IS KAHAN’S "UNCONVENTIONAL" METHOD CONVENTIONAL?

Subtitle: Can one teach an old dog new tricks?

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An “unconventional” discretization of quadratic vector fields was introduced by Kahan.

In 1993 he wrote:

“I have used these methods for 24 years without quite understanding why they work so well as they do, when they work.”

In 2000, Hirota and Kimura independently rediscovered Kahan’s method, applied it to two integrable quadratic vector fields, and showed that the discretization preserved the integrability.

This was generalized in 2011 to a large number of integrable quadratic vector fields by Petrera, Pfadler and Suris, but again without a real explanation.
Our aim in this talk is to explain some of the good properties of the KHK discretization, both in the integrable and in the non-integrable case, as well as uncover large classes of new integrable examples.
WHAT IS THE KHK DISCRETIZATION?

Example:

\[
\begin{align*}
\frac{dx}{dt} &= y \\
\frac{dy}{dt} &= xy + c
\end{align*}
\]

Introducing the short-hand notation:

\[
\begin{align*}
x &:= x(n), \quad x' := x(n+1), \\
y &:= y(n), \quad y' := y(n+1),
\end{align*}
\]

the KHK discretization is

\[
\begin{align*}
\frac{x' - x}{h} &= \frac{y' + y}{2} \\
\frac{y' - y}{h} &= \frac{xy' + x'y}{2} + c
\end{align*}
\]

Note: KHK is *implicit*, but it is *linearly* implicit!
More generally

\[
\frac{dx_i}{dt} = \sum_{j,k} a_{ijk} x_j x_k + \sum_j b_{ij} x_j + c_i
\]

\[
\rightarrow
\]

\[
\frac{x'_i - x_i}{h} = \sum_{j,k} a_{ijk} \frac{x'_j x_k + x_j x'_k}{2} + \sum_j b_{ij} \frac{x_j + x'_j}{2} + c_i
\]

This is the definition of Kahan’s “unconventional” method.
Proposition 1 The Kahan method coincides with the Runge-Kutta methods

\[ \frac{X' - X}{h} = (1+2c) f \left( \frac{X + X'}{2} \right) - cf(\beta X + (1-\beta)X') - cf((1-\beta)X + \beta X') \]

with \( c := \frac{1}{8(\beta - \frac{1}{2})^2} \), restricted to quadratic vector fields \( f \).
\[ \beta = 0 \rightarrow c = \frac{1}{2} \]

\[ \frac{X' - X}{h} = 2 f \left( \frac{X + X'}{2} \right) - \frac{1}{2} f(X') - \frac{1}{2} f(X) \]

\[ \frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ xy + c \end{pmatrix} \]

\[ \frac{1}{h} \begin{pmatrix} x' - x \\ y' - y \end{pmatrix} = 2 \left( \frac{\frac{y+y'}{2} + c}{x+x'} \cdot \frac{y+y'}{2} + c \right) - \frac{1}{2} \left( x' y' + c \right) - \frac{1}{2} \left( xy + c \right) \]

\[ = \begin{pmatrix} \frac{y+y'}{x y' + \frac{xy}{2} + c} \\ \frac{y+y'}{xy' + \frac{xy}{2} + c} \end{pmatrix} \]
Notes:

1. As all Runge-Kutta (R-K) methods, (1) is defined for all vector fields $f$.

2. The R-K method (1) corresponding to Kahan’s method is not unique.

3. The R-K method (1) is self-adjoint, i.e. invariant under $X \leftrightarrow X', \ h \leftrightarrow -h$. 
PROPERTIES AUTOMATICALLY INHERITED FROM BEING A RUNGE-KUTTA METHOD

I. All R-K methods preserve all affine symmetries, integrals, and foliations.

Consequence:

Kahan’s method preserves all affine symmetries, integrals, and foliations

II. All *self-adjoint* R-K methods preserve all affine *reversing* symmetries

Consequence:

Kahan’s method preserves all affine reversing symmetries
Proposition 2 Let $H(X)$ be a general cubic Hamiltonian, and let the corresponding Hamiltonian/Poisson vector field be

$$
\frac{dX}{dt} = f(X) = K \nabla H(X),$$

(2)

where $K$ is a constant skew matrix.

