Discrete integrable systems with self-adaptive moving mesh

Kenichi Maruno

Department of Mathematics, The University of Texas - Pan American

Follow-up meeting, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK
July 8-12. 2013

Joint work with Bao-Feng Feng (UTPA), Yasuhiro Ohta (Kobe University)

Discrete integrable systems with self-adaptive moving mesh

Kenichi Maruno

Department of Mathematics, The University of Texas - Pan American

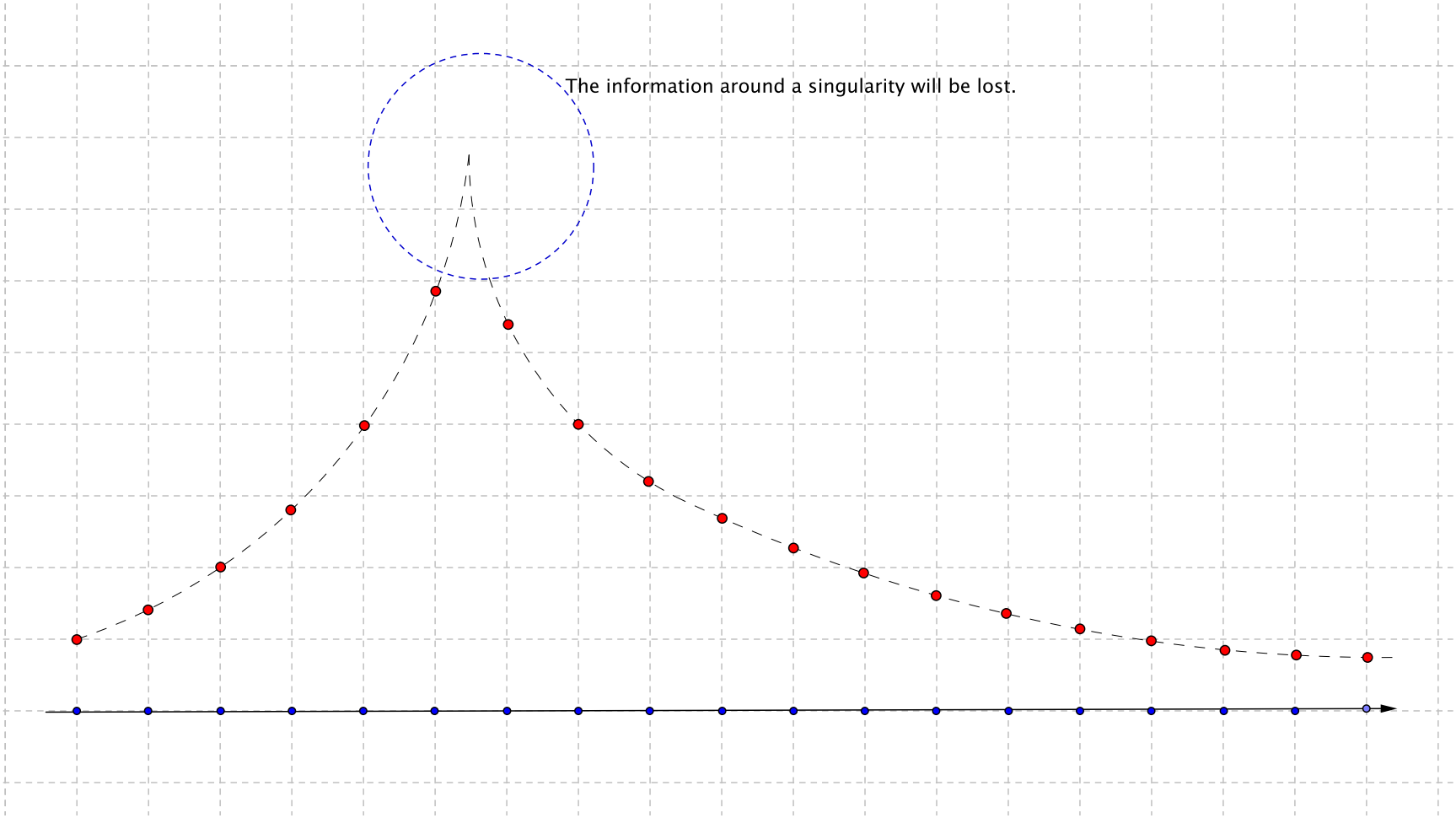
Follow-up meeting, Isaac Newton Institute for Mathematical Sciences, Cambridge, UK
July 8-12. 2013

Joint work with Bao-Feng Feng (UTPA), Yasuhiro Ohta (Kobe University)

Geometric approach is joint work with Kenji Kajiwara (Kyushu University), Jun-ichi Inoguchi (Yamagata University)

1. Introduction: Self-adaptive moving mesh difference equations
2. Discretization of the short pulse equation
3. Discretization of the coupled short pulse equation
4. Discretization of the Hunter-Saxton equation
5. Discretization of the Camassa-Holm equation
6. Discretization of the WKI elastic beam equation and the curve shortening equation by a geometric approach

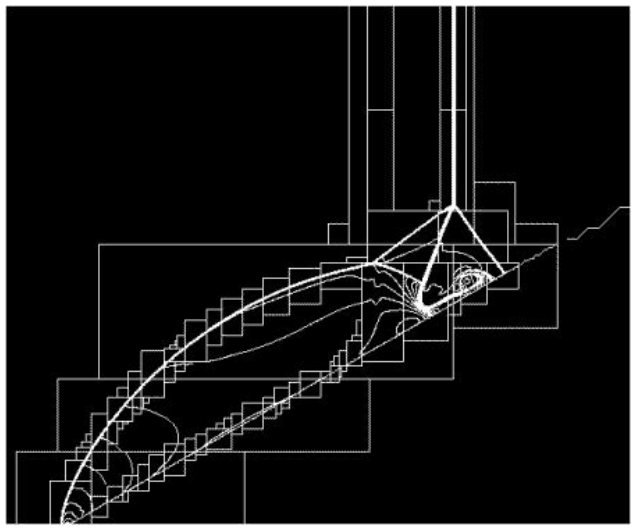
A difficulty of numerical computations for large deformation phenomena



