A New Theory of Micro-Robots: multiple scales & distinguished limits Newton Institute, June 28th, 2013

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Everything in Nature, Technology, Economics is oscillating, including our bodies: breathing, heartbeating, walking, swimming; also molecules, atoms, stock markets, bridges, etc.

We develop Vibrodynamics

since 2003-4. It contains:

averaging methods, multi-scale methods, asymptotics, distinguished limits, . . . + many applications; our recent applications: MHD-dynamo, acoustics, micro-robots

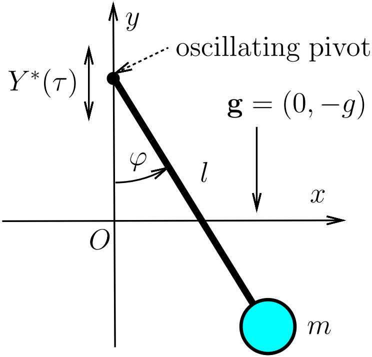
Mathematically, *Vibrodynamics* represents a theory of ODEs and PDEs with time-oscillating coefficients.

Partially similar (but more complex) theory of PDEs and ODEs with coefficients which are oscillating in space is known as *Homogenisation*.

1. We use an example of Inverted Pendulum to expose two basic elements of *Vibrodynamics*:

- the two-timing method and
- the distinguished limit theorem

2. We consider linear, triangle, and dumbbell Micro-Robots



The equation of pendulum

$$l\ddot{\varphi} = -(g + \ddot{Y}^*)\sin\varphi$$

$Y(\tau)$ - given function of $\tau \equiv \omega^* t^*$

Basic frequency $\omega_0 \equiv \sqrt{g/l}$

Small parameter $\varepsilon \equiv \omega_0/\omega^*$

e.g. $Y^* = a \sin \omega^* t^*$

Dimensionless: $t \equiv \omega_0 t^*,\ Y \equiv Y^*/l$

$$\varphi_{tt} = -(1 + Y_{tt})\sin\varphi$$

The dimensionless equation for the pendulum is

$$\varphi_{tt} + (1 + Y_{tt})\sin\varphi = 0$$

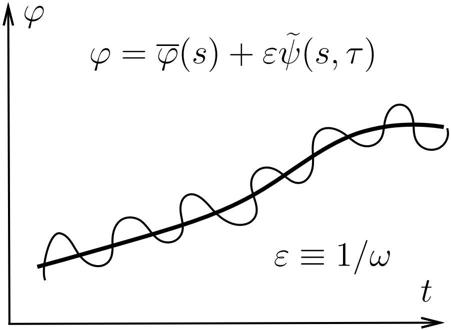
The presence of two different time-scales is enforced:

$$s \equiv t \qquad \text{and} \qquad \tau \equiv \omega t, \qquad \omega \gg 1$$

For the vibrations of pivot we accept (why?)

$$Y = \widetilde{Y} = \frac{1}{\omega}\widetilde{\xi}(\tau), \quad \text{for example,} \quad \widetilde{\xi}(\tau) = \sin\tau$$

which means that the oscillations of the pivot are asymptotically small



$\varphi(s,\tau) = \varphi(s,\tau + 2\pi)$

Dependent variables:

$s \equiv t$ and $\tau \equiv \omega t$

$$\frac{d}{dt} \equiv \left(\frac{\partial}{\partial s} + \omega\frac{\partial}{\partial \tau}\right)$$

Averaged operation

$$\langle f\,\rangle \equiv \frac{1}{2\pi}\int_0^{2\pi} f(s,\tau)\,d\tau$$

$$\langle \tilde{\psi}\,\rangle = 0$$

TEMPORARY ASSUMPTION

s and τ are two independent variables

We move from ODE to PDE

The Auxiliary PDE is

$$\left(\frac{\partial}{\partial s}+\omega\frac{\partial}{\partial \tau}\right)^2\varphi+(1+\omega\widetilde{\xi}_{\tau\tau})\varphi=0$$

which can be rewritten as

$$\varphi_{\tau\tau}+\varepsilon(2\varphi_{s\tau}+\widetilde{\xi}_{\tau\tau}\sin\varphi)+\varepsilon^2(\varphi_{ss}+\sin\varphi)=0,\quad \varepsilon\equiv 1/\omega$$

