Line defects in a variational model

Radu IGNAT
Department of Mathematics
University Paris-Sud 11 (Orsay)

joint work with Benoit Merlet (Ecole Polytechnique)

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Line-energy Ginzburg-Landau model.

Let $\Omega \subset \mathbb{R}^2$ be an open bounded set.

Our configurations: $2D$ vector fields $u : \Omega \to \mathbb{R}^2$ with $\nabla \cdot u = 0$ in Ω.

Rescaled $2D$ Ginzburg-Landau energy:

$$E_\varepsilon(u) = \varepsilon \int_\Omega |\nabla u|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega (1 - |u|^2)^2 \, dx,$$

where $\varepsilon > 0$ is a small parameter.

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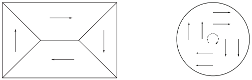
Main features of the model as $\varepsilon \to 0$:

- rigidity of (finite energy) limit configurations:

$$|u_0| = 1 \text{ and } \nabla \cdot u_0 = 0 \text{ in } \Omega.$$

Locally, $u_0 = \nabla^\perp \psi_0 : |\nabla \psi_0| = 1$.



$\psi_0 = \mathrm{dist}(\cdot, K)$, K rectangle or circle.

- energy concentration on line-defects;
- vortex defects are zero energy states.

Outline of the talk.

- Physical motivation.
- Asymptotics as $\varepsilon \to 0$.
- Entropies for the eikonal equation.
- Study of the reduced model. Minimizing configurations.

Motivation. Liquid crystals. 1/2

$\Omega \subset \mathbb{R}^3$ nematic liquid crystal, $u : \Omega \to \mathbb{S}^2$ optical axis.
The Oseen-Frank free-energy density:

$$W(\nabla u, u) = k_1(\nabla \cdot u)^2 + k_2(u \cdot \nabla \times u)^2 + k_3|u \times (\nabla \times u)|^2 + (k_2 + k_4)\Big(\operatorname{tr}(\nabla u)^2 - (\nabla \cdot u)^2 \Big).$$

$$\text{AIM:} \quad \min\Big\{ \int_\Omega W(\nabla u, u)\, dx \ : \ \text{boundary conditions on } u \Big\}.$$

Special case:

$$W(\nabla u, u) = |\nabla u|^2 \quad \text{if } k_1 = k_2 = k_3 \text{ and } k_4 = 0$$

$\Rightarrow$ minimizing harmonic maps $u : \Omega \to \mathbb{S}^2$ $\Rightarrow$ vortex point defects
(Shoen-Uhlenbeck '82, Brezis-Coron-Lieb '86)

For general $W(\nabla u, u)$, solutions u have defects of dimension less than 1.
(Hardt-Kinderlehrer-Lin '86)

Motivation. Liquid crystals. 2/2

Smectic state: $u = \frac{\nabla\varphi}{|\nabla\varphi|}$ with $\varphi : \Omega \to \mathbb{R}$.
Level sets of φ form a system of parallel surfaces.

Relaxation problem (Aviles-Giga model for gradient fields):

$$AG_\varepsilon(\varphi) \;:=\; \int_\Omega \left[\varepsilon^2|\nabla\nabla\varphi|^2 + (|\nabla\varphi|^2 - 1)^2\right] dx$$

with $\varepsilon > 0$ a small parameter.

Expected behavior as $\varepsilon \downarrow 0$: If φ_ε minimizes AG_ε (under some BC), then $\nabla\varphi_\varepsilon \to \nabla\varphi_0$, $|\nabla\varphi_0| = 1$.

Question: What is the limit (rescaled) energy

$$\frac{1}{\varepsilon}AG_\varepsilon(\varphi) = \int_\Omega \left[\varepsilon|\nabla\nabla\varphi|^2 + \frac{1}{\varepsilon}(|\nabla\varphi|^2 - 1)^2\right] dx \quad \text{as } \varepsilon \to 0?$$

Motivation. Micromagnetics. 1/2

$\Omega \subset \mathbb{R}^3$, $m : \Omega \to \mathbb{S}^2$ magnetization,

$$E^{3D}(m) = \underbrace{d^2 \int_\Omega |\nabla m|^2}_{\text{Exchange energy}} + \underbrace{\int_\Omega \varphi(m)}_{\text{Anisotropy}} + \underbrace{\int_{\mathbb{R}^3} |H(m)|^2}_{\text{Stray field energy}}$$

