Continuum Models of Two Phase Flow in Porous Media

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Applications

Oil recovery: water flood

Water infiltration under gravity (DiCarlo 2004)

Carbon Sequestration in Deep Saline Aquifers
Framework: Conservation of mass; Darcy’s law; incompressibility

Buckley-Leverett flux $f(u)$ (1942) $u$ is saturation
non-convex conservation law $u_t + f(u)_x = 0$

Modeling: pressure difference $p^c$ across interfaces: interfacial energy

Classical approach: $p^c = p^e(u)$ equilibrium pressure; decreasing in $u$
$u_t + f(u)_x = -(H(u)p^e(u)_x)_x$

Netherlands group: undercompressive shocks; sharp shocks
Pop, van Duijn, Peletier, Cuesta, Fan (2006-2012)
• 2-d stability/instability for Lax shocks: hyperbolic/elliptic system

• Phase field modeling: Juanes (2008), Witelski (1998), based on polymer mixtures: introduce stored energy function with capillary effects

  Capillary tube gas/liquid finger: degenerate fourth order PDE

  Dynamical systems trick for traveling waves

• Carbon sequestration: Huppert, Neufeld, Hesse, ... (2008-2011). Discontinuous dependence in conservation law to allow for deposition of CO$_2$ bubbles as plume propagates in brine (deep saline aquifer)

  Track interface; non-standard wave interactions
2-D Model

\( u(x, y, t) \) : **saturation** (vol. fraction) of water, \((1 - u)\) : oil saturation

\( p(x, y, t) \) : **pressure** in water

**Conservation of mass** with Darcy’s law: velocity \( \mathbf{v} = -\lambda(u)\nabla p \):

\[
\phi \frac{\partial u}{\partial t} - \nabla \cdot (\lambda(u)\nabla p) = 0
\]

\( \phi = \text{porosity}, \lambda(u) = \frac{Kk(u)}{\mu} \)

\( K = \text{absolute permeability}, \ k(u) = \text{relative permeability}, \ \mu = \text{viscosity} \)

**Incompressibility**: \( \nabla \cdot \mathbf{v}^{\text{Total}} = 0 \)

\[
\nabla \cdot \left( \Lambda(u)\nabla p + \lambda^{\text{oil}}(u)\nabla p^c(u) \right) = 0
\]

\( \Lambda = \lambda^{\text{water}} + \lambda^{\text{oil}}, \quad p^c = p^{\text{oil}} - p : \text{capillary pressure}; \)

*For simplicity, neglect gravity, set \( \phi = 1 \).*
One-Dimensional Equation for \( u(x, t) : u_t - (\lambda p_x)_x = 0 \)

\[
\nabla \cdot \mathbf{v}^{\text{Total}} = 0 \quad \Rightarrow \quad \mathbf{v}^{\text{Total}} = (v^T, 0) = \text{constant, after rescaling } t
\]

Thus,

\[
\Lambda p_x + (\Lambda - \lambda)p^c_x = -v^T
\]

Eliminate \( p_x \):

\[
p_x = -\frac{v^T}{\Lambda(u)} - \left(1 - \frac{\lambda(u)}{\Lambda(u)}\right)p^c_x
\]

Then

\[
 u_t + f(u)_x = -(H(u)p^c_x)_x
\]

\[
f(u) = v^T \frac{\lambda(u)}{\Lambda(u)}, \quad H(u) = \frac{\lambda(u)}{\Lambda(u)}(\Lambda(u) - \lambda(u))
\]
\[ u_t + f(u)_x = - (H(u) \rho^c_x)_x \]

\[ f(u) = \nabla^T \frac{\lambda(u)}{\Lambda(u)}, \quad H(u) = \frac{\lambda(u)}{\Lambda(u)} (\Lambda(u) - \lambda(u)) \]

**Fractional flow rate**  
**Dissipation coefficient**

Recall: \( \Lambda(u) \) is total mobility: \( \Lambda(u) - \lambda(u) = \lambda^{oil}(1 - u) \)

\( \lambda, \lambda^{oil} \) are positive, increasing convex functions, with \( \lambda(0) = \lambda^{oil}(0) = 0 \).
$u_t + f(u)_x = -(H(u)p^c_x)_x$, but what to do about $p^c$?