The successive approximations are:

$$\varphi=\varphi_0+\varepsilon\varphi_1+\varepsilon\varphi_2+\ldots,\quad \sin\varphi=\sin\varphi_0+\varepsilon\varphi_1\cos\varphi_0+\ldots$$

$$\varphi_{0\tau\tau}=0$$

$$\varphi_{1\tau\tau}+2\varphi_{0s\tau}+\widetilde{\xi}_{\tau\tau}\sin\varphi_0=0$$

$$\varphi_{2\tau\tau}+2\varphi_{1s\tau}+\widetilde{\xi}_{\tau\tau}\varphi_1\cos\varphi_0+\varphi_{0ss}+\sin\varphi_0=0$$

The zeroth and first approximations give

$$\varphi_0 = \overline{\varphi}_0(s), \quad \varphi_1 = \overline{\varphi}_1(s) - \widetilde{\xi}\sin\overline{\varphi}_0$$

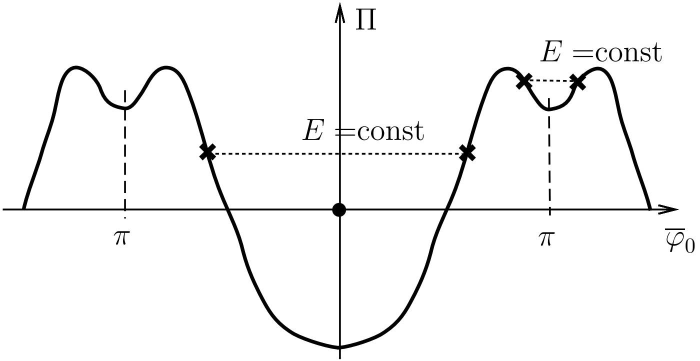
then the second approximation (after its averaging) gives

$$\overline{\varphi}_{0ss} + \sin\overline{\varphi}_0 + 2\Omega^2\sin(2\overline{\varphi}_0) = 0, \quad \Omega^2 \equiv \langle\widetilde{\xi}_\tau^2\rangle/4$$

It can be converted to the 'conservation of energy' form

$$E \equiv K + \Pi = \text{const}$$
$$K \equiv \overline{\varphi}_{0s}^2/2, \quad \Pi \equiv -\cos\overline{\varphi}_0 - \Omega^2\cos(2\overline{\varphi}_0)$$

Vibrogenic Potential Energy $\Pi = \Pi(\overline{\varphi}_0)$



$$\Pi \equiv -\cos \overline{\varphi}_0 - \Omega^2 \cos(2\overline{\varphi}_0)$$

Question 1: How the use of an auxiliary PDE instead of the original ODE can be justified?

Answer: One can justify it *a posteriori* by the substitution of $s \mapsto t$, $\tau \mapsto \omega t$ into $\varphi(s, \tau) \mapsto \varphi(t)$. The substitution of such $\varphi(t)$ into the original ODE leads to a small remainder (the right-hand side) instead of zero.

Question 2: Why have we considered only the asymptotically small vibrations of the pivot: $\widetilde{Y} = \widetilde{\xi}/\omega$ with $\xi = O(1)$?