1. Exchange energy penalizes variations.
2. Anisotropy ($\varphi(m) = Qm_3^2$) favors in-plane magnetizations.

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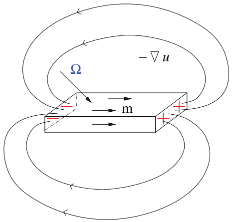
1. Exchange energy penalizes variations.
2. Anisotropy ($\varphi(m) = Qm_3^2$) favors in-plane magnetizations.
3. Stray-field energy penalizes the divergence $\nabla \cdot (m\mathbf{1}_\Omega)$.

$$\begin{cases} \nabla \times H(m) & = 0 \\ \nabla \cdot H(m) & = -\nabla \cdot (m\,\mathbf{1}_\Omega) \end{cases} \quad \text{in} \quad \mathbb{R}^3,$$

i.e., $H(m) = \nabla(-\Delta)^{-1}\nabla \cdot (m\mathbf{1}_\Omega)$.



Assumptions:

- $\Omega = \omega \times \mathbb{R}$ with $\ell = \mathrm{diam}(\omega)$,
- m is x_3-invariant, i.e., $(m', m_3) = m(x_1, x_2) : \omega \to \mathbb{S}^2$.

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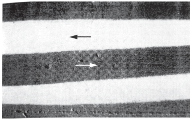
Regime: $Q \ll 1, \ \frac{d}{\ell} \ll \sqrt{Q}$.



Bloch walls

$\Rightarrow$ the stray field energy is strongly penalized $\Rightarrow \nabla \cdot m' = 0$ in ω.

The rescaled $2D$ energy per unit length ($\omega := \frac{\omega}{\ell}$):

$$F_\varepsilon(m) := \varepsilon \int_\omega |\nabla m|^2 + \frac{1}{\varepsilon} \int_\omega m_3^2$$

with $m : \omega \subset \mathbb{R}^2 \to \mathbb{S}^2$, $\nabla \cdot m = 0$ in ω and $\varepsilon = \frac{1}{\sqrt{Q}} \frac{d}{\ell} \ll 1$.

Asymptotics of the line-energy Ginzburg-Landau model.

Context: $\Omega \subset \mathbb{R}^2$, $m_\varepsilon : \Omega \to \mathbb{R}^2$ with $\nabla \cdot m_\varepsilon = 0$ in Ω.

Rescaled Ginzburg-Landau functional:

$$E_\varepsilon(m_\varepsilon) = \varepsilon \int_\Omega |\nabla m_\varepsilon|^2 \, dx + \frac{1}{\varepsilon} \int_\Omega g\left(\left|1 - |m_\varepsilon|^2\right|\right) dx.$$

Here, $\varepsilon \downarrow 0$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is an increasing function with $g(0) = 0$.

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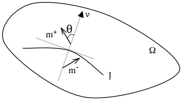
Expectation: if $E_\varepsilon(m_\varepsilon) \leq C$ then $m_\varepsilon \overset{L^2}{\to} m$ with $|m| = 1$ & $\nabla \cdot m = 0$ and

E_ε $\Gamma-$converges to $\mathcal{I}_f$ as $\varepsilon \downarrow 0$.



The line-energy $\mathcal{I}_f$ is defined on jump line-singularities J of m:

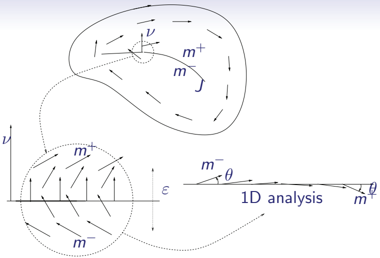
$$\mathcal{I}_f(m) = \int_J f(|m^+ - m^-|) \, d\mathcal{H}^1,$$

where $f : [0, 2] \to \mathbb{R}_+$ is a cost function.

One-dimensional analysis.

Question: What is the relation between the cost function f and the potential g?

Ansatz: $E_\varepsilon(m_\varepsilon)$ concentrates on $1D$ transition layers of length scale ε corresponding to line singularities of m.