- Buckley-Leverett (1942): *ignore it*: $p^c = 0$
  
  scalar non-convex conservation law $u_t + f(u)_x = 0$
  
  Lax-Oleinik construction: *rarefaction-shock interface*

- Classical approach: equilibrium capillary pressure depends on saturation: $p^c = p^e(u)$, a decreasing function: $\frac{d}{du}p^e(u) < 0$.

  The PDE is parabolic but degenerate at $u = 0, 1$.

  Max principle: $0 \leq u \leq 1$ is positively invariant.

  Lax-Oleinik solutions replaced by smooth approximation.
Hassanizadeh and Gray (1991, 1993): rate dependence in $p^c$:

$$p^c = p^e(u) - \tau u_t$$

Modified Buckley-Leverett equation:

$$u_t + f(u)_x = - (H(u) \frac{\partial}{\partial x} p^e(u))_x + \tau (H(u) u_{tx})_x$$

Equation is dissipative-dispersive. No maximum principle, but positive invariance of $0 \leq u \leq 1$ (Yabin Fan thesis 2012)

Analogous equation: modified BBM-Burgers:

$$u_t + (u^3)_x = \alpha u_{xx} - \beta u_{xxt}$$

Traveling waves for undercompressive shocks (Spayd thesis, 2012)

Explicit construction similar to modified KdV-Burgers:

$$u_t + (u^3)_x = \alpha u_{xx} - \beta u_{xxx}$$

(Jacobs, McKinney, Shearer, 1995; LeFloch book, 2002)
Traveling Wave Solutions

\[ u(\xi) = u(x - st) \Rightarrow -su' + (f(u))' = [H(u)u']' - s\tau [H(u)u'']' \]

Integrating with boundary conditions \( u(\pm\infty) = u_{\pm} \) leads to second order ODE: 
\[ -s(u - u_{-}) + f(u) - f(u_{-}) = H(u)u' - s\tau H(u)u'' \]

Rescale \( u = u\left(\frac{x-st}{\sqrt{s\tau}}\right) \) and write as first order system of ODEs:

\[
\begin{align*}
    u' &= v \\
    v' &= \frac{1}{\sqrt{s\tau}}v + \frac{1}{H(u)} [s(u - u_{-}) - f(u) + f(u_{-})]
\end{align*}
\]

Singular at \( u = 0, \ u = 1 \), where \( H(u) = 0 \)
Saddle-Saddle Connections

Phase portraits of ODE system: up to three equilibria
\((u_e, 0): (u_\pm, 0), (u_{\text{mid}}, 0), s = (f(u_e) - f(u_-))/(u_e - u_-)\)

Analyze phase plane with separation function \(R(u_-, u_+, \tau)\)

Saddle-saddle connection: \(R = 0\): heteroclinic orbit from \((u_-, 0)\) to \((u_+, 0)\) undercompressive shock
Numerical procedure

For \( \tau < \infty \), fix \( u_\text{--} \) and determine \( u_\text{+} = u_\Sigma(u_\text{--}) \) for which
\[
R(u_\text{--}, u_\text{+}, \tau) = 0
\]
Monotonicity of \( u_\Sigma(u_\text{--}) \) from Melnikov integral: derivatives of \( R \)
For \( \tau = \infty \), determine \((u_\text{--}, u_\Sigma)\) by using exact integral:

\[
\frac{1}{2} \frac{d}{du} v^2 = \frac{1}{H(u)} [s(u - u_\text{--}) - f(u) + f(u_\text{--})] \equiv G(u; u_\text{--}, u_\text{+})
\]

Thus: connection when
\[
\int_{u_\text{--}}^{u_\text{+}} G(u; u_\text{--}, u_\text{+}) \, du = 0 : \quad u_\text{+} = u_\Sigma(u_\text{--})
\]

