Positive Semidefinite Rank of Polytopes

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Joint work with

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Let $S^f_+$ be the cone of $r \times r$ psd matrices. Given a polytope $P$ with $f$ facets, we can always write $P$ as the projection of an affine slice of $S^f_+$. In fact, we can write it as a spectrahedron.

\[ P = \left\{ x \mid a_i^T x + b_i \geq 0 \text{ for all } i = 1, \ldots, f \right\} \]

\[ = \left\{ x \mid \text{diag}(a_1^T x + b_1, \ldots, a_f^T x + b_f) \succeq 0 \right\} \]

**Question:** Given $P$, what is the smallest psd cone that admits an affine slice that projects to $P$?
For a graph $G = ([n], E)$, consider the stable set polytope $\text{STAB}(G)$.

Define $\text{TH}(G)$ to be $x \in \mathbb{R}^n$ such that

$$
\begin{pmatrix}
1 & x^T \\
x & U
\end{pmatrix} \succeq 0
$$

for some $U$ with $\text{diag}(U) = x$ and $U_{ij} = 0$ for $\{i, j\} \in E$. 

Properties:
- $\text{TH}(G) \supseteq \text{STAB}(G)$ since $(1, x) \cdot (1, x)^T \succeq 0$.
- The containment is tight iff $G$ is perfect.
- $\text{TH}(G)$ is a projection of an affine slice of the PSD cone of dimension $n + 1$. 

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- The containment is tight iff $G$ is perfect
- $\text{TH}(G)$ is a projection of an affine slice of the psd cone of dimension $n + 1$
A **psd lift** of $P$ is an affine space $L$ and a linear map $\pi$ with $P = \pi(L \cap S^r_+)$. 

**Example**

Let $P = [0, 1]^2$. Then:

$$P = \left\{ (x, y) : \exists z \text{ with } \begin{bmatrix} 1 & x & y \\ x & x & z \\ y & z & y \end{bmatrix} \succeq 0 \right\}$$
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We define $xc_{psd}(P) = \min(r \mid P = \pi(L \cap S^r_+))$. This measures how well semidefinite programming can express $P$. 
Given $P$, we would like to know $x_{c_{psd}}(P)$.
slack matrices

Given $P$, we would like to know $xc_{psd}(P)$

Let $P$ be a polytope defined by the $f$ inequalities $b_j - A_j^T x \geq 0$ and with $v$ vertices $p_1, \ldots, p_v$. Then we define the slack matrix to be $S \in \mathbb{R}^{v \times f}_+$ with $S_{ij} = b_j - A_j^T p_i$. 

Example
This is a slack matrix for the displayed trapezoid.

\[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 \\
2 & 0 & 0 & 1 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]
slack matrices

Given $P$, we would like to know $xc_{psd}(P)$

Let $P$ be a polytope defined by the $f$ inequalities $b_j - A_j^T x \geq 0$ and with $v$ vertices $p_1, \ldots, p_v$. Then we define the **slack matrix** to be $S \in \mathbb{R}_+^{v \times f}$ with $S_{ij} = b_j - A_j^T p_i$.

**Example**

This is a slack matrix for the displayed trapezoid.

\[
\begin{array}{cccc}
& F1 & F2 & F3 & F4 \\
(1,0) & 0 & 1 & 1 & 0 \\
(2,1) & 0 & 0 & 2 & 1 \\
(0,1) & 2 & 0 & 0 & 1 \\
(0,0) & 1 & 1 & 0 & 0 \\
\end{array}
\]
psd rank of matrices

Recall the matrix inner product \( \langle A, B \rangle = \text{tr}(A^T B) \).

Given a nonnegative matrix \( M \in \mathbb{R}_{+}^{m \times n} \), the **psd rank** of \( M \) is the smallest number \( r \) such that there exist \( A_1, \ldots, A_m \) and \( B_1, \ldots, B_n \) in the cone \( S_+^r \) with \( M_{ij} = \langle A_i, B_j \rangle \).
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**Example**

\[
\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -0.5 \\ -0.5 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}
\]
Yannakakis’ result on nonnegative rank can be generalized to the following:

**Theorem (Fiorini, et. al; Gouveia, et. al ’12)**

Given a polytope $P$ with slack matrix $S_P$, we have $\text{xc}_{\text{psd}}(P) = \text{rank}_{\text{psd}}(S_P)$

Difficulty of calculating psd rank of a matrix usually prevents us from applying the theorem directly. Can we find bounds?
nonnegative rank

If we replace $S^r_+$ with $\mathbb{R}^r_+$ in the definition of $xc_{psd}$ and $\text{rank}_{psd}$, we get Yannakakis’ result on the nonnegative rank of a polytope, $xc(P) = \text{rank}_+ (S_P)$.