$1D$ transition layer: $m_\varepsilon(x+t\nu) = \cos\theta\ \nu + u(\frac{t}{\varepsilon})\ \nu^\perp$ with $u \overset{t\to\pm\infty}{\longrightarrow} \pm\sin\theta$

$$\min\left\{\int_{\mathbb{R}}\left[(u'(s))^2 + g\left(|\sin^2\theta - u^2(s)|\right)\right] ds \ :\ u \overset{t\to\pm\infty}{\longrightarrow} \pm\sin\theta\right\}$$

$$= 4\int_0^{\sin\theta}\sqrt{g(\sin^2\theta - u^2)}\, du = f(|2\sin\theta|) = f(|m^+ - m^-|).$$

In particular, if $g(t) = t^p$ then $f(t) = c(p)t^{p+1}$.

Γ-convergence.

Goal: Prove that E_ε Γ-converges to $\mathcal{I}_f$ as $\varepsilon \downarrow 0$ in the L^2-topology, i.e.,

- Lower bound: if $m_\varepsilon \to m$ in $L^2(\Omega)$ then

$$\liminf_{\varepsilon \downarrow 0} E_\varepsilon(m_\varepsilon) \geq \mathcal{I}_f(m).$$

- Upper bound: For any limit m, there exists $m_\varepsilon \to m$ in $L^2(\Omega)$ s.t.

$$\limsup_{\varepsilon \downarrow 0} E_\varepsilon(m_\varepsilon) \leq \mathcal{I}_f(m).$$

- Compactness: If $E_\varepsilon(m_\varepsilon) \leq C$ then $m_\varepsilon \to m$ in $L^2(\Omega)$ (for a subsequence).

In particular, minimisers of E_ε converge to minimisers of $\mathcal{I}_f$.

Technics (for our model):
1) Upper bound: Construction (recovery sequence) based on the $1D$ analysis.
2) Compactness & Lower bound: Entropy method for eikonal equation.

Particular case: $g(t) = t^2$.

Aviles-Giga model: $m_\varepsilon : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$, $\nabla \cdot m_\varepsilon = 0$ in Ω,

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Compactness (Ambrosio-DeLellis-Mantegazza, DKMO):

If $E_\varepsilon(m_\varepsilon) \leq C$ then (for a subsequence) $m_\varepsilon \overset{L^2(\Omega)}{\to} m$,

$$\nabla \cdot m = 0 \quad \textit{and} \quad |m| = 1 \quad \textit{in } \Omega.$$

Remark: In general, m is not BV (i.e., ∇m is not finite Radon-measure).

However, m has an $\mathcal{H}^1$ rectifiable jump set J (DeLellis-Otto).

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$\Gamma-$convergence (Aviles-Giga, Jin-Kohn, Conti-DeLellis): $\nabla \cdot m_\varepsilon = 0$ in Ω,

$$E_\varepsilon(m_\varepsilon) = \int_\Omega \varepsilon |\nabla m_\varepsilon|^2 + \frac{(1 - |m_\varepsilon|^2)^2}{\varepsilon} \quad \overset{\Gamma}{\to} \quad \mathcal{I}_f(m) = \frac{1}{3}\int_J |m^+ - m^-|^3 \, d\mathcal{H}^1$$

$$\text{with} \quad f(t) = \frac{t^3}{3}.$$

The case of general potential g.

Question: Is the Γ-convergence result true for general potential g?
Do we have matching of lower and upper bound,
i.e., the $2D$ lower bound of $E_\varepsilon \overset{???}{=}$ the $1D$ upper bound $\mathcal{I}_f$?

Counterexamples: any cost function f such that $\mathcal{I}_f$ is not lower semicontinuous (in L^2).

Ambrosio-DeLellis-Mantegazza: If $f(t) = t^p$, $p > 3$ ($\Rightarrow g(t) = c(p)t^{p-1}$) then $\mathcal{I}_f$ is not l.s.c. and therefore, $\mathcal{I}_f$ is not a Γ-limit in L^2-topology.

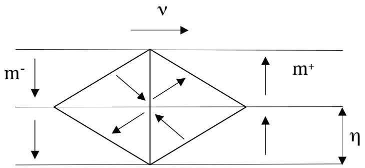

Figure: Microstructure: as $\eta \to 0$, it tends to the jump (m^-, m^+) in direction ν.