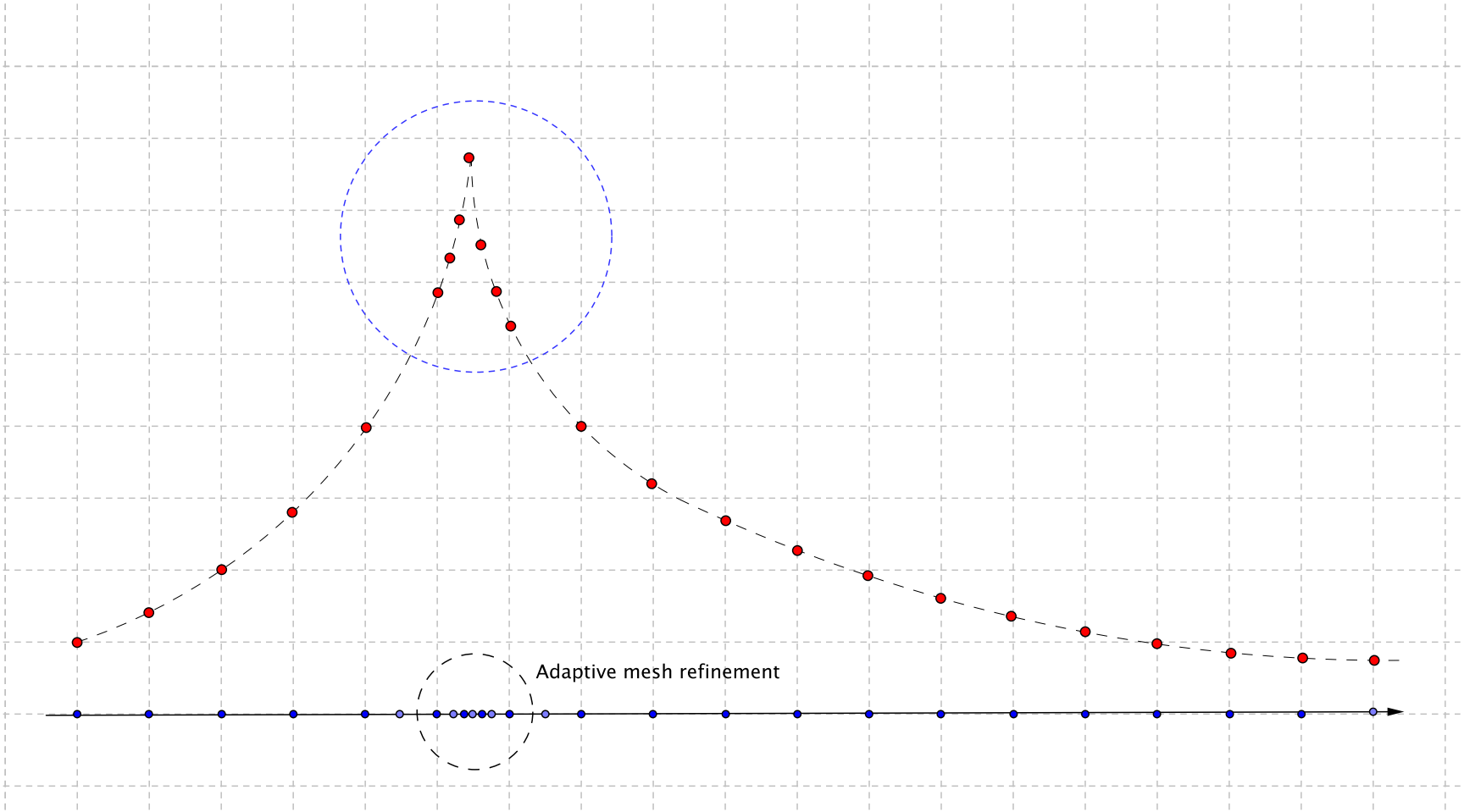
Adaptive mesh refinement

Adaptive mesh refinement: A method of changing the spacing of grid points in certain regions, while the solution is being calculated. The grid might in general be spaced coarsely, but it could be adaptively refined to have many grid points in places where there are large waves.

1. Static mesh refinement (static gridding schemes)
2. Dynamic mesh refinement (dynamic gridding schemes): increase computational savings, increased storage savings, control grid resolution



Adaptive mesh refinement



Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!

Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!
Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Integrable PDEs having singularities (loop, cusp, etc.) in their solutions, e.g. the Camassa-Holm equation, the short pulse equation, the coupled short pulse equation, the WKI elastic beam equation, the Hunter-Saxton equation, the Dym equation, etc.

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!
Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Integrable PDEs having singularities (loop, cusp, etc.) in their solutions, e.g. the Camassa-Holm equation, the short pulse equation, the coupled short pulse equation, the WKI elastic beam equation, the Hunter-Saxton equation, the Dym equation, etc.
Their integrable discretizations are unknown.

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!
Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Integrable PDEs having singularities (loop, cusp, etc.) in their solutions, e.g. the Camassa-Holm equation, the short pulse equation, the coupled short pulse equation, the WKI elastic beam equation, the Hunter-Saxton equation, the Dym equation, etc.
Their integrable discretizations are unknown.
It is also very difficult to make numerical difference schemes for these equations.

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!
Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Integrable PDEs having singularities (loop, cusp, etc.) in their solutions, e.g. the Camassa-Holm equation, the short pulse equation, the coupled short pulse equation, the WKI elastic beam equation, the Hunter-Saxton equation, the Dym equation, etc.
Their integrable discretizations are unknown.
It is also very difficult to make numerical difference schemes for these equations.

Objective: Develop methods to discretize these PDEs keeping the integrability

Finding integrable difference schemes with dynamic mesh refinement

Finding good algorithm of dynamic mesh refinement is important!
Are there integrable dynamic gridding schemes, i.e., discrete integrable systems with automatic adaptive mesh refinement?

Integrable PDEs having singularities (loop, cusp, etc.) in their solutions, e.g. the Camassa-Holm equation, the short pulse equation, the coupled short pulse equation, the WKI elastic beam equation, the Hunter-Saxton equation, the Dym equation, etc.
Their integrable discretizations are unknown.
It is also very difficult to make numerical difference schemes for these equations.

Objective: Develop methods to discretize these PDEs keeping the integrability

$$\Downarrow$$

Goal: Find and develop effective numerical methods for nonlinear PDEs (including nonintegrable systems) which describe large deformation

Hirota's discretization method

1. Transform a nonlinear differential equation to bilinear equations.

Hirota's discretization method

1. Transform a nonlinear differential equation to bilinear equations.
2. Discretize bilinear equations (keep solutions).

Hirota's discretization method

1. Transform a nonlinear differential equation to bilinear equations.
2. Discretize bilinear equations (keep solutions).
3. Transform discrete bilinear equations into a nonlinear form.

Hirota's discretization method

1. Transform a nonlinear differential equation to bilinear equations.
2. Discretize bilinear equations (keep solutions).
3. Transform discrete bilinear equations into a nonlinear form.

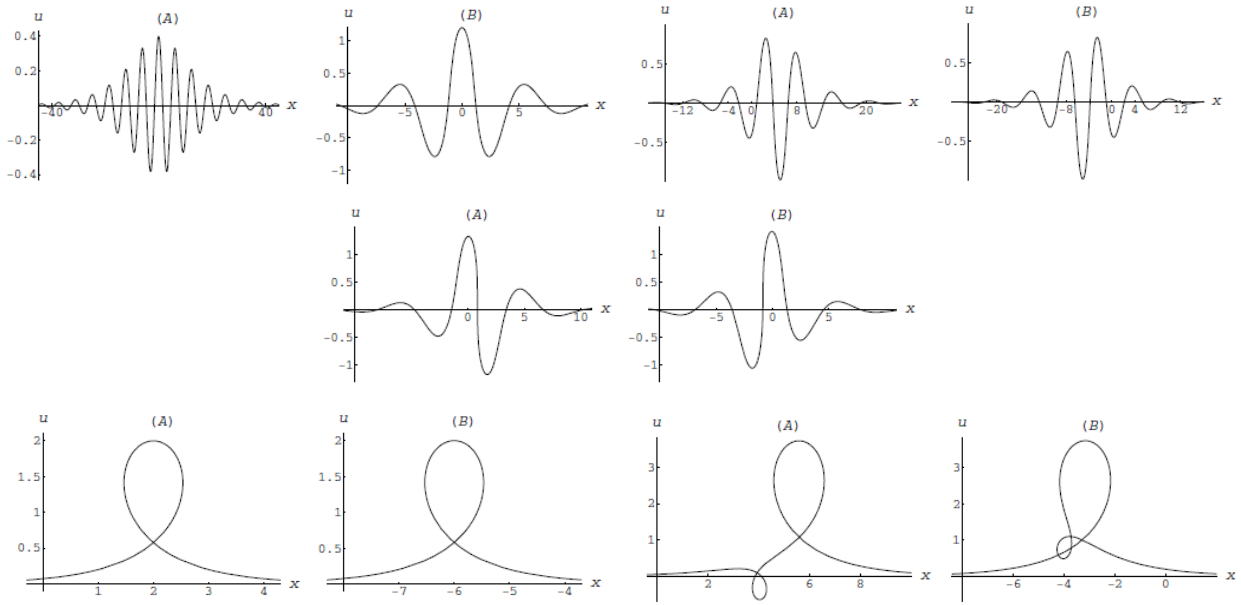
x Examples: Discrete KdV equation, discrete sine-Gordon equation, discrete modified KdV equation, discrete time Toda, etc.