Answer: This is the only available possibility. For all other cases asymptotic solution can not be built. It can be proved with the use of the distinguished limit theorem (see below)

Let us consider an abstract ODE for unknown function $x(t)$

$$\ddot{x} = f(x) + \omega^{\alpha}\widetilde{g}(x,\tau), \quad \alpha = \text{const}, \quad \tau \equiv \omega t$$

which represents a generalization of the pendulum equation. Again, there are two enforced time scales

$$s \equiv t, \quad \tau \equiv \omega t: \quad \frac{d}{dt} \equiv \frac{\partial}{\partial s} + \omega\frac{\partial}{\partial \tau}$$

We look for a solution as the sum of slow changes + fast oscillations:

$$x = \overline{x}(s) + \frac{1}{\omega^{\beta}}\widetilde{x}(s,\tau), \quad \beta = \text{const} > 0$$

hence we have two indefinite constants α and β

The substitution of the solution into the PDE gives

$$\underline{\omega^{2-\beta}\widetilde{x}_{\tau\tau}}+2\omega^{1-\beta}\widetilde{x}_{s\tau}+\overline{x}_{ss}=f(\overline{x})+\omega^{-\beta}\widetilde{x}f_x(\overline{x})+\underline{\omega^{\alpha}\widetilde{g}(\overline{x},\tau)}+\omega^{\alpha-\beta}\widetilde{x}\widetilde{g}_x(\overline{x},\tau)$$

where the oscillating terms and non-oscillating terms give two different equalities

It is clear that the underlined terms must balance each other. Then

$$2-\beta=\alpha,\quad \widetilde{x}_{\tau\tau}=\widetilde{g}(\overline{x},\tau);\quad \Rightarrow\quad \widetilde{x}=\widetilde{g}^{\tau\tau}(\overline{x},\tau)$$

Taking the average $\langle\rangle$ yields

$$0=\alpha-\beta,\quad \overline{x}_{ss}=f(\overline{x})+\langle\widetilde{x}\widetilde{g}_x\rangle=f(\overline{x})-\langle\widetilde{g}^{\tau}\widetilde{g}_x^{\tau}\rangle$$

This equation gives the only available evolution of $\overline{x}$, while

$$\alpha=\beta=1$$

The values $\alpha = \beta = 1$ and the correspondent $x(s, \tau)$ are called the *distinguished limit solution*.

Distinguished Limit Theorem (DLT)*:*
$\alpha = \beta = 1$ is the *only available possibility* for building of asymptotic solution. All other values of α and β produce either secular growing terms (say, $x(s, \tau) \sim s$) or lead to a controversial (unsolvable) system of equations for the successive approximations.

The majority of researchers ignore the existence of *DLT*. Hence in each particular case *DLT* should be considered as an important item to be studied.

It is a device of microscopic size (say, from 1 to 10 microns) able to controlled self-propulsion in a fluid.

1. Micro-robot is an interesting example of Vibrodynamics, including the two-timing asymptotic theory and the distinguished limit theorem.

2. Micro-robots are aimed to be used in medicine, biology, oil industry, and technology. Possiblee applications: the delivering of medical drugs to a required part of human body or the obtaining of key information about the oil reservoirs in Nature (porous medium filled with oil), *etc.*

Viscous incompressible fluid:

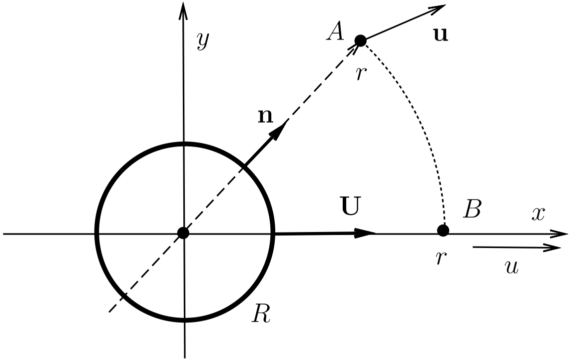
$$\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{u}, \quad \mathrm{div}\,\mathbf{u} = 0$$

