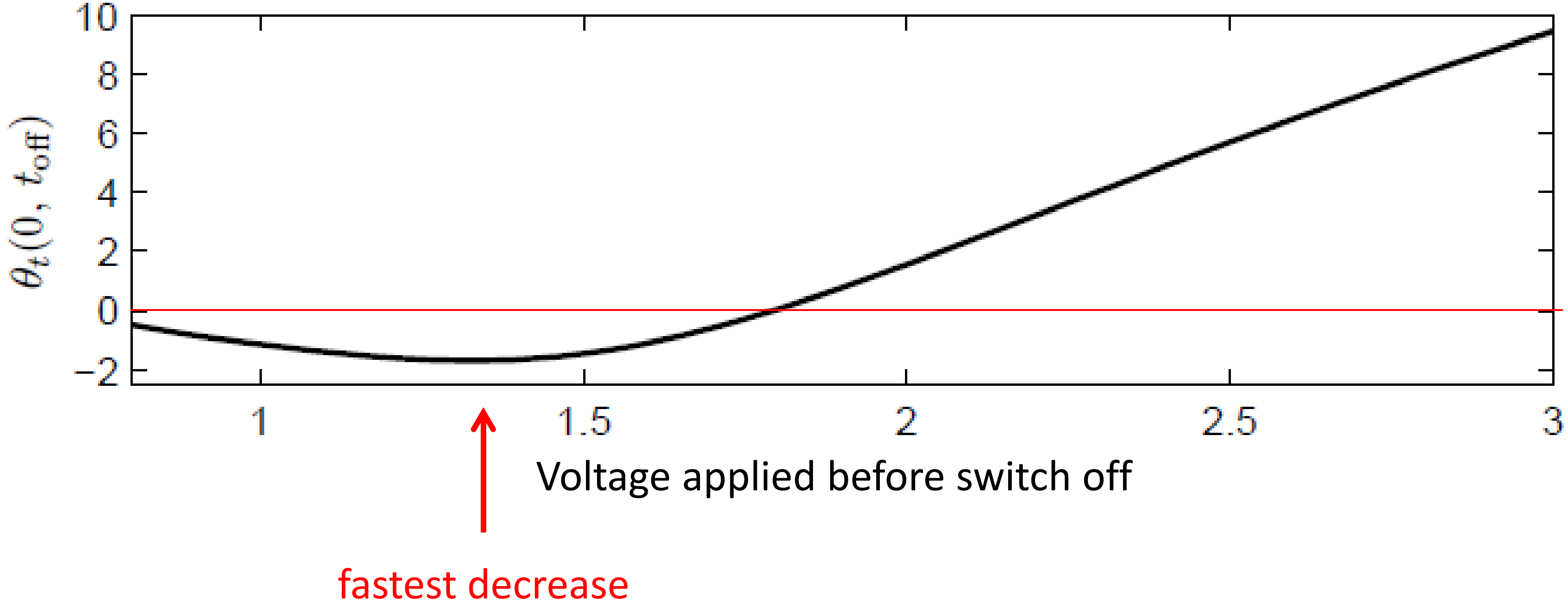


Antiparallel rubbed anchoring

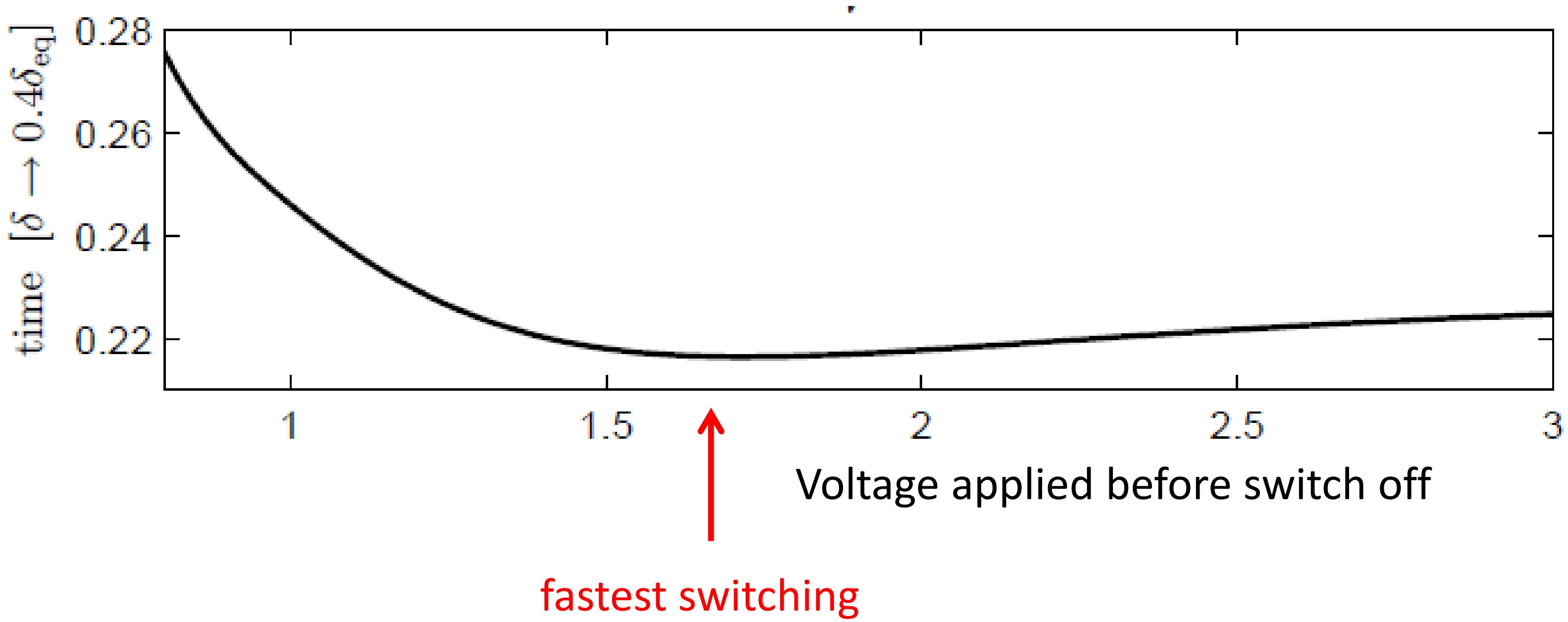
- From this simple evaluation of the right-hand side of the evolution equation we can find the fastest decrease in director angle



Antiparallel rubbed anchoring

- This compares well to a full simulation of switching

Time taken for optical retardation in the cell to reduce by 60%



Linearisation

- If we assume that we start from a state where the director is aligned with the electric field, $\theta=\pi/2$, then we can perturb about this state to obtain

$$u_\tau = u_{xx} - \frac{\alpha}{2}\int_{-1}^{1} u_{xx}\, dx$$

(τ is now time and x is space coordinate, u is the perturbation)

- It is then possible to prove that, for sufficiently large applied voltages, there will be kickback.

Linearisation

- For sufficiently large applied voltages there will be kickback
- Need to show that the initial state for high voltages is in the set

$$S_\alpha := \left\{ u \in Y \, | u_{xx}(0) - \frac{\alpha}{2} \int_{-1}^{1} u_{xx} \, dx > 0 \right\}$$

(i.e. $u_\tau(0, \tau) > 0$)

- a regular perturbation expansion for large V gives an initial state in which
  - $u_{xx}(0)$ is exponentially small
  - $\int_{-1}^{1} u_{xx} \, dx$ is negative and of order V (large)

- What else is possible?

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with boundary conditions $\theta(\pm 1,\, t)=\sigma^{\pm}$