Short pulse equation

$$u_{xt} = u + \frac{1}{6}(u^3)_{xx}$$

Schäfer & Wayne(2004): Derived from Maxwell equation on the setting of ultra-short optical pulse in silica optical fibers. The pulse spectrum is not narrowly localized around the carrier

Variety of short pulse solitons (Sakovich & Sakovich 2006)



Short pulse equation

Sakovich & Sakovich (2005): A Lax pair of WKI type, relationship with sine-Gordon equation; Sakovich & Sakovich (2006): Exact solutions Matsuno (2007): Systematic construction of multisoliton solutions through a hodograph (reciprocal) transformation

Parametric form of the soliton solution

The soliton solution $u(x,t)$:

$$u(x_1,x_{-1}) = \frac{\partial}{\partial x_{-1}}\left(2i\ln\frac{F^*(x_1,x_{-1})}{F(x_1,x_{-1})}\right) = \frac{\partial\theta(x_1,x_{-1})}{\partial x_{-1}}$$

through the hodograph transformation

$$x(x_1,x_{-1}) = x_{1,0} - 2\left(\ln F^*F\right)_{x_{-1}}, \quad t(x_1,x_{-1}) = x_{-1}$$

A function $\theta = 2i\ln\frac{F^*}{F}$ satisfies the sine-Gordon equation $\theta_{x_1x_{-1}} = \sin\theta$.

A conservation form of the short pulse equation

The short pulse equation $u_{xt} = u + \frac{1}{6}(u^3)_{xx}$ can be written in the form of

$$\rho_{x_{-1}} = -\left(\frac{u^2}{2}\right)_{x_1},$$
$$u_{x_{-1}x_1} = \rho u,$$

with the hodograph transformation

$$\begin{aligned} x &= x_{1,0} + \int_{x_{1,0}}^{x_1} \rho(x_1', x_{-1}) dx_1' \\ t &= x_{-1}. \end{aligned}$$

Remark: The above system is obtained from the AKNS scheme. Setting $\rho = \cos\theta$ and $u = \theta_{x_{-1}}$ gives the sine-Gordon equation $\theta_{x_1 x_{-1}} = \sin\theta$.

Bilinear equations of the short pulse equation

Bilinear equations of the short pulse equation $u_{xt} = u + \frac{1}{6}(u^3)_{xx}$

$$F F_{x_{-1}x_1} - F_{x_{-1}} F_{x_1} = \frac{1}{4}(F^2 - F^{*2})$$
$$F^* F^*_{x_{-1}x_1} - F^*_{x_{-1}} F^*_{x_1} = \frac{1}{4}(F^{*2} - F^2)$$

through the hodograph transformation
$x = x_{1,0} + \int_{x_{1,0}}^{x_1} \rho(x_1', x_{-1}) dx_1', \quad t = x_{-1}$ and the dependent variable transformation $u = \frac{\partial}{\partial x_{-1}} \left(2i \ln \frac{F^*(x_1, x_{-1})}{F(x_1, x_{-1})} \right)$, the τ functions $F^* = \tau_1$, $F = \tau_0$

$$\tau_n(x_1, x_{-1}) = \left| \psi_i^{(n+j-1)}(x_1, x_{-1}) \right|_{1 \le i,j \le N} ,$$

where

$$\psi_i^{(n)} = p_i^n e^{(p_i/2)x_1 + 1/(2p_i)x_{-1} + \eta_{0i} - i\pi/4} + (-p_i)^n e^{-(p_i/2)x_1 - 1/(2p_i)x_{-1} + \eta'_{0i} + i\pi/4} .$$

Semi-discretization

Semi-discretization of bilinear equations

$$\frac{2}{a}\partial_{x_{-1}} f(k+1)f(k) - \frac{2}{a} f(k+1)\partial_{x_{-1}} f(k) - f(k+1)f(k) + \bar{f}(k+1)\bar{f}(k) = 0\,,$$

$$\frac{2}{a}\partial_{x_{-1}} \bar{f}(k+1)\bar{f}(k) - \frac{2}{a} \bar{f}(k+1)\partial_{x_{-1}} \bar{f}(k) - \bar{f}(k+1)\bar{f}(k) + f(k+1)f(k) = 0\,,$$

where

$$\bar{f} = \tau_{n-1}\,, \quad f = \tau_n\,, \quad \bar{f} = \tau_{n+1}.$$

and

$$\tau_n(k, x_{-1}) = \left|\psi_i^{(n+j-1)}(k, x_{-1})\right|_{1\le i,j\le N}\,,$$

and

$$\psi_i^{(n)}(k, x_{-1}) = p_i^n (1 - a p_i)^{-k} e^{\xi_i} + (-p_i)^n (1 + a p_i)^{-k} e^{\eta_i}\,,$$

$$\xi_i = \frac{1}{p_i} x_{-1}/2 + \xi_{i0}\,, \quad \eta_i = -\frac{1}{p_i} x_{-1}/2 + \eta_{i0}\,,$$

Discretization of a hodograph transformation

Continuous

The soliton solution $u(x,t)$:

$$u(x_1,x_{-1}) = \frac{\partial}{\partial x_{-1}}\left(2i\ln\frac{F^*(x_1,x_{-1})}{F(x_1,x_{-1})}\right) = \frac{d\phi}{dx_{-1}}$$

through the hodograph transformation

$$x = x_{1,0} + \int_{x_{1,0}}^{x_1} \rho(x_1', x_{-1})dx_1', \quad t = x_{-1}$$

$$\Downarrow$$

Semi-discrete

$$u_k(x_{-1}) = \frac{\partial}{\partial x_{-1}}\left(2i\ln\frac{\bar{f}_k(x_{-1})}{f_k(x_{-1})}\right) = \frac{d\phi_k}{dx_{-1}}$$

through the hodograph transformation

$$X_k = x_{1,0} + \sum_{j=0}^{k-1} a\rho_j, \quad t = s.$$

Integrable self-adaptive moving mesh scheme of the short pulse equation

$$\partial_{x_{-1}}\rho_k = -\frac{u_{k+1}^2 - u_k^2}{2a},$$

$$\partial_{x_{-1}}(u_{k+1} - u_k) = \frac{1}{2}(X_{k+1} - X_k)(u_{k+1} + u_k).$$

($\rho_k = (X_{k+1} - X_k)/a$) with the hodograph transformation

$$\begin{aligned} X_k &= ka + \sum_{j=0}^{k-1} a\rho_j \\ t &= x_{-1}, \end{aligned}$$

$X_k \equiv X(k, x_{-1})$, $u_k \equiv u(k, x_{-1})$, $\rho_k \equiv \rho(k, x_{-1})$. A set of (X_k, u_k) gives a solution of the semi-discrete analogue of the short pulse equation.