Another counterexample: The cross-tie wall microstructure.

Alouges-Rivière-Serfaty: $f(2\sin\theta) = \sin\theta - \theta\cos\theta$ for $0 \leq \theta \leq \pi/2$
$\Rightarrow \mathcal{I}_f$ is not l.s.c. (in L^2)

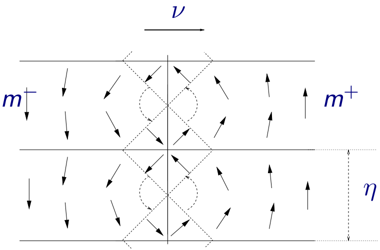

Figure: A "cross tie" microstructure. When $\eta \downarrow 0$, we obtain a jump between m^- and m^+ in direction ν.

Goal: Characterise the cost function f leading to a l.s.c. line-energy $\mathcal{I}_f$?

Relaxation. Lower semicontinuity (l.s.c.).

We extend $\mathcal{I}_f : L^2(\Omega, \mathbb{R}^2) \to \mathbb{R}_+ \cup \{+\infty\}$:

$$\mathcal{I}_f(m) = \begin{cases} \int_J f(|m^+ - m^-|)\, d\mathcal{H}^1 & \text{if } m \in BV_{div}(\Omega, \mathbb{S}^1), \\ +\infty & \text{if } m \in L^2(\Omega, \mathbb{R}^2) \setminus BV_{div}(\Omega, \mathbb{S}^1), \end{cases}$$

with $BV_{div}(\Omega, \mathbb{S}^1) = \big\{ m \in L^2 \,:\, \nabla m \text{ is a measure }, \, |m| = 1, \, \nabla \cdot m = 0 \big\}$.

Relaxed functional: $\overline{\mathcal{I}_f} : L^2(\Omega, \mathbb{R}^2) \to \mathbb{R}_+ \cup \{+\infty\}$ is the l.s.c. hull of $\mathcal{I}_f$:

$$\overline{\mathcal{I}_f}(m) = \inf_{\{m_k\}} \left\{ \liminf_{k \to \infty} \mathcal{I}_f(m_k) \,:\, m_k \to m \text{ in } L^2 \right\}, \quad \forall m \in L^2(\Omega, \mathbb{R}^2).$$

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Definition: We call $\mathcal{I}_f$ l.s.c. (in L^2) if

$$\mathcal{I}_f(m) = \overline{\mathcal{I}_f}(m) \quad \text{for every } m \in BV_{div}(\Omega,\mathbb{S}^1).$$

Question: Characterise the cost functions f such that $\mathcal{I}_f$ is l.s.c.

Idea: Use the concept of entropy.

Entropies.

Starting point: $m : \Omega \to \mathbb{S}^1$ and $\nabla \cdot m = 0$ in $\Omega \subset \mathbb{R}^2$.
Scalar conservation law:

$$m = (u, \pm\sqrt{1-u^2}) = (u, h(u)),\ (x, y) \to (t, s),$$

$$\nabla \cdot m = 0 \qquad \text{writes} \qquad \partial_t u + \partial_s [h(u)] = 0.$$

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$\to$ concept of entropy solution;

(entropy, entropy-flux) pair: (η, q) such that $q' = \eta' h'$.

If u is a smooth solution, then $\underbrace{\nabla \cdot [\eta(u), q(u)]}_{entropy\ \ production} = \partial_t[\eta(u)] + \partial_s[q(u)] = 0.$

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Definition of ENTROPY (DKMO) : $\Phi \ : \ \mathbb{S}^1 \to \mathbb{R}^2$ smooth s.t.

$$m \text{ smooth }, |m| = 1 \text{ and } \nabla \cdot m = 0 \Longrightarrow \nabla \cdot [\Phi(m)] \ = \ 0.$$

Lemma: Φ is an entropy iff $\frac{d}{d\theta}\Phi(z) \cdot z \ = \ 0, \ z = e^{i\theta} \in \mathbb{S}^1$

$$\text{iff there exists } \varphi \in C^\infty_{per}[0, 2\pi] \text{ s.t. } \Phi(z) = \varphi(\theta) z + \frac{d}{d\theta}\varphi(\theta) z^\perp.$$

Good cost functions.