Asymptotics in corners: e.g., \( u_\text{--} \to 0, \ u_\text{+} \to 1 : \)

\[
u_\text{+} = u_\Sigma(u_\text{--}) \sim 1 - u_\text{--}^M
\]
Fix $\tau > 0$, plot $(u_-, u_+)$ with saddle-saddle connection $u_- \rightarrow u_+$
Scalar conservation law: \( u_t + f(u)_x = 0 \)

**Idealization:** no capillary pressure; characteristic speed \( f'(u) \)

Scale invariant solutions: building blocks for solving initial value problem

**Rarefactions**

\[
 u(x, t) = \begin{cases} 
 u_- & \text{if } x < f'(u_-)t \\
 r\left(\frac{x}{t}\right) & \text{if } f'(u_-)t \leq x \leq f'(u_+)t \\
 u_+ & \text{if } x > f'(u_+)t 
\end{cases}
\]

**Shocks**

\[
 u(x, t) = \begin{cases} 
 u_- & \text{if } x < st \\
 u_+ & \text{if } x > st 
\end{cases}
\]

Rankine-Hugoniot condition: shock speed

\[
 s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}
\]
Admissible Shocks

\( f'(u_+) < s < f'(u_-) \)  \text{[Lax]} \\

A shock is admissible if there is a traveling wave from \( u_- = u_{\text{mid}} \) (unstable node) to \( u_+ \) (saddle point).

\( s > f'(u_\pm) \)  \textbf{PLUS:} shock has corresponding traveling wave:

\textit{undercompressive} shock \( \Sigma \).

\textit{Jacobs, McKinney, Shearer (1995), LeFloch Book (2002)}
Buckley-Leverett solution 1942

Solve conservation law with initial jump from all water $u_- = 1$ to all oil $u_+ = 0$: water flooding:

$$u_t + f(u)_x = 0, \quad u(x,0) = \begin{cases} 1 & \text{if } x < 0 \\ 0 & \text{if } x > 0 \end{cases}$$

Solution: rarefaction from $u_- = 1$ to $u^*$; Lax shock from $u^*$ to $u_+ = 0$:

*rarefaction-shock.*
The Riemann Problem: Classical Solution

Solve conservation law with initial jump from $u_\ell$ to $u_r$

$$u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} u_\ell & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{cases}$$

R: Rarefaction Wave
S: Lax Shock
RS : Rarefaction - Shock
The Riemann Problem; dynamic capillary pressure

\[(RP): \quad u_t + f(u)_x = 0, \quad u(x, 0) = \begin{cases} 
  u_\ell & \text{if } x < 0 \\
  u_r & \text{if } x > 0 
\end{cases}\]

Solutions of (RP): leading order approximations to solutions of modified Buckley-Leverett equation

\[\text{R: Rarefaction Wave} \]
\[\text{S: Admissible Lax Shock} \]
\[\Sigma : \text{Undercompressive Shock} \]

For every \((u_\ell, u_r) \in (0, 1)\), there is a solution that stays in \((0, 1)\) even though \(\Sigma\) solutions are nonmonotonic.
PDE Simulations

Full PDE: \[ u_t + f(u)_x = (H(u)u_x)_x + \tau (H(u)u_{tx})_x \]

\[ u_\ell \in (0, u_r) : \text{rarefaction wave} \quad u_\ell \in (u_r, u_{mid}) : \text{Lax shock} \]
\( u_\ell \in (u_{\text{mid}}, u_\Sigma) : \)

- **Lax shock** \( u_\ell \) to \( u_\Sigma \)
- **undercompressive shock** \( u_\Sigma \) to \( u_r \)

\[ \begin{align*}
\text{Graph 1:} & \quad \begin{array}{c}
\text{Graph 2:}
\end{array}
\end{align*} \]

\( u_\ell \in (u_\Sigma, 1) : \)