Animation: 1 loop soliton, 2 loop solitons, 1 loop soliton and 1 breather

Soliton equations which can be expressed in the form of

$$\partial_{x_{-1}}\rho = \partial_{x_1} f(u)$$

$+$ an integrable system (which belongs to AKNS system) $+$ a hodograph transformation $x = x(x_1, x_{-1})$, $t = x_{-1}$. $\rightarrow \rho = \frac{\partial x}{\partial x_1}$.

$\Downarrow$

$$\partial_{x_{-1}}\rho_k = \frac{f(u_{k+1}) - f(u_k)}{a}.$$

$+$ an discrete integrable system (which belongs to Ablowitz-Ladik system) $+$ a (discrete) hodograph transformation $X_k = X_k(x_{-1})$, $t = x_{-1}$. $\rightarrow \rho_k = \frac{\Delta X_k}{\Delta x_{1,k}} = \frac{X_{k+1} - X_k}{a}$ where $\Delta X_k = X_{k+1} - X_k$, $\Delta x_{1,k} = x_{1,k+1} - x_{1,k} = (k+1)a - ka = a$.

$$\partial_{x_{-1}}\rho_k = \frac{f(u_{k+1}) - f(u_k)}{a}, \quad \Leftarrow \text{Conservation law}$$

$$\text{Conserved density} \quad \rho_k = \frac{\Delta X_k}{\Delta x_{1,k}} = \frac{X_{k+1} - X_k}{a}, \quad X_k : \text{grid points.}$$

$$\Downarrow$$

$$\partial_{x_{-1}}\left(\frac{X_{k+1} - X_k}{a}\right) = \frac{f(u_{k+1}) - f(u_k)}{a}.$$

The conserved density ρ_k (the mesh interval is $\delta_k = a\rho_k$) gives the lattice interval between X_k and X_{k+1}. Since ρ_k satisfies the conservation law, ρ_k will increase if $f(u_{k+1}) > f(u_k)$, and ρ_k will decrease if $f(u_{k+1}) < f(u_k)$. Thus semi-discrete analogues of the short pulse equation have a property of self-adaptive moving mesh schemes.

Fully discrete short pulse equation

$$
\begin{aligned}
&(X_{k+1}^{l+1} - X_{k+1}^{l} - X_{k}^{l+1} + X_{k}^{l})(X_{k+1}^{l} - X_{k}^{l+1}) \\
&\quad -(u_{k+1}^{l+1} + u_{k+1}^{l} - u_{k}^{l+1} - u_{k}^{l})(u_{k+1}^{l} + u_{k}^{l+1}) = 0\,, \\
&(X_{k+1}^{l} - X_{k}^{l+1})(u_{k+1}^{l+1} + u_{k}^{l}) + (X_{k+1}^{l+1} - X_{k}^{l})(u_{k+1}^{l} + u_{k}^{l+1}) = 0\,.
\end{aligned}
$$

u_k^l and X_k^l satisfy the following relations

$$
\begin{aligned}
&\left(\frac{u_{k+1}^{l} - u_{k}^{l}}{a_k}\right)^2 + \left(\frac{X_{k+1}^{l} - X_{k}^{l}}{a_k}\right)^2 = 1\,, \\
&\left(\frac{u_{k}^{l+1} + u_{k}^{l}}{\frac{1}{c_l}}\right)^2 + \left(\frac{X_{k}^{l+1} - X_{k}^{l}}{\frac{1}{c_l}}\right)^2 = 1\,.
\end{aligned}
$$

Coupled short pulse equation

The coupled short pulse equation (Matsuno 2011)

$$u_{xt}^{(l)} = u^{(l)} + \frac{1}{2}\left(\left(\sum_{1\le j<k\le n} C_{j,k}u^{(j)}u^{(k)}\right)u_x^{(l)}\right)_x, \quad l = 1, 2, \cdots, n\,.$$

can be written in the form of

$$\partial_{x_{-1}}\rho = -\partial_{x_1}\left(\frac{1}{2}\sum_{1\le j<k\le n} C_{j,k}u^{(j)}u^{(k)}\right),$$

$$\partial_{x_1}\partial_{x_{-1}}u^{(l)} = \rho u^{(l)}, \quad l = 1, 2, \cdots, n\,.$$

with the hodograph transformation

$$\begin{aligned} x &= x_{1,0} + \int_{x_{1,0}}^{x_1} \rho(x_1', x_{-1})dx_1' \\ t &= x_{-1}\,. \end{aligned}$$

Bilinear equations of the coupled short pulse equation

$$D_s D_y f \cdot g_l = f g_l \,,$$
$$D_s^2 f \cdot f = \frac{1}{2} \sum_{1 \le j < k \le n} C_{j,k} g_j g_k \,.$$

$x = x_{1,0} + \int_{x_{1,0}}^{x_1} \rho(x_1', x_{-1}) dx_1'$, $t = x_{-1}$ and $u^{(l)} = \frac{g_l}{f}$.

$$f = \mathrm{pf}(a_1, \cdots, a_{2N}, b_1, \cdots, b_{2N})$$
$$g_l = \mathrm{pf}(\beta_l, d_0, a_1, \cdots, a_{2N}, b_1, \cdots, b_{2N})$$

where

$$\mathrm{pf}(a_j, a_k) = \frac{p_j - p_k}{p_j + p_k} e^{\xi_j + \xi_k} \,, \quad \mathrm{pf}(a_j, b_k) = \delta_{j,k} \,,$$
$$\mathrm{pf}(b_j, b_k) = \frac{1}{4} \frac{c_{\mu\nu}}{p_j^{-2} - p_k^{-2}} \,, \quad (b_j \in B_\mu, b_k \in B_\nu) \,,$$
$$\mathrm{pf}(d_l, a_k) = \frac{\partial^l e^{\xi_k}}{\partial y^l} = p_k^l e^{\xi_k} \,, \; \mathrm{pf}(b_j, \beta_\mu) = \begin{cases} 1 & b_j \in B_\mu \\ 0 & b_j \notin B_\mu \end{cases} \,.$$

$\xi_j = p_j y + p_j^{-1} s + \xi_{i0}$. Sets B_μ, $\mu = 1, 2, \cdots, n$ satisfy the following condition,

$$B_\mu \cap B_\nu = \emptyset, \;\; \text{if} \;\; \mu \neq \nu, \qquad \cup_{\mu=1}^n B_\mu = \{b_1, b_2, \cdots, b_{2N}\}.$$