AIM: Define cost functions $f := c_S$ associated to subsets of entropies $S \subset ENT$ such that $\mathcal{I}_f$ is l.s.c.

Lemma (I. - Merlet): If $m \in BV_{div}(\Omega, \mathbb{S}^1)$, then the entropy production generated by an entropy $\Phi \in ENT$ is given by

$$\nabla \cdot [\Phi(m)] \; = \; [\Phi(m^+) - \Phi(m^-)] \cdot \nu(x) \, \mathcal{H}^1 \llcorner J.$$

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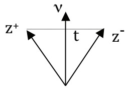
For $S \subset ENT = \{entropies\}$ we define the cost function $c_S : [0,2] \to \mathbb{R}_+$ by



$$c_S(t) \;:=\; \sup \Big\{ [\Phi(z^+) - \Phi(z^-)] \cdot \nu \;:\; \Phi \in S, \; z^+, z^-, \nu \in \mathbb{S}^1, \; (z^+ - z^-) \cdot \nu = 0, \; |z^+ - z^-| = t \Big\}.$$

Assumptions on $S \subset ENT$:

- S symmetric ($S = -S$),
- S equivariant ($S = R \circ S \circ R^{-1}$ for every rotation $R \in SO(2)$).

($\Rightarrow$ Invariance of line-energies over all isometries.)

Theorem (I. - Merlet): Let $S \subset ENT$ be symmetric and equivariant. Then $c_S : [0,2] \to \mathbb{R}_+$ is a good cost function, i.e., $\mathcal{I}_{c_S}$ is l.s.c.

$$\overline{\mathcal{I}}_{c_S}(m) = \mathcal{I}_{c_S}(m) = \int_{J(m)} c_S\big(|m^+ - m^-|\big)\, d\mathcal{H}^1, \ \forall m \in BV_{div}(\Omega, \mathbb{S}^1).$$

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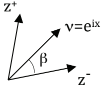
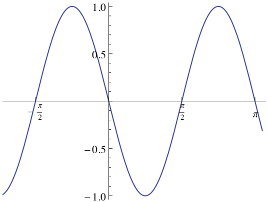
Particular case: if S is generated by only one entropy $\Phi \in ENT$, i.e.,

$$S = \langle \Phi \rangle = \{\pm R \circ \Phi \circ R^{-1} \ : \ R \in SO(2)\},$$

set $\lambda(\theta) = \frac{d}{d\theta}\Phi(z) \cdot z^\perp$ for every $z = e^{i\theta} \in \mathbb{S}^1$ and

$$c_{\langle \Phi \rangle}(2 \sin \beta) = \sup_{x \in [0, 2\pi]} \left| \int_{-\beta}^{\beta} \lambda(x - \theta) \sin \theta \, d\theta \right|, \ \beta \in [0, \frac{\pi}{2}].$$





Remark: Above, **sup** is achieved at $x = 0$ if λ is $\pi-$periodic, odd and convex on $(0, \frac{\pi}{2})$ and symmetric at $\frac{\pi}{4}$.

Examples of good cost functions.

• Jin-Kohn (for the Aviles-Giga model): $f(t) = t^3$ is a good cost function c_S where S is generated by only one entropy:

$$\Phi(m_1, m_2) = 8(m_2^3,\ m_1^3) \quad \text{with} \quad \frac{d}{d\theta}\Phi(z) \cdot z^\perp = \lambda(\theta) = -6\sin(2\theta).$$

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$$\{\Phi_\theta,\ 0 \le \theta \le \pi/2\}.$$

Conjecture (Ambrosio-DeLellis-Mantegazza): if $p \in [1,3]$ and $f(t) = t^p$ for $t \in [0,2]$, then $\mathcal{I}_f$ is l.s.c.

Theorem (I. - Merlet): For every $p \in [1,3]$, there exists $S \subset ENT$ symmetric and equivariant s.t. $c_S(t) = t^p$ for $t \in [0,\sqrt{2}]$.

Minimising the reduced model of line-energy.

Goal: Study $\inf_{m \cdot n = 0 \text{ sur } \partial\Omega} \mathcal{I}_f(m)$.