$xc(P)$ measures how efficiently $P$ can be represented as the projection of a higher dimensional polyhedron.

\[
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
0 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
1 \\
0.5
\end{pmatrix}
\begin{pmatrix}
2 \\
1 \\
1
\end{pmatrix}
\begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix}
\begin{pmatrix}
2.5 \\
1
\end{pmatrix}
\]

Many results have been shown in this context (optimal lifts for permutahedron and regular n-gons, nonexistence of small lifts of the cut polytope, NP-hardness of nonnegative rank factorization).
For general nonnegative matrices $M$, we have:

$$\frac{1}{2} \sqrt{1 + 8 \text{rank}(M)} - \frac{1}{2} \leq \text{rank}_{\text{psd}}(M)$$
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**Theorem (Gouveia, R., Thomas), (Lee, Theis ’12)**

Let $P \subset \mathbb{R}^n$ be a full-dimensional polytope. Then
\[ \text{rank}_{\text{psd}}(S_P) \geq \text{rank}(S_P) = n + 1. \]

So the lift of the square we saw earlier was optimal!
psd rank lower bound

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So the lift of the square we saw earlier was optimal!

**Question**: Which polytopes have psd rank $n + 1$?
A nonnegative matrix $M$ has many possible entrywise square roots

$$
\begin{pmatrix}
1 & 2 \\
0 & 1 \\
\end{pmatrix} \rightarrow \begin{pmatrix}
-1 & \sqrt{2} \\
0 & 1 \\
\end{pmatrix}, \begin{pmatrix}
-1 & -\sqrt{2} \\
0 & 1 \\
\end{pmatrix}, ...
$$

**Definition**

The **square root rank** of $M$ is $\text{rank} \sqrt{M} = \min \{ \text{rank} \sqrt{M} \}$. 
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Definition

The **square root rank** of $M$ is $\sqrt{\text{rank}} \ (M) = \min \{ \text{rank} \sqrt{M} \}$.

By writing a rank one psd matrix as $uu^T$ and noting that

$$
\langle uu^T, vv^T \rangle = \langle u, v \rangle^2,
$$

we see that $\sqrt{\text{rank}} \ (M)$ is equivalent to psd rank restricted to using only rank one matrices. Hence,

$$
\text{rank}_{\text{psd}} \ (M) \leq \sqrt{\text{rank}} \ (M).
$$
square root rank

Example

Let $M$ be the derangement matrix:

$$
\begin{bmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
$$

Then we have $\text{rank}(M) = 3$ but $\text{rank}(\sqrt{M}) = 2$ due to the square root

$$
\begin{bmatrix}
0 & -1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0 \\
\end{bmatrix}
$$
psd minimal polytopes

Theorem (Gouveia, R., Thomas)

An $n$-dimensional polytope $P$ has psd rank $n + 1$ if and only if $\text{rank} (\sqrt{S_P}) = n + 1$. 

The psd minimal polytopes in $\mathbb{R}^2$ and $\mathbb{R}^3$ are the triangles, quadrilaterals, tetrahedra, triangular bipyramids, quadrilateral pyramids, triangular prisms, biplanar octahedra, and biplanar cuboids.

psd rank is not constant over combinatorial type or oriented matroid.

Can we come up with a geometric classification?

All-positive square root has been sufficient, is this true in general?
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psd rank is not constant over combinatorial type or oriented matroid.

- Can we come up with a geometric classification?
- All-positive square root has been sufficient, is this true in general?
For a polytope $P$ with $f$ facets, we know that $\text{rank}_+(S_P) \geq \log(f)$. Our current psd bound only grows with dimension.

Theorem (Gouveia, R., Thomas)
For a generic polytope $P \subset \mathbb{R}^n$ with $v$ vertices, the psd rank is bounded below by $\left(\frac{v}{n}\right)^{1/4}$.

So a generic 129-gon has psd rank at least five. This is the simplest example of a polygon with psd rank five that is known!

Question: Can we show a 7 or 8-gon with psd rank five?
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**Theorem (Gouveia, R., Thomas)**

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psd lower bound for generic polytopes

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upper bound in $\mathbb{R}^2$

No upper bound (other than the trivial $v$) on the psd rank of a general $v$-gon was known until recently.

Shitov showed that the nonnegative rank of a $v$-gon was bounded above by $6 \lceil \frac{v}{7} \rceil$.

**Theorem (Gouveia, R., Thomas)**

All pentagons and hexagons have psd rank four. Hence, any $v$-gon has psd rank at most $4 \lceil \frac{v}{6} \rceil$. 
Example

The $n \times n$ matrix with entries $(i - j)^2$ has nonnegative rank $\Theta(\log(n))$ and psd rank two. (Beasley, Laffey, and Hrubeš) and psd rank two.

\[
\begin{pmatrix}
0 & 1 & 4 & 9 \\
1 & 0 & 1 & 4 \\
4 & 1 & 0 & 1 \\
9 & 4 & 1 & 0 \\
\end{pmatrix}
\]
The $n \times n$ matrix with entries $(i - j)^2$ has nonnegative rank $\Theta (\log(n))