Semi-discretization

Semi-discretization of bilinear equations

$$
\begin{cases}
\dfrac{1}{a} D_s (g_{k+1}^{(l)} \cdot f_k - g_k^{(l)} \cdot f_{k+1}) = g_{k+1}^{(l)} f_k + g_k^{(l)} f_{k+1}\,, \quad l = 1, 2, \cdots, n\,, \\
D_s^2 f_k \cdot f_k = \dfrac{1}{2} \displaystyle\sum_{1 \le \mu < \nu \le n} g_k^{(\mu)} g_k^{(\nu)}\,,
\end{cases}
$$

$$
\begin{aligned}
f_k &= \mathrm{pf}(a_1, \cdots, a_{2N}, b_1, \cdots, b_{2N})_k\,, \\
g_k^{(l)} &= \mathrm{pf}(\beta_l, d_0, a_1, \cdots, a_{2N}, b_1, \cdots, b_{2N})_k\,,
\end{aligned}
$$

Semi-discretization

$$\mathbf{pf}(a_i, a_j)_k = \frac{p_i - p_j}{p_i + p_j}\varphi_i^{(0)}(k)\varphi_j^{(0)}(k)\,, \quad \mathbf{pf}(a_i, b_j)_k = \delta_{i,j}\,,$$

$$\mathbf{pf}(b_i, b_j)_k = \frac{1}{4}\frac{c_{\mu\nu}}{p_i^{-2} - p_j^{-2}}\,, (b_i \in B_\mu, b_j \in B_\nu)\,,$$

$$\mathbf{pf}(d_l, a_i)_k = \varphi_i^{(l)}(k)\,, \quad \mathbf{pf}(a_i, d^k)_k = \varphi_i^{(0)}(k+1)\,,$$

$$\mathbf{pf}(b_j, \beta_\mu)_k = \begin{cases} 1 & b_j \in B_\mu \\ 0 & b_j \notin B_\mu \end{cases}$$

$$\mathbf{pf}(d_0, d^k)_k = 1, \quad \mathbf{pf}(d_{-1}, d^k)_k = -a\,,$$

where

$$\varphi_i^{(n)}(k) = p_i^n \left(\frac{1 + ap_i}{1 - ap_i}\right)^k e^{\xi_i}, \quad \xi_i = p_i^{-1}s + \xi_{i0}\,.$$

Here $i, j = 1, 2, \cdots, 2N$, $\mu, \nu = 1, 2, \cdots, n$ and k, l are arbitrary integers.

Integrable self-adaptive moving mesh scheme of the coupled short pulse equation

$$\partial_{x_{-1}}\rho_k = -\frac{\sum_{1\le l<m\le n} C_{l,m}(u_{k+1}^{(l)}u_{k+1}^{(m)} - u_k^{(l)}u_k^{(m)})}{2a},$$

$$\partial_{x_{-1}}(u_{k+1}^{(l)} - u_k^{(l)}) = \frac{1}{2}(X_{k+1} - X_k)(u_{k+1}^{(l)} + u_k^{(l)})\,.$$

$(\rho_k = (X_{k+1} - X_k)/a)$ with the hodograph transformation

$$\begin{aligned} X_k &= ka + \sum_{j=0}^{k-1} a\rho_j \\ t &= x_{-1}\,, \end{aligned}$$

$X_k \equiv X(k, x_{-1})$, $u_k \equiv u(k, x_{-1})$, $\rho_k \equiv \rho(k, x_{-1})$. A set of (X_k, u_k) gives a solution of the semi-discrete analogue of the coupled short pulse equation.

Hunter-Saxton equation

The Hunter-Saxton equation (a short wave limit of the Camassa-Holm equation)

$$w_{txx} - 2w_x + 2w_x w_{xx} + w w_{xxx} \ ,$$

Soliton solutions are expressed by

$$w = -2 \left(\ln f\right)_{x_{-1}x_{-1}}$$

through the hodograph transformation

$$\begin{aligned} x(x_1, x_{-1}) &= 2x_1 + \int_{-\infty}^{x_{-1}} w(x_1, x'_{-1}) dx'_{-1} \\ &= 2x_1 - 2(\ln f)_{x_{-1}} \ , \\ t(x_1, x_{-1}) &= x_{-1} \ , \end{aligned}$$

A conservation form of the Hunter-Saxton equation

The Hunter-Saxton equation can be written in the form of

$$\rho_{x_{-1}} = w_{x_1}\,,$$
$$\frac{1}{2}(\ln\rho)_{x_1 x_{-1}} = \frac{\rho}{2} - \frac{2}{\rho}\,,$$

with the hodograph transformation

$$\begin{aligned} x &= 2x_1 + \int_{-\infty}^{x_{-1}} w(x_1, x'_{-1})dx'_{-1} \\ t &= x_{-1}\,. \end{aligned}$$

Note that the second equation is the sinh-Gordon equation.

Bilinear equations

The Hunter-Saxton equation is decomposed into the bilinear equations

$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right) f \cdot f + g^2 = 0\,,$$
$$\left(\frac{1}{2}D_{x_1}D_{x_{-1}} - 1\right) g \cdot g + f^2 = 0\,,$$

through the dependent variable transformation $w = -2\,(\ln f)_{x_{-1}x_{-1}}$ and the hodograph transformation

$$\begin{aligned} x &= 2x_1 + \int_{-\infty}^{x_{-1}} w(x_1, x'_{-1})dx'_{-1} \\ &= 2x_1 - 2(\ln f)_{x_{-1}}\,, \\ t &= x_{-1}\,. \end{aligned}$$

τ-functions

The previous bilinear equations have a determinant solution

$$f = \det(\psi_i^{(j-1)})_{1\leq i,j\leq N}\ ,\ \ g = \det(\psi_i^{(j)})_{1\leq i,j\leq N}\ .$$

where

$$\psi_i^{(j)} = p_{2i-1}^j e^{p_{2i-1}x_1+p_{2i-1}^{-1}x_{-1}+\theta_{2i-1,0}} + (-p_{2i-1})^j e^{-p_{2i-1}x_1-p_{2i-1}^{-1}x_{-1}+\theta_{2i,0}}\ .$$

Bilinear equations

Consider the following τ-functions:

$f_k = \det(\psi_i^{(j-1)}(k))_{1\le i,j\le N}$, $\quad g_k = \det(\psi_i^{(j)}(k))_{1\le i,j\le N}$ where

$$\psi_i^{(j)}(k) = p_{2i-1}^j(1-ap_{2i-1})^{-k}e^{p_{2i-1}x_1+p_{2i-1}^{-1}x_{-1}+\theta_{2i-1,0}} + (-p_{2i-1})^j(1+ap_{2i-1})^{-k}e^{-p_{2i-1}x_1-p_{2i-1}^{-1}x_{-1}+\theta_{2i,0}}.$$

These τ-functions satisfy the bilinear equations

$$\left(\frac{1}{a}D_{x_{-1}}-1\right) f_{k+1}\cdot f_k + g_{k+1}g_k = 0,$$
$$\left(\frac{1}{a}D_{x_{-1}}-1\right) g_{k+1}\cdot g_k + f_{k+1}f_k = 0.$$

Self-adaptive moving mesh semi-discrete integrable systems

The system of differential-difference equations

$$\partial_{x_{-1}}\rho_k = \frac{w_{k+1}-w_k}{a},$$

$$\frac{1}{a^2}\left(\frac{w_{k+1}-w_k}{\rho_k}-\frac{w_k-w_{k-1}}{\rho_{k-1}}\right)=\frac{\rho_k}{2}-\frac{2}{\rho_k}+\frac{\rho_{k-1}}{2}-\frac{2}{\rho_{k-1}}.$$

($\rho_k=(X_{k+1}-X_k)/a$) with the hodograph transformation

$$X_k = 2x_{1,k}+\int_{-\infty}^{x_{-1}} w_k(x'_{-1})dx'_{-1}\,,\quad t=x_{-1}\,,$$

gives a discrete analogue of the Hunter-Saxton equation. $X_k \equiv X(k,x_{-1})$, $w_k \equiv w(k,x_{-1})$, and $\rho_k \equiv \rho(k,x_{-1})$.