Theorem (I. - Merlet): If $f = c_S$ is a good cost function associated to $S \subset ENT$ such that

$$\inf_{t>0} \frac{f(t)}{t^3} > 0,$$

then $\{\overline{\mathcal{I}}_f(m) < \alpha\}$ is relatively compact in $L^2 \Rightarrow$ existence of a minimiser of $\overline{\mathcal{I}}_f$, i.e., $\min_{B.C.} \overline{\mathcal{I}}_f$.

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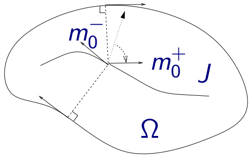
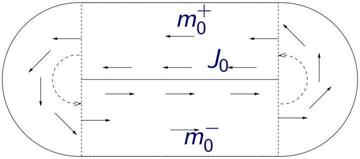
Stronger boundary condition:
$m = n^\perp$ on $\partial\Omega$.
Natural candidate (viscosity solution):
$m_0(x) := \nabla^\perp \operatorname{dist}(x, \partial\Omega)$.

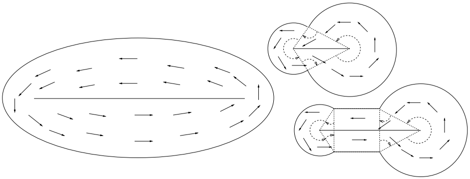


Conjecture (AG, ADM): If $f(t) = t^p$, $1 \le p \le 3$ and Ω is convex then m_0 is a minimiser of $\mathcal{I}_f$.



Proposition (I.-Merlet) If Ω is a stadium and c_S is a good cost function, then m_0 is a minimiser of $\mathcal{I}_{c_S}$.

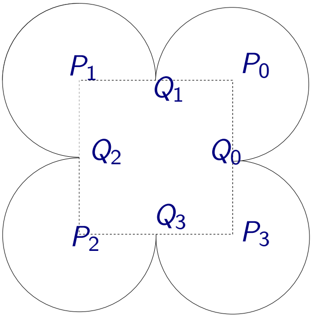
The result also holds if $f = c_{\langle \Phi \rangle}$ for certain entropies $\Phi \in ENT$ and Ω is an ellipse, or a union of two disks...



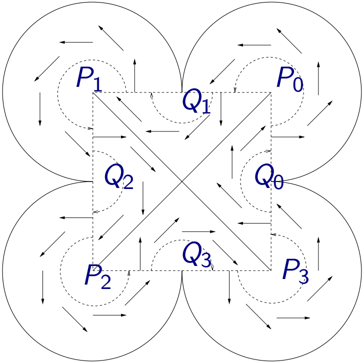
Counterexamples

Proposition (I.-Merlet): There exists a universal non-convex domain Ω s.t. m_0 is NOT a minimiser of $\mathcal{I}_f$ for every $f(t) \gneqq 0$ on $[\frac{\sqrt{2}}{2}, 2]$.

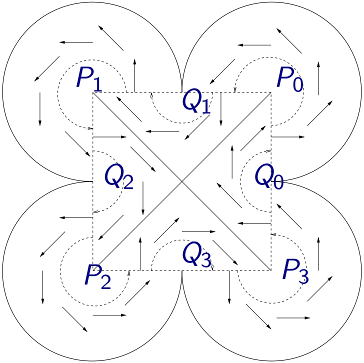
Proposition bis : If f is fixed then we can find such a domain Ω that is smooth $C^{1,1}$.



