A simple model of topographic Rossby waves in the ocean

V. Teeluck, S.D. Griffiths, C.A. Jones

University of Leeds

April 18, 2013
1. Outline of talk

1. Introduction and motivation

2. Analytical model for topographic Rossby waves

3. Comparison of analytical and numerical work
2. Surface signature of topographic Rossby waves

Fig 2d from Wright and Xu (2004, A-O)
3. Motivation

Environmental conditions

- Rapid depth change from continental shelf through continental slope to deep ocean
- Temperature variations (warm sea surface), salinity variations, sea ice...
- Tidal forcing, wind forcing, discharge from rivers

With the aid of applied mathematics

- Start from equations of motion describing underlying physics (conservation of mass, conservation of momentum, etc.)
- Derive a reduced model incorporating what is thought to be the essential physics
- Derive wave-like solutions in this model, which can be compared to observations and used to predict behaviour elsewhere
- Mathematical approach gives understanding of the structure of the solutions
4. Navier-Stokes Equations

\[
\begin{align*}
\frac{\partial u}{\partial t} + (u \cdot \nabla) u + 2\Omega \times u &= -\frac{1}{\rho} \nabla p + g + \nu \nabla^2 u \\
\nabla \cdot u &= 0
\end{align*}
\]

where \( u = (u, v, w) \).

Figure: local plane called the f-plane at a fixed latitude \( \theta_0 \)
4. Navier-Stokes Equations

\[
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} + 2\Omega \times \mathbf{u} = -\frac{1}{\rho} \nabla p + \mathbf{g} + \nu \nabla^2 \mathbf{u} \\
\nabla \cdot \mathbf{u} = 0
\]

where \( \mathbf{u} = (u, v, w) \).

Figure: local plane called the f-plane at a fixed latitude \( \theta_0 \)
Hydrostatic in $z$, so $p = p_0 - \rho g(z - h(x, y, t))$ giving

$$\frac{\partial u}{\partial t} - fv = -g \frac{\partial h}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g \frac{\partial h}{\partial y}$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (Hu) = 0$$

where $f = 2\Omega \sin \theta_0$, $\theta_0$ (constant) is latitude, $H$ is undisturbed depth.
6. The topographic Rossby wave (TRW) mechanism

Motion of columns of water

\[ \frac{DQ}{Dt} = 0 \text{ where } Q = \frac{f + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}}{H + h} \]

Step profile

\[ \omega = \frac{f|H_1 - H_2|}{H_1 + H_2} \]

Longuet-Higgins, JFM (1969)
7. Scaling analysis

Scale lengths by $L$, velocities by $U$, and depth by $H_0$. To target TRWs, we scale time by $f^{-1}$ and $h$ by $fUL/g$:

$$
\frac{\partial u}{\partial t} - v = - \frac{\partial h}{\partial x}
$$

$$
\frac{\partial v}{\partial t} + u = - \frac{\partial h}{\partial y}
$$

$$
\left(\frac{L}{L_R}\right)^2 \frac{\partial h}{\partial t} + \left(\frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y}\right) = 0
$$

where $L_R = \sqrt{gH_0/|f|}$ is the Rossby radius of deformation and a typical value is 2000km. Since $L \ll L_R$, we drop the $\partial h/\partial t$ term.
8. Wave-like solutions

\[
\begin{align*}
\frac{\partial u}{\partial t} - fv &= -g \frac{\partial h}{\partial x}, \\
\frac{\partial v}{\partial t} + fu &= -g \frac{\partial h}{\partial y}, \\
\frac{\partial (Hu)}{\partial x} + \frac{\partial (Hv)}{\partial y} &= 0.
\end{align*}
\]

So we introduce a streamfunction

\[
Hu = -\psi_y \quad Hv = \psi_x.
\]

When \( H = H(y) \), we seek wave-like solutions of the form

\[
\psi(x, y, t) = \text{Re} \left( H^{1/2} \phi(y) e^{i(kx - \omega t)} \right).
\]

Eliminating \( h \) gives

\[
\phi'' + \left[ \frac{1}{2} \frac{H''}{H} - \frac{3}{4} \left( \frac{H'}{H} \right)^2 + \frac{fk}{\omega} \left( \frac{H'}{H} \right) - k^2 \right] \phi = 0. \tag{1}
\]

This is an eigenvalue problem for \( \omega \) given \( k \) and suitable boundary conditions.
9. A model slope: \( H = H_0 + \Delta H \tanh(y/L_s) \)

\[
\frac{d^2 \phi}{d\hat{y}^2} + \left[ -k^2 L_s^2 - \frac{\Delta H}{H} \tanh \hat{y} + \frac{3}{4} \left( \frac{\Delta H}{H} \right)^2 \sech^2 \hat{y} - \frac{f k \Delta H L_s}{\omega H} \right] \phi = 0. \quad (2)
\]
10. Small topography assumption: $\Delta H \ll H_0$

$$\frac{d^2 \phi}{d\hat{y}^2} + \left( - k^2 L_s^2 + \frac{f k \Delta H L_s}{\omega H_0} \text{sech}^2 \hat{y} \right) \phi = 0,$$

to leading order.