The *Reynolds number Re* and *Stokes number St* are small

$$Re = \frac{LU}{\nu} \ll 1, \qquad St = \frac{L^2}{\nu T} \ll 1$$

$\Rightarrow$ inertia and unsteadiness are neglected $\Rightarrow$ Stokes's equations:

$$\mu\nabla^2\mathbf{u} = \nabla p, \qquad \mathrm{div}\,\mathbf{u} = 0 \quad \text{in} \quad \mathcal{D}$$
$$(\mathbf{u}-\mathbf{U})\cdot\mathbf{n} = 0 \quad \text{on} \quad \partial\mathcal{D}$$



Velocity of fluid at a generic point A and at a special point B is

$$\mathbf{u}(\mathbf{x}) = \frac{3R}{4r}\{\mathbf{U} + (\mathbf{n}\cdot\mathbf{U})\mathbf{n}\}, \quad \mathbf{u}(\mathbf{x}) = \left(\frac{3R}{2r}U, 0\right)$$

The Stokes drag force exerted on a sphere is

$$\mathbf{F} = -6\pi\eta R\mathbf{U}$$

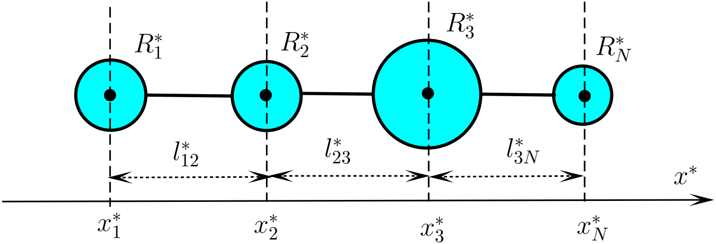


Figure : N spheres, linked by bars/rods of periodically changing lengths.

Total force exerted to each sphere is zero (Stokes' approximation):

$$F_i + f_i = 0, \quad i = 1, 2, \dots, N$$

where F_i– Stokes's friction force, f_i– reactions force from the bars (constraints)

The dimensionless equations of motion

$$R_i \dot{x}_i - \delta \sum_{k \neq i} R_{ik} \dot{x}_k / l_{ik} = f_i, \qquad R_{ik} \equiv R_i R_k$$

$$l_{ik} = L_{ik} + \varepsilon \widetilde{l}_{ik}, \quad \sum_i f_i = \mathbf{f} \cdot \mathbf{i} = 0, \quad \mathbf{i} \equiv (1, 1, \ldots, 1)$$

$$\varepsilon \equiv a/L \ll 1, \quad \delta \equiv 3R/(2L) \ll 1$$

where the characteristic scales are L, R, a, $T \equiv 1/\omega$.

Two independent small parameters ε and δ are present.

The matrix form of equations:

$$\mathbb{A}\dot{\mathbf{x}} = \mathbf{f} \quad \text{or} \quad \sum_{k=1}^{N} A_{ik}\dot{x}_k = f_i$$

$$\mathbb{A} = A_{ik} = \begin{cases} R_i & \text{for} \quad i = k, \\ -\delta R_{ik}/l_{ik} & \text{for} \quad i \neq k \end{cases}$$

where

$$\mathbb{A} = \overline{\mathbb{C}}_0 + \varepsilon\delta\widetilde{\mathbb{A}}'_0 + \dots, \quad \overline{\mathbb{C}}_0 \equiv \overline{\mathbb{A}}_0 + \delta\overline{\mathbb{B}}_0$$

$$\overline{\mathbb{A}}_0 \equiv \mathrm{diag}\{R_1, R_2, ..., R_N\}, \quad \widetilde{\mathbb{A}}'_0 \equiv \begin{cases} 0 & \text{for } i = k, \\ R_{ik}\widetilde{l}_{ik}/L^2_{ik} & \text{for } i \neq k \end{cases}$$