- **rarefaction wave** \( u_\ell \) to \( u_\Sigma \)
- **undercompressive shock** \( u_\Sigma \) to \( u_r \)
‘Sharp’ Traveling waves TW \( u = u(x - st) \)

\[
\begin{align*}
  u_t + f(u)_x &= -\left( H(u) \frac{\partial}{\partial x} p^e(u) \right)_x + \tau (H(u) u_{tx})_x \\
  -s(u - u_-) + f(u) - f(u_-) &= -H(u) p^e(u)' - s\tau H(u) u'', \quad u(\pm\infty) = u_\pm
\end{align*}
\]

after one integration

Singular at \( u = 0, 1 \)  Can have connections up to \( u = 1 \), or down to \( u = 0 \) depending on relative permeability functions at \( u = 0, 1 \)

Example: \( \lambda(u) = u^{3/2}; \quad \lambda^{oil}(u) = (1 - u)^{3/2} \)

Pop, Cuesta, van Duijn (2011): sharp shock TW with corner at \( u = 1 \)
Relative permeabilities

Integrable case:
$K(u) = u^{3/2}$

Phase Portraits:
Traveling waves

$u = 0.95, u_{\star} = 1$

$Lax$ shock profile

$u = 1, u_{\star} = 0.025$

$sharp$ shock
Planar Lax shock $u_t + f(u)_x = 0$

 Interface $\bar{x} = st$; velocity $v_\pm = -\lambda_\pm \partial_x p_\pm$; mobility $\lambda_\pm = Kk(\bar{u}_\pm)/\mu$

 Continuous pressure $p = \bar{p}_\pm (x - st) = -\frac{v_\pm}{\lambda_\pm} z = -\frac{v_T}{\lambda_\pm} z$, $z = x - st$;

 shock location: $\bar{z} = \bar{x} - st = 0$

 Shock speed 
 
 \[ s = \frac{f(\bar{u}_+) - f(\bar{u}_-)}{\bar{u}_+ - \bar{u}_-} \]

 Lax condition: 
 
 \[ f'(\bar{u}_+) < s < f'(\bar{u}_-) \]

 Characteristic speeds $f'(\bar{u}_\pm)$
2-dimensional linearized stability of Lax shocks

2-d equations with \( p_c \equiv 0 \) variables \( u, p \) saturation, pressure:

\[
\begin{align*}
\frac{\partial u}{\partial t} - \nabla \cdot (\lambda(u) \nabla p) &= 0 \\
\nabla \cdot (\Lambda(u) \nabla p) &= 0
\end{align*}
\] (1)

Interface \( x = \hat{x}(y, t) \), normal in \( t, x, y : (−\hat{x}_t, 1, −\hat{x}_y) \)
Jump condition at shock: \( ([q] = q_+ − q_-) \)

\[
\begin{align*}
−\hat{x}_t[u] − [\lambda(u)p_x] + \hat{x}_y[\lambda(u)p_y] &= 0 \\
−[\Lambda(u)p_x] + \hat{x}_y[\Lambda(u)p_y] &= 0
\end{align*}
\] (2)

Base shock: \( u = \bar{u}_\pm, p = \bar{p}_\pm(x − st), \hat{x} = st \), constants \( \bar{u}_\pm, \bar{p}_\pm, V \)

\[
s = \frac{f(\bar{u}_+ − f(\bar{u}_-))}{\bar{u}_+ − \bar{u}_-}, \quad f(u) = v^T \frac{\lambda(u)}{\Lambda(u)}, \quad \bar{p}_\pm = −\frac{v^T}{\Lambda(\bar{u}_\pm)}
\]
Equations \( \frac{\partial u}{\partial t} - \nabla \cdot (\lambda(u)\nabla p) = 0, \ \nabla \cdot (\Lambda(u)\nabla p) = 0 \)

\[
\frac{\partial u}{\partial t} - \lambda'(u) \nabla u \cdot \nabla p - \lambda(u) \Delta p = 0
\]

\[
\Lambda'(u) \nabla u \cdot \nabla p + \Lambda(u) \Delta p = 0
\]

(1)

Linearize about shock on each side:

\( u = \bar{u} + U(x, y, t), p = \bar{p}(x - st) + P(x, y, t) \)

\[
U_t - \lambda'(\bar{u}) \bar{p} U_x - \lambda(\bar{u}) \Delta P = 0
\]

\[
\Lambda'(\bar{u}) \bar{p} U_x + \Lambda(\bar{u}) \Delta P = 0
\]

(2)

Thus, \( \Delta P = -\frac{\Lambda'(\bar{u})}{\Lambda(\bar{u})} \bar{p} U_x \). Subs. into \( U_t \) equation:

\[
U_t - \bar{p} \left( \lambda'(\bar{u}) - \lambda(\bar{u}) \frac{\Lambda'(\bar{u})}{\Lambda(\bar{u})} \right) U_x = 0
\]
CLAIM: Lax shock $\implies U$ has to be zero.

$$U_t - \bar{p} \left( \lambda'(\bar{u}) - \lambda(\bar{u}) \frac{\Lambda'(\bar{u})}{\Lambda(\bar{u})} \right) U_x = 0, \quad \bar{u} = \bar{u}_\pm$$

Let $U(x, y, t) = w(x - st)e^{i\alpha y}e^{\sigma t}$. Then, with $z = x - st$, $w' = \frac{dw}{dz}$:

$$\sigma w - sw' - \bar{p} \lambda(\bar{u}) \left( \frac{\lambda'(\bar{u})}{\lambda(\bar{u})} - \frac{\Lambda'(\bar{u})}{\Lambda(\bar{u})} \right) w' = 0$$

But $f(u) = v^T \lambda(u)/\Lambda(u)$, $\bar{p} \lambda(\bar{u}) = -f(\bar{u})$ so we have

$$\sigma w = (s - f'(\bar{u})) w', \quad \text{with solutions } w(z) = a_\pm e^{\beta_\pm z},$$

$$\sigma = \beta_\pm (s - f'(\bar{u}_\pm))$$

For a Lax shock, $f'(\bar{u}_+) < s < f'(\bar{u}_-)$. Thus, $Re \sigma \geq 0$ implies $w(z)$ does not decay at $z = \pm \infty$ unless $w \equiv 0$. Consequently, $U \equiv 0$
2-d stability: perturb pressure in linearized equations

\[ u = \bar{u}_\pm; \quad p = \bar{p}_\pm z + P; \quad P(z, y, t) = q_\pm(z)e^{i\alpha y + \sigma t} \]

\[ \hat{z} = \hat{x} - Vt = a e^{i\alpha y + \sigma t}, \quad z = x - \hat{x} \]

We seek \( \sigma = \sigma_1 \alpha, \ \alpha > 0. \) Linearized equations: \( \Delta P = 0 : \)

\[ q''_\pm - \alpha^2 q_\pm = 0 \quad (1) \]

Decaying solutions: \( q_\pm = b_\pm e^{\mp \alpha z}, \ \pm z > 0, \)

Next: linearize jump conditions: two equations for three coefficients

\( a, b_-, b_+ \)
Linearized jump conditions (no perturbation on \( u \))

1. \[ \sigma a[\bar{u}] + [\lambda(\bar{u})q'] = 0 \]  
2. \[ [\Lambda(\bar{u})q'] = 0 \]

Equation (2):

\[ \Lambda(\bar{u}_+)b_+ = -\Lambda(\bar{u}_-)b_- \]  

Then (1) implies

\[ b_-\Lambda(\bar{u}_-)(f(\bar{u}_+) - f(\bar{u}_-)) = -\frac{\sigma}{\alpha}a(\bar{u}_+ - \bar{u}_-)v^T \]

But \( (f(\bar{u}_+) - f(\bar{u}_-))/(\bar{u}_+ - \bar{u}_-) = s \), the shock speed, so

\[ b_-\Lambda(\bar{u}_-)s = -a\frac{\sigma}{\alpha}v^T \]
2-dimensional stability: third equation for $a, b_{\pm}$