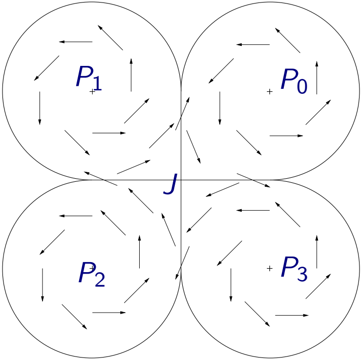
m_0

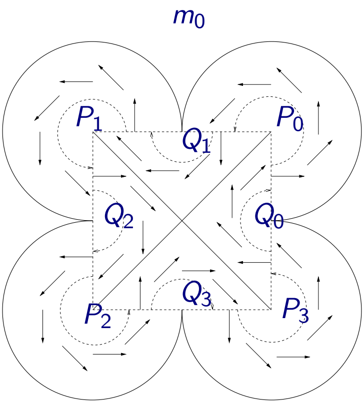


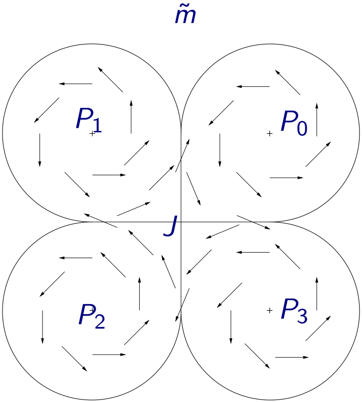
m_0



$\tilde{m}$

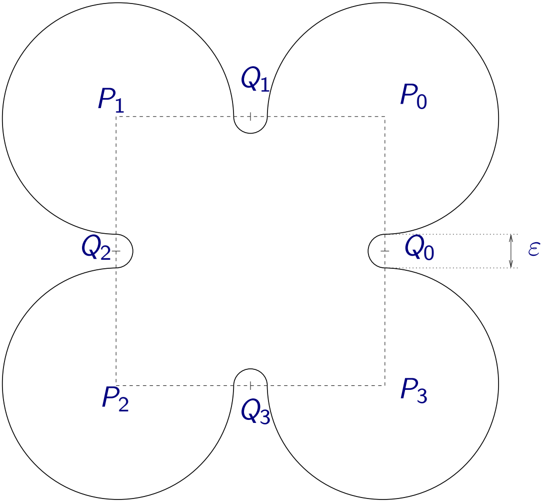




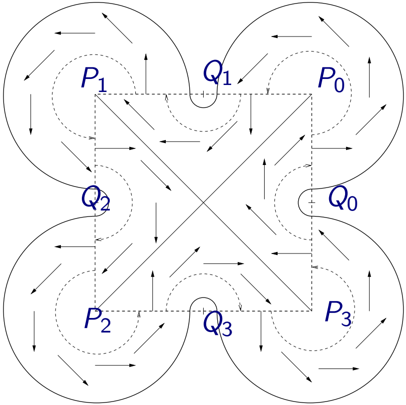


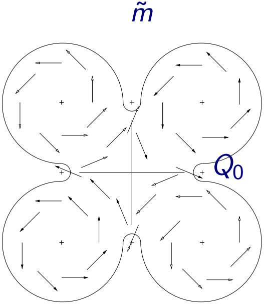
$$\mathcal{I}_f(\tilde{m}) \;=\; \frac{\sqrt{2}}{2}\mathcal{I}_f(m_0) < \mathcal{I}_f(m_0).$$

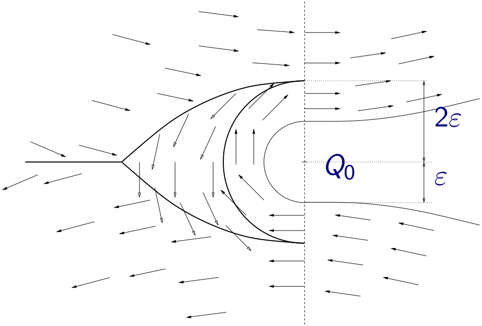
Construction of $\Omega_\varepsilon := \Omega \setminus \{\partial\Omega + B(0,\varepsilon)\}$

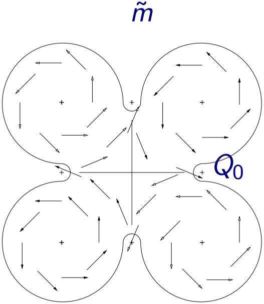


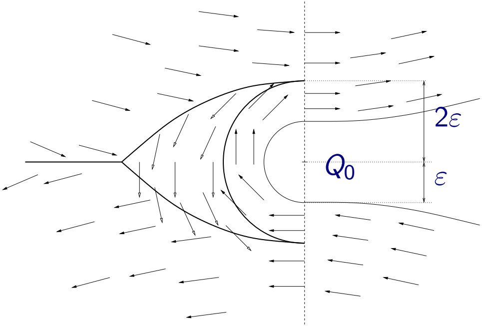
m_0




$\tilde{m}$
Q_0


2ε
Q_0
ε





$$\mathcal{I}_f(\tilde{m}) \;=\; \frac{\sqrt{2}}{2}\,\mathcal{I}_f(m_0) + O(\varepsilon).$$

Thank you for your attention!