Self-adaptive moving mesh fully discrete integrable systems

$$\frac{\rho_k^{l+1}-\rho_k^l}{b}+\frac{1}{4}\rho_k^{l+1}\left(x_{k+1}^{l+1}+x_k^{l+1}-2x_k^l\right)$$
$$+\frac{1}{4}\rho_k^l\left(x_{k+1}^l+x_k^l-2x_{k+1}^{l+1}\right)=\frac{1}{2}\left(w_{k+1}^{l+1}+w_{k+1}^l-w_k^{l+1}-w_k^l\right),$$
$$\frac{1}{a^2}\left(\frac{w_{k+1}^l-w_k^l}{\rho_k^l}-\frac{w_k^l-w_{k-1}^l}{\rho_{k-1}^l}\right)=\frac{\rho_k^l}{2}-\frac{2}{\rho_k^l}+\frac{\rho_{k-1}^l}{2}-\frac{2}{\rho_{k-1}^l}.$$

($\rho_k^l=(X_{k+1}^l-X_k^l)/a$) with the hodograph transformation

$$X_k^l=2x_{1,k}+\int_{-\infty}^{x_{-1}}w_k(x'_{-1})dx'_{-1}=2ka+\sum_{j=0}^{k-1}a\rho_j^l,$$

gives a fully discrete analogue of the Hunter-Saxton equation.

Camassa-Holm equation

The Camassa-Holm equation

$$w_t + 2\kappa^2 w_x - w_{txx} + 3ww_x = 2w_x w_{xx} + ww_{xxx} \,,$$

Soliton and cusped soliton solutions are expressed by

$$w = \left(\ln \frac{g(x_1, x_{-1})}{h(x_1, x_{-1})} \right)_{x_{-1}}$$

through the hodograph transformation

$$\begin{aligned} x(x_1, x_{-1}) &= 2cx_1 + \int_{-\infty}^{x_{-1}} w(x_1, x'_{-1}) dx'_{-1} \\ &= 2cx_1 + \ln \frac{g}{h} \,, \\ t(x_1, x_{-1}) &= x_{-1} \,, \end{aligned}$$

A conservation form of the Camassa-Holm equation

The Camassa-Holm equation can be written in the form of

$$\rho_{x_{-1}} = w_{x_1}\,,$$
$$\frac{1}{2}(\ln\rho)_{x_1 x_{-1}} = \frac{\rho}{2c} - \frac{2c}{\rho} + \frac{w\rho}{2}\,,$$

with the hodograph transformation

$$\begin{aligned} x &= 2cx_1 + \int_{-\infty}^{x_{-1}} w(x_1, x'_{-1})dx'_{-1} \\ t &= x_{-1}\,. \end{aligned}$$

Note that the second equation is the deformed sinh-Gordon equation.

Self-adaptive moving mesh semi-discrete integrable systems

The system of differential-difference equations

$$\partial_{x_{-1}}\rho_k = \frac{w_{k+1}-w_k}{a}, \quad \delta_k = 2\frac{(1+ac)e^{a(\rho_k-2c)}-(1-ac)}{(1+ac)e^{a(\rho_k-2c)}+(1-ac)},$$

$$\frac{2}{\delta_k}(w_{k+1}-w_k)-\frac{2}{\delta_{k-1}}(w_k-w_{k-1}) = \frac{\delta_k}{2}(w_{k+1}+w_k)$$

$$+\frac{\delta_k}{c}\left(1-\frac{4a^2c^2}{\delta_k^2}\right)+\frac{\delta_{k-1}}{2}(w_k+w_{k-1})+\frac{\delta_{k-1}}{c}\left(1-\frac{4a^2c^2}{\delta_{k-1}^2}\right).$$

($\rho_k = (X_{k+1}-X_k)/a$) with the hodograph transformation

$$X_k = 2cx_{1,k} + \int_{-\infty}^{x_{-1}} w_k(x'_{-1})dx'_{-1}, \quad t = x_{-1},$$

gives a discrete analogue of the Camassa-Holm equation. $X_k \equiv X(k, x_{-1})$, $w_k \equiv w(k, x_{-1})$, and $\rho_k \equiv \rho(k, x_{-1})$.

Discretization of the WKI elastic beam equation

WKI (Wadati-Konno-Ichikawa) elastic beam equation
(Wadati-Konno-Ichikawa 1979)

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}}\right)_x$$

Discretization of the WKI elastic beam equation

WKI (Wadati-Konno-Ichikawa) elastic beam equation
(Wadati-Konno-Ichikawa 1979)

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}}\right)_x$$

This PDE has N-loop soliton and N-breather solutions.
This PDE is obtained as compatibility conditions of
WKI(Wadati-Konno-Ichikawa) type linear eigenvalue problem

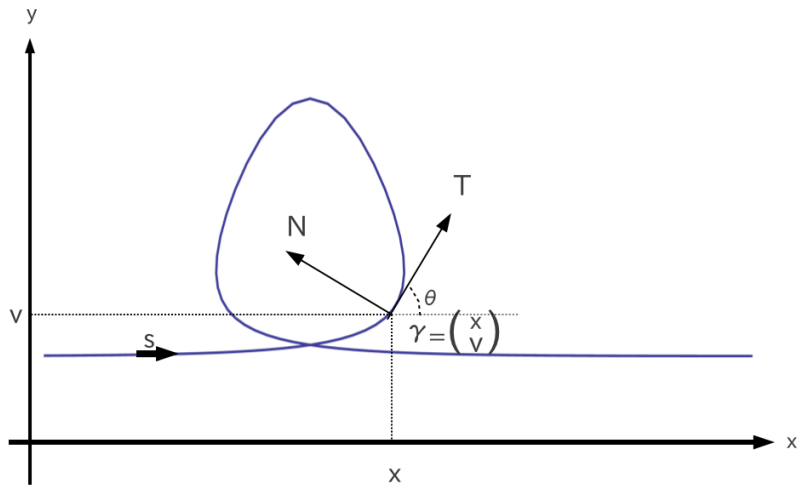
A geometric approach: motions of a plane curve

Goldstein & Petrich 1991, Doliwa-Santini 1994, Kajiwara's talk

A curve on $\gamma(s)$: Euclidean plane $\mathbb{R}^2$, s: arc length parameter.

Tangent vector $\mathbf{T} = \frac{\partial \gamma}{\partial s} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$, $|\mathbf{T}| = 1$. Normal vector $\mathbf{N} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathbf{T}$, $\theta = \theta(s)$: an angle function.