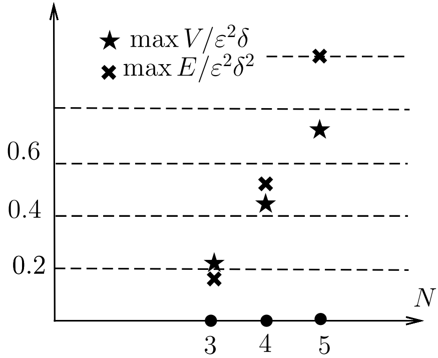
The explicit formula for self-propulsion velocity is

$$\overline{V}_0 = \frac{\delta}{\Delta^2} \sum_{i<k<l} \overline{G}_{ikl}, \quad \Delta \equiv \sum_{\alpha=1}^{N} R_\alpha$$

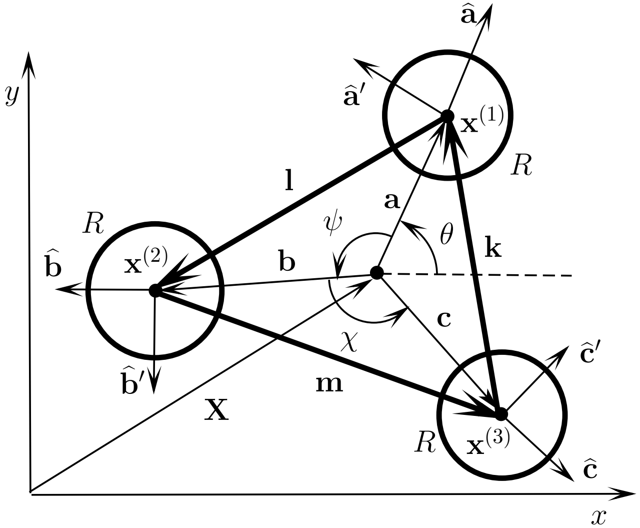
$$\overline{G}_{ikl} \equiv R_i R_k R_l \left(\frac{1}{L_{ik}^2} + \frac{1}{L_{kl}^2} - \frac{1}{L_{il}^2} \right) \langle \widetilde{l}_{ik} \widetilde{l}_{kl\tau} - \widetilde{l}_{ik\tau} \widetilde{l}_{kl} \rangle$$

where the sum is taken over all possible triplets $(i, k, l): \ 1 \le i < k < l \le N$.

- For the N-swimmer the number of triplets is $N!/[(N-3)!3!]$:
- For 3-swimmer the only triplet is $(1, 2, 3)$ (Golestanian),
- For 4-swimmer we already have four triplets $(1, 2, 3), (1, 2, 4), (1, 3, 4), (2, 3, 4)$,
- For 5-swimmer we already have 10 triplets,
- For 10-swimmer the number of triplets is up to 120.



Both the self-propulsion speed and the efficiency are rapidly growing with N. Hence the use of chains is beneficial.



$\hat{\mathbf{a}}$
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Motion and rotation are given by expressions:

$$\overline{\mathbf{V}} \equiv \overline{\mathbf{X}}_t = \frac{\delta\varepsilon^2}{9}(A\widehat{\mathbf{a}} + B\widehat{\mathbf{a}}'), \quad \Omega \equiv \overline{\theta}_t = \varepsilon^2 D$$

$$\overline{\theta} = \varepsilon^2 D s + \theta_0$$

$$A \equiv \frac{\sqrt{3}}{2}\langle(\widetilde{l} + \widetilde{k})\,\widetilde{m}_\tau\rangle, \quad B \equiv \frac{1}{2}\langle 2\widetilde{k}\widetilde{l}_\tau - \widetilde{l}\widetilde{m}_\tau - \widetilde{m}\widetilde{k}_\tau\rangle$$

$$D \equiv -\frac{4}{3\sqrt{3}}\langle\widetilde{l}\,\widetilde{m}_\tau + \widetilde{k}\,\widetilde{l}_\tau + \widetilde{m}\,\widetilde{k}_\tau\rangle$$

$$\widehat{\mathbf{a}} \equiv \begin{pmatrix} \cos\overline{\theta} \\ \sin\overline{\theta} \end{pmatrix}, \qquad \widehat{\mathbf{a}}' \equiv \begin{pmatrix} -\sin\overline{\theta} \\ \cos\overline{\theta} \end{pmatrix}$$

with constants A, B, and D.