Then Kahan’s method applied to (2) preserves a (rational) modified Hamiltonian $\tilde{H}_h(X)$, and also a modified measure $\tilde{m}_h(X)$, where

$$
\tilde{H}_h(X) := H(X) + \frac{1}{3} h \nabla H^t(X) \left( I - \frac{1}{2} hf'(X) \right)^{-1} f(X)
$$

$$
\tilde{m}_h(X) := \frac{dx_1 \wedge \cdots \wedge dx_n}{\det \left( I - \frac{1}{2} hf'(X) \right)}
$$
Crucial ingredient of the proof:

Kahan’s method can be written:

\[
\frac{X' - X}{h} = \left( I - \frac{1}{2} hf'(X) \right)^{-1} f(X)
\]
Today will discuss only $D = 2$:

**Proposition 3.** Let $X := \begin{pmatrix} x \\ y \end{pmatrix}$,

Let

$$H(X) = ax^3 + bx^2y + cxy^2 + dy^3 + ex^2 + fxy + gy^2 + hx + iy \quad (3)$$

and let

$$f(X) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial H}{\partial x} \\ \frac{\partial H}{\partial y} \end{pmatrix}.$$ 

Then Kahan’s map:

$$\frac{X' - X}{h} = \left( I - \frac{1}{2} hf'(X) \right)^{-1} f(X) \quad (4)$$

is an integrable map of the plane.
Pf.

From Prop 2, we know that (4) with (3) preserves an integral and a measure. In 2D this is sufficient for integrability.
Figure: Top left: Level sets of $H = \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$ (the so-called Hénon–Heiles potential). Same level sets of the conserved quantity $\tilde{H}$ of Kahan’s method for $h = 1/3$ (top right); $h = 2/3$ (bottom left) (the jagged circle $x^2 + y^2 = \frac{1}{4} + \frac{1}{h^2} = 1.58$ indicates $\tilde{H} = \infty$, on which initial conditions are mapped to infinity—for $h = 1/3$ the circle has radius 3.04 and is out of view); and $h \to \infty$ (bottom right). Note that Kahan’s method preserves the 3-fold discrete symmetry of $H$, because as a Runge–Kutta method it preserves all affine symmetries.
Figure: Left: Level sets of $H = y - y^3 + x^2 - x^3$. Right: Level sets of the conserved quantity $\tilde{H}$ of Kahan’s method for $h = 0.3$. 
Figure: Left: Level sets of $H = a(q^3 + p^3) + b(q^2p + qp^2)$, with $a = 0.33$, $b = 0.17$. Right: Level sets of the conserved quantity $\tilde{H}$ of Kahan’s method for $h = 0.89$. 
CONCLUSIONS

Our propositions explain the following results of Petrera, Pfadler & Suris:

(1) The integrability of the KHK discretization in their

- Eq.(4.2) \( H = \frac{\gamma^2}{2} - 3x^2 + \alpha x \)
- Eq.(5.4) \( H = y(3x^2 - y) \)
- Eq.(8.1) (Volterra chain in \( \mathbb{R}^3 \), \( H = x_1x_2x_3 \), constant \( K \))

(2) The invariant measure and cubic integral of their eq(1.1.1),
(Three-wave system in \( \mathbb{C}^3 \), \( H = z_1z_2z_3 + \bar{z}_1\bar{z}_2\bar{z}_3 \))

(3) The invariant measure for the family of systems in their Prop.1.
CONCLUSIONS (cont)

(4) The linear integrals throughout their paper.

(5) Our results also imply that Kahan’s application of this method to the KdV equation preserves a measure and a modified energy.

In addition to explaining previous results, we uncovered a number of new results:

(6) Our result that Kahan’s method preserves a modified energy and volume for cubic Hamiltonians was not only previously unknown, but the very possibility of a Runge-Kutta method having such properties was never even even suspected.

(7) We have constructed much more general classes of integrable KHK discretizations than exist in the literature.
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