The mKdV equation $\kappa_t + \frac{3}{2}\kappa^2\kappa_s + \kappa_{sss} = 0$, $\kappa = \theta_s$, describes a motion of a plane curve.



Describe the motion of a plane curve in the rectangular coordinates (x, v)

A geometric approach: An Eulerian description of the motion of a plane curve

x and v are described by an angle function θ:

$$\gamma(s,t) = \begin{bmatrix} x(s,t) \\ v(s,t) \end{bmatrix} = \int_0^s \begin{bmatrix} \cos\theta(s',t) \\ \sin\theta(s',t) \end{bmatrix} ds' + \begin{bmatrix} x_0 \\ v_0 \end{bmatrix}.$$

Thus consider the hodograph transformation from (s,t) to (x,t'):

$$(x,t') = \left(\int_0^s \cos\theta(s',t)\, ds' + x_0,\ t \right).$$

Find an equation in (x,t'). (we use t for t'.)
WKI elastic beam equation

$$v_t = -\left(\frac{v_{xx}}{(1+v_x^2)^{\frac{3}{2}}} \right)_x, \quad \left(\text{or} \quad u_t = -\left(\frac{u_x}{(1+u^2)^{\frac{3}{2}}} \right)_{xx}, \quad u = v_x \right).$$

A geometric approach: the motion of a discrete curve

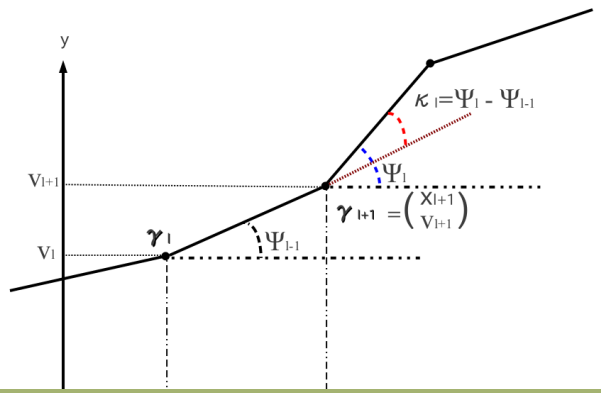
Hisakado-Nakayama-Wadati 1995, Doliwa-Santini 1995, Nishinari 1998, Kajiwara's talk

Tangent vector

$$\mathbf{T}_l = \frac{\boldsymbol{\gamma}_{l+1} - \boldsymbol{\gamma}_l}{a_l}\,, \quad \left|\frac{\boldsymbol{\gamma}_{l+1} - \boldsymbol{\gamma}_l}{a_l}\right| = 1.$$

$s_l = \sum_{k=0}^{l-1} a_k$, ψ_l and κ_l are given by

$$\frac{\boldsymbol{\gamma}_{l+1} - \boldsymbol{\gamma}_l}{a_l} = \begin{bmatrix} \cos\psi_l \\ \sin\psi_l \end{bmatrix}\,, \quad \mathbf{T}_{l+1}\cdot\mathbf{T}_l = \cos\kappa_l\,,$$



A geometric approach: the motion of a discrete curve

The Frenet frame

$$F_l = (\mathbf{T}_l, \mathbf{N}_l), \quad \mathbf{T}_l = \frac{\gamma_{l+1} - \gamma_l}{a_l},$$

The discrete Frenet equation

$$F_{l+1} = F_l \begin{bmatrix} \cos\kappa_l & -\sin\kappa_l \\ \sin\kappa_l & \cos\kappa_l \end{bmatrix},$$

The time evolution of a discrete plane curve:

$$\frac{\partial}{\partial t}\gamma_l = g_l \mathbf{T}_l + f_l \mathbf{N}_l \,.$$

A non-streching condition $g_{l+1}\cos\kappa_l - g_l = f_{l+1}\sin\kappa_l$ gives

$$\frac{\partial}{\partial t}F = F \begin{bmatrix} 0 & \frac{g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l}{a_l} \\ -\frac{g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l}{a_l} & 0 \end{bmatrix}.$$

A geometric approach: the motion of a discrete curve

The compatibility condition gives

$$\frac{d}{dt}U_l + V_l U_l - U_l V_{l+1} = 0,$$

$$U = \begin{bmatrix} \cos\kappa_l & -\sin\kappa_l \\ \sin\kappa_l & \cos\kappa_l \end{bmatrix}, \quad V = \begin{bmatrix} 0 & A_l \\ -A_l & 0 \end{bmatrix},$$

$$A_l = (g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l)/a_l.$$

The evolution equation of motion of a discrete plane curve

$$\frac{d\kappa_l}{dt} = \frac{A_{l+1}}{a_{l+1}} - \frac{A_l}{a_l},$$

$$g_{l+1}\cos\kappa_l - g_l = f_{l+1}\sin\kappa_l .$$

The compatibility condition gives

$$\frac{d}{dt}U_l + V_l U_l - U_l V_{l+1} = 0,$$

$$U = \begin{bmatrix} \cos\kappa_l & -\sin\kappa_l \\ \sin\kappa_l & \cos\kappa_l \end{bmatrix}, \quad V = \begin{bmatrix} 0 & A_l \\ -A_l & 0 \end{bmatrix},$$