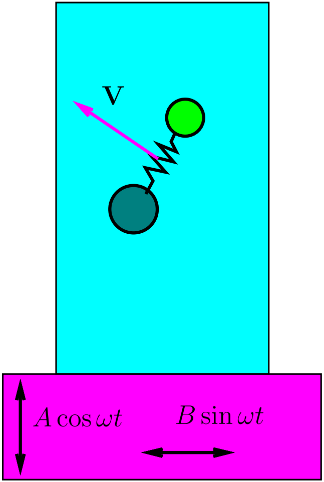
Purely translational motion: $\Omega = 0$

$$|\overline{\mathbf{V}}| = \delta\varepsilon^2\sqrt{A^2 + B^2}/9$$

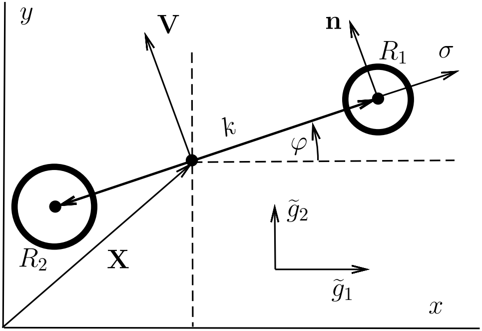
$$\langle \widetilde{l}\,\widetilde{m}_\tau + \widetilde{k}\,\widetilde{l}_\tau + \widetilde{m}\,\widetilde{k}_\tau \rangle = 0 \quad \text{or} \quad \langle (\widetilde{l} - \widetilde{k})\widetilde{\varphi}_\tau \rangle = 0$$

For example, $\widetilde{k} = \widetilde{l} \neq \widetilde{m}$ gives

$$\Omega = 0, \quad A = \sqrt{3}\langle \widetilde{k}\widetilde{m}_\tau \rangle = 3\langle \widetilde{l}\widetilde{\varphi}_\tau \rangle/4, \quad B = 0$$



Rigid-body oscillations of a fluid can drive an elastic dumbbell micro robot!



$$\overline{\mathbf{V}} = -\frac{\overline{\mathbf{n}}}{2}\varepsilon^2\delta\mu AB, \quad \overline{\varphi} = \frac{1}{2}\arcsin\Phi = \text{const}$$

$$\Phi \equiv \frac{2AB}{A^2 - B^2}\left(\frac{2}{K} + K\right), \; K \equiv 2k/R_1R_2, \; \mu = \alpha(R_1 - R_2);$$

$$\widetilde{g}_1 = -A\sin\tau, \; \widetilde{g}_2 = B\cos\tau, \; Mg/F = \varepsilon\alpha$$

Lecture is based on the papers by V.A. Vladimirov:

(2013) On the self-propulsion velocity of an *N*-sphere micro-robot. *JFM Rapids*, 716, R1-1.

(2013) Dumbbell micro-robot driven by flow oscillation. *JFM Rapids*, 717, R8-1.

(2012) Acoustic-drift equation. *ArXiv*: 1206.1297v1.

(2012) MHD drift equation: from Langmuir circulations to MHD dynamo? *J.Fluid Mech.* **698**, 51-61.

(2011) Theory of non-degenerate oscillatory flows. E-print: arXiv: 1110,3633v2

(2010) Admixture and drift in oscillating fluid flows. E-print: arXiv: 1009,4058v1

(2008) Viscous flows in a half space caused by tangential vibrations on its boundary. *Studies in Appl. Math.* **121**, 337

(2005) Vibrodynamics of pendulum and submerged solid. *J. Math. Fluid Mech.*, **7**, 397-412.