Third equation comes from continuity of pressure (linearized) at $z = \hat{z}(y, t) = ae^{i\alpha y + \sigma t}$

$$p = \bar{p}_{\pm} z + q_{\pm}(z)e^{i\alpha y + \sigma t}, q_{\pm} = b_{\pm} e^{\mp\alpha z} \quad (\pm z > 0)$$

Consequently,

$$\bar{p}_{+} a + b_{+} = \bar{p}_{-} a + b_{-}$$

(5)

Thus,

$$\left( \bar{p}_{+} - \bar{p}_{-} \right) a = -\frac{\sigma}{\alpha} s a \left( \frac{1}{\Lambda(\bar{u}_{+})} + \frac{1}{\Lambda(\bar{u}_{-})} \right), \quad \text{(from (3,4))}$$

Since $\bar{p}_{\pm} = -\frac{\nu^{T}}{\Lambda(\bar{u}_{\pm})}$, we obtain, for $a \neq 0$:

$$\frac{\sigma}{\alpha} = s \frac{\Lambda(\bar{u}_{-}) - \Lambda(\bar{u}_{+})}{\Lambda(\bar{u}_{-}) + \Lambda(\bar{u}_{+})}$$
Interpretation of stability condition: quadratic relative permeabilities: \( k(u) = \kappa u^2 \)

Lax shocks for \( u_+ < u_- \leq u^* \), \( f'(u^*) = (f(u_+) - f(u^*))/(u_+ - u^*) \)

Stability boundary: \( u_+ = -u_- + \frac{2M}{M+1} \)

\( M = 0.2 \quad \frac{2M}{M + 1} = \frac{1}{3} \)

Inflection point \( I \) of \( f(u) \) at \( u_I = 0.2591 \)

S: Stable Lax shocks

U: Unstable Lax shocks
Fingering Instability - Zhengzheng's Simulations

Full parabolic/elliptic system with capillary pressure $p^c = -u + \tau u_t$

Crank-Nicolson time-step; upwind advection; periodic side boundary conditions, moving frame, plot middle contour $u = \frac{1}{2}(u_- + u_+)$

$\Delta t = O(10^{-3})$, $\Delta x = \Delta y = O(10^{-2})$

Initial condition: $u = \text{randomly perturbed hyperbolic tangent}$

$u_- = 0.2, u_+ = 0, M = 0.05$
Numerical Simulations - Stable case

\[ M = 0.2 \]

\[ u_- = 0.25, \ u_+ = 0 \]

Oil-water mixture displaces oil

Initial perturbation decays

Zoomed-in contour
$M = 0.2$

$u_\neg = 0.25, u_\neg = 0.15$

weak Lax shock

Initial perturbation grows $\Rightarrow$ fingering instability
Conclusions: stability/fingering instability in 2-d

Analysis of hyperbolic/elliptic system linearized around 1-d shock
Linear dependence of growth rate $\sigma$ on transverse wave number $\alpha$
distinguishes stable waves from unstable

Numerical simulations of full parabolic/elliptic system confirm results;
(Riaz and Tchelepi (2006) also conducted numerical experiments)

Weak Lax shocks may be stable or unstable

Oil/water mixture displacing oil can be stable.

Back to basics: capillary pressure in a capillary tube

\[ \gamma = 1/Ca = \sigma/U\eta \]

\( Ca =: \) capillary number

balance between surface tension \( \sigma \) and viscosity \( \eta \).

Saturation \( u \): area fraction

\( u(x, t) = r(x)^2/R^2. \)

Write equation in form

\[ u_t + f(u)_x = -\gamma \partial_x [f(u)\lambda(u)\partial_x\psi] \]

\[ f(u) = u/(u + (1 - u)^3) \]

Fractional flow rate for air/water (Bear 1972)

\[ \lambda(u) = (1 - u)^3 \]

\[ \psi = \frac{\delta F}{\delta u}, \quad F: \text{free energy} \]

\[ F = F^0(u) + \kappa(u)F^1(u_x) \]