In an unbounded domain, this has solutions of the form

$$\phi = \text{sech} |k L_s| \hat{y} \sum_{n=0}^{p} C_n \tanh^n \hat{y}$$

When this series terminates, we obtain an eigenvalue relation for $\omega$:

$$\omega = \frac{4 f k L_s \Delta H}{H_0 \left[ \left( 2n + 1 + |2k L_s| \right)^2 - 1 \right]} \quad \text{where} \quad n \in \mathbb{N}_0.$$ 

So

- there are infinitely many eigenvalues
- as $n \to \infty$, $\omega \to 0$
- when $n = 0$ and $L_s \to 0$, we obtain Longuet-Higgins (1969) solution

$$\omega \to \frac{f \Delta H \text{sgn}(k)}{H_0}. $$
11. Cross-shore structure

- Modes with even/odd $n$ are symmetric/anti-symmetric in $\hat{y}$

$$n = 0, \phi = \text{sech} |k_L| \hat{y}$$

$$n = 1, \phi = \text{sech} |k_L| \hat{y} \tanh \hat{y}$$

$$n = 2, \phi = \left[1 - \frac{3}{4} (4 + 3|k_L|) \tanh^2 \hat{y}\right] \text{sech} |k_L| \hat{y}$$

Figure: Fixed $H_1$ with $\Delta H = 0.5\text{km}$ and $\Delta H = 3\text{km}$
12. Comparison of analytical/numerical work

Figure: $kL_s = 1\ H_1 = 4\text{km}\ L_s = 50\text{km}$. 

- $n=0$: numerical
- $n=0$: analytical
- $n=0$: second order correction
- $n=1$: numerical
- $n=1$: analytical
- $n=2$: numerical
- $n=2$: analytical
13. Why do analytics/numerics match so well?

\( n = 0 \) mode

Rearranging (2) so that we have the leading order solution on the left gives,

\[
\frac{d^2 \phi}{d \hat{y}^2} + \left[ -k^2 L_s^2 + \frac{f k \Delta H L_s}{\omega H_0} \text{sech}^2 \hat{y} \right] \phi
\]

\[
= \left( \frac{\Delta H}{H_0} \tanh \hat{y} + \frac{3(\Delta H)^2}{4 H_0^2} \text{sech}^2 \hat{y} \right) \text{sech}^2 \hat{y} \phi + \cdots
\]

Expanding \( \omega \) and \( \phi \) in powers of \( \Delta H / H_0 \) we have

\[
\omega = \omega_0 \left[ 1 + a_1 \left( \frac{\Delta H}{H_0} \right) + a_2 \left( \frac{\Delta H}{H_0} \right)^2 + \cdots \right]
\]

we find

\[
a_1 = 0 \quad \text{and} \quad a_2 = \frac{|k L_s| (2 + |k L_s|)^2}{2 (3 + 2|k L_s|) (1 + |k L_s|)^2}
\]
14. Comparing analytical/numerical results for $a_2$

\[ a_{2\text{analytical}} = \frac{|kL_s|(2 + |kL_s|)^2}{2(3 + 2|kL_s|)(1 + |kL_s|)^2}, \quad a_{2\text{numerical}} = \left(\frac{\omega}{\omega_0} - 1\right)/\left(\frac{\Delta H}{H_0}\right)^2. \]

**Figure**: $a_2$ against $kL_s$ for step values 50m to 400m

Note that $|a_2| < 0.27$. 
15. TRWs with a shelf and coast


\[ \omega = \frac{f\left|\left(H_2 - H_1\right)\left(1 - e^{-2|k|L}\right)\right|}{H_2\left(1 - e^{-2|k|L}\right) + H_1\left(1 + e^{-2|k|L}\right)}. \]

Figure: Comparison of TRW dispersion relations
16. Cross-shore structure with a coastline

TRW \( n=0, \ H_0 = 2150 \text{ m}, \ \Delta H = 1850 \text{ m}, \ k = 2 \times 10^{-5} \text{ m}^{-1} \)
17. Summary

- Topographic Rossby waves are likely to be excited in the ocean above steep topographic slopes.
- We studied the form of these waves above a smooth slope with a tanh profile.
- There exist infinitely many topographic Rossby waves.
- Analytical predictions (based on a small topographic amplitude expansion) for frequency and cross-slope structure are excellent.
- These same predictions also work well in the presence of a coastline, providing the wave is sufficiently trapped above the continental slope.
- Future work will be directed towards understanding how these trapped barotropic waves may generate radiating internal waves, at tidal frequencies.