$$A_l = (g_{l+1}\sin\kappa_l + f_{l+1}\cos\kappa_l - f_l)/a_l.$$

The evolution equation of motion of a discrete plane curve

$$\frac{d\kappa_l}{dt} = \frac{A_{l+1}}{a_{l+1}} - \frac{A_l}{a_l},$$

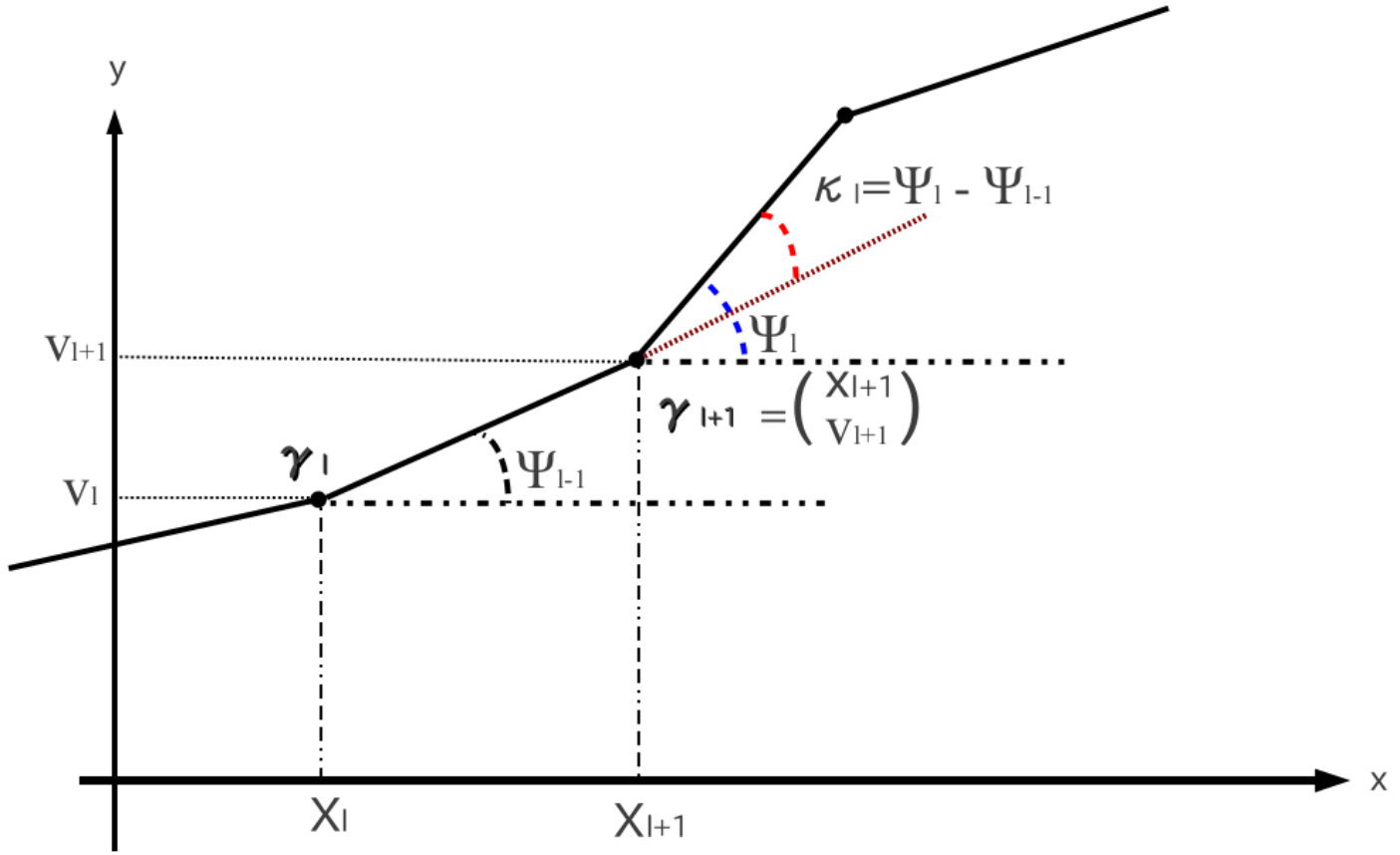
$$g_{l+1}\cos\kappa_l - g_l = f_{l+1}\sin\kappa_l\,.$$

Setting $f_l = -u_{l-1} = -\tan\frac{\kappa_n}{2}$, $g_l = 1$, $a_l = a$, we obtain the semi-discrete mKdV equation:

$$\frac{du_l}{dt} = \frac{1}{2a}(1+u_l^2)(u_{l+1} - u_{l-1}).$$

A geometric approach: the discrete WKI elastic beam equation

Describe the motion of a discrete plane curve in the rectangular coordinates $(\boldsymbol{X_l}, \boldsymbol{v_l})$ (An Eulerian description of the motion of a discrete plane curve):



A geometric approach: the discrete WKI elastic beam equation

Consider the following coordinate transformation:

$$\gamma_l(t) = \begin{bmatrix} X_l(t) \\ v_l(t) \end{bmatrix} = \sum_{j=0}^{l-1} \begin{bmatrix} \epsilon \cos(\psi_j) \\ \epsilon \sin(\psi_j) \end{bmatrix} + \begin{bmatrix} X_0 \\ v_0 \end{bmatrix}.$$

The semi-discrete WKI elastic beam equation

$$\frac{d}{dt}\delta_l = -\frac{v_{l+1} - v_l}{a}(G_{l+1} + G_l),$$

$$\frac{d}{dt}(v_{l+1} - v_l) = \frac{\delta_l}{a}(G_{l+1} + G_l),$$

$$\delta_l \equiv X_{l+1} - X_l, \quad G_l \equiv \frac{v_{l+1} - 2v_l + v_{l-1}}{\delta_l - \delta_{l-1}}, \quad \delta_l = a\rho_l.$$

δ_l: lattice spaces

A geometric approach: the curve shortening equation

The evolution equation for a motion of a continuous plane curve

$$\kappa_t = (f_s + g\kappa)_s \,,$$
$$g_s = f\kappa \,.$$

Setting $f = \kappa$, we obtain $g = \int_0^s \kappa^2 ds'$. Thus

$$\kappa_t = \kappa_{ss} + \kappa_{sss} + \kappa_s \int_0^s \kappa^2 ds'.$$

Write down geometric quantities in terms of x, v, t. By $\frac{\partial}{\partial t}\gamma = \kappa \mathbf{N} + \left(\int_0^s \kappa^2 ds'\right) \mathbf{T}$, we obtain

$$v_t \quad = \quad \kappa\sqrt{1+v_x^2}\,.$$

By the formula of κ, we obtain the curve shortening equation

$$v_t = \frac{v_{xx}}{1+v_x^2}.$$

This equation appears in pattern formations such as viscous fingering (Nakayama-Iizuka-Wadati 1994). Nonintegrable.

A geometric approach: the discrete curve shortening equation

The semi-discrete curve shortening equation

$$\frac{d}{dt}\delta_l = -\frac{v_{l+1}-v_l}{a}\left[(g_{l+1}+H_{l+1})G_{l+1}-G_l\right],$$

$$\frac{d}{dt}\left(v_{l+1}-v_l\right) = \frac{\delta_l}{a}\left[(g_{l+1}+H_{l+1})G_{l+1}-G_l\right],$$

$$g_{l+1} = \frac{G_{l+1}^2+g_l}{H_{l+1}},$$

$$G_l \equiv ((v_{l+1}-v_l)(X_l-X_{l-1})-(X_{l+1}-X_l)(v_l-v_{l-1}))/a^2,$$

$$H_l \equiv ((X_{l+1}-X_l)(X_l-X_{l-1})+(v_{l+1}-v_l)(v_l-v_{l-1}))/a^2.$$

$$\delta_l \equiv X_{l+1}-X_l, \quad \rho_l = \frac{\delta_l}{a}$$

δ_l: lattice spaces

Summary

- Integrable self-adaptive moving mesh (semi-discrete and fully discrete) schemes for Camassa-Holm, Hunter-Saxton, short pulse, WKI elastic beam equation, Harry Dym equation, coupled short pulse equation, modified magma equation.
- A geometric construction of self-adaptive moving mesh schemes can be applied for nonintegrable geometric PDEs such as the curve shortening equation.

- B-F. Feng, J. Inoguchi, K. Kajiwara, KM and Y. Ohta: “Discrete Integrable Systems and Hodograph Transformations Arising from Motions of Discrete Plane Curves” J. Phys. A: Math. Theor., **44** (2011) 395201 (19pp)
- B-F. Feng, KM and Y. Ohta: “Integrable discretizations for the short-wave model of the Camassa-Holm equation” J. Phys. A: Math. Theor., **43** (2010) 265202 (14pp)
- B-F. Feng, KM and Y. Ohta: “A self-adaptive moving mesh method for the Camassa-Holm equation” J. Comp. Appl. Math., **235** (2010) 229-243
- B-F. Feng, KM and Y. Ohta: “Integrable discretizations of the short pulse equation ” J. Phys. A: Math. Theor., **43** (2010) 085203 (14pp)
- Y. Ohta, KM, B-F. Feng: “An integrable semi-discretization of the Camassa-Holm equation and its determinant solution” J. Phys. A: Math. Theor., **41** (2008) 355205 (30pp).