\( F^0: \) bulk free energy

\( F^1(u_x): \) interfacial energy

\( \kappa(u): \) interfacial energy weight
Phase field model: \[ F = F^0(u) + \kappa(u)F^1(u_x) \]

Cueto-Felgueroso and Juanes, PRL (2008, 2011)

Simple choices:

\[
F^0(u) = -\frac{\Sigma}{R}(1-u)^2u^2 + 2\frac{\gamma_{wg}\cos\theta}{R}(1-u)^2
\]

Double-well potential; tangent construction (like equal area rule)

\[ \Sigma = \gamma_{sw} - \gamma_{sg} - \gamma_{wg} < 0 \quad \text{spreading parameter} \]

\[ \theta: \text{static contact angle} \quad \text{Young-Laplace law at contact line} \]

\[ \gamma_{sw} - \gamma_{sg} = \gamma_{wg}\cos\theta, \quad \text{set} \quad \gamma = \gamma_{wg} \]

\[
F^1(u_x) = \Gamma u_x^2
\]

\[ \Gamma = C(\theta)\gamma R \]
Phase field model:  \( u_t + f(u)_x = -\gamma \partial_x [f(u)\lambda(u)\partial_x \psi] \)

Variational derivative:  \( \psi = \frac{\delta F}{\delta u} \)

\[
\psi = -C_1 u(1 - u)(1 - 2u) + C_2 \sqrt{\kappa(u)} \partial_x (\sqrt{\kappa(u)} \partial_x u), \quad C_1 > 0, \ C_2 > 0
\]

Chemical potential for polymer mixture:


\[
\partial_t u + \partial_x f(u) = \gamma \partial_x [H(u)Q(u)\partial_x u] - \gamma \partial_x \left[ H(u)\partial_x C_2 \sqrt{\kappa(u)} \partial_x (\sqrt{\kappa(u)} \partial_x u) \right]
\]

\( Q(u) = C_1(6u^2 - 6u + 1), \quad H(u) = f(u)\lambda(u) \)

Cahn-Hilliard type equation
Now choose $\kappa(u)$ so that a static bubble is an equilibrium solution

$$u(x) = r^2 = 1 - x^2, \quad 0 \leq x \leq 1. \quad \text{Thus} \quad u' = -2x = -2\sqrt{1-u}, \quad u'' = -2.$$ 

Ends of bubble are spherical caps (take contact angle $\theta = 0$, or $\pi$):

Here, tube radius $R = 1$, $\theta = \pi$, $'= \partial_x$

$$\partial_x [f(u)\lambda(u)\partial_x\psi] = 0$$

Consequently,

$$f(u)\lambda(u)\psi' = c_0 = 0, \quad \text{since} \quad f(0) = 0$$
\( \psi(u) = -C_1 G(u) + C_2 \sqrt{\kappa} \partial_x(\sqrt{\kappa} \partial_x u) \); \quad G(u) = u(1 - u)(1 - 2u) \\
\quad f(u) \lambda(u) \psi' = 0 \implies \psi = c_1 = 0, \quad \text{since } \kappa(0) = 0 \\

Thus,

\[ m(u)(m(u)u')' = Ku(1 - u)(1 - 2u), \quad K = C_2 / C_1, \quad m(u) = \sqrt{\kappa(u)}. \]

\( u(x) = 1 - x^2 \implies x = \sqrt{1 - u} : \) ODE for \( m(u) : \)

\[ m(u) \left( \frac{dm}{du} 4(1 - u) - 2m(u) \right) = KG(u) = Ku(1 - u)(1 - 2u). \]

But \( \kappa(u) = m(u)^2. \) Then

\[ \frac{d\kappa}{du}(1 - u) - \kappa = \frac{K}{2} u(1 - u)(1 - 2u). \]

LHS = \( \frac{d}{du}((1 - u)\kappa) \), so \( (1 - u)\kappa = \frac{K}{2} \left( \frac{1}{2} u^2 - u^3 + \frac{1}{2} u^4 \right) \). That is,

\[ \kappa(u) = \frac{K}{4}(1 - u)u^2 \]
General contact angles $\theta$ : Coeff. of interfacial energy $\kappa$

Similar result for general contact angle $\theta$, $0 < \theta < \pi$:

$$ u = \sec^2 \theta - x^2/R^2 $$

$$ \kappa(u) = \frac{K(\theta)u^2(1-u)^2}{4(\sec^2 \theta - u)}, \quad 0 \leq u \leq 1. $$

Idea works for more general constitutive laws

$$ \psi(u) = -C_1G(u) + C_2\sqrt{\kappa} \partial_x (\sqrt{\kappa} \partial_x u); \quad G(u) \text{ bistable} $$

$$ \frac{d}{du} \left( \kappa(u)(\sec^2 \theta - u) \right) = \frac{K(\theta)}{2} G(u) $$

$$ \kappa(u) = \frac{K(\theta) \int G(u) \, du}{4(\sec^2 \theta - u)}, \quad 0 \leq u \leq 1 $$
structure: rarefaction and leading (undercompressive) shock/TW from plateau $u = u_-$ to $u = 0$ at $x = st + x_0 < \infty$. 
Traveling Waves

\[ \partial_t u + \partial_x f(u) = \gamma \partial_x [H(u)G(u)\partial_x u] - \gamma C \partial_x [H(u)\partial_x m(u)\partial_x (m(u)\partial_x u)] \]

\[ u = u(x - st), \quad u(-\infty) = u_-, \quad u(0) = 0, \quad s = f(u_-)/u_- \]

\[-su + f(u) = \gamma H(u)G(u)u' - \gamma CH(u)(m(u)(m(u)u')')' \]

Note: constant of integration is zero
Write as first order system, \( \beta = 1/(C\gamma) \) (multiply 3rd eqn by \( m(u) \))

\[ m(u)u' = v \]
\[ m(u)v' = w \]
\[ m(u)w' = \frac{1}{C} G(u)v + \beta \frac{m(u)}{H(u)}(su - f(u)) \]

Eliminate \( m(u) \) to remove singularity at origin
ODE system - nondegenerate vector field

Eliminate $m(u)$: \[ m(u) \frac{d}{d\xi} = \frac{d}{d\eta}, \ u(\xi) = U(\eta), \ldots \]

\[
U' = V \\
V' = W \\
W' = \frac{1}{C} G(U)V + \beta \frac{m(U)}{H(U)}(sU - f(U))
\]

$(U, V, W) = (0, 0, 0)$ now a regular equilibrium

Seek solutions with $-\infty < \eta < \infty$

$(U, V, W)(-\infty) = (u_-, 0, 0), \ (u, v, w)(\infty) = (0, 0, 0)$

Need real eigenvalues at origin: $\gamma > \gamma^* : \ Ca < Ca^*$

Shooting: 1-d unstable manifold from $u_-$ to 2-d stable manifold at origin: $u_- = u_-(\beta)$. 
Recover original variable \( u(\xi) \to 0, \xi \to 0^- \)

Transform back: \( m(U(\eta)) \, d\eta = d\xi \)

\[ U(\eta) \sim ae^{\mu \eta} \text{ as } \eta \to \infty, (\mu < 0 : \text{eigenvalue}) \text{ implies} \]

\[ \xi = -\int_{\eta}^{\infty} U(y) \, dy = ae^{\mu \eta} / \mu \]

Thus, \( u(\xi) = U(\eta) \sim \mu \xi \), as \( \xi \to 0^- \), slope \( \mu < 0 \).
Many applications with two-phase porous media flow

Oil recovery context: oil/water interface; Lax shocks may be stable or unstable to 2-D perturbations; hyperbolic analysis gives condition for linear stability

Yortsos and Hickernell (1989) traveling wave stability; most unstable wave number from dispersion relation tested by Zhengzheng Hu

Phase field analysis for flow in capillary tube; undercompressive shocks, stationary compactons; degenerate diffusion traveling waves

Hassanizadeh-Gray model
degenerate diffusion-dispersion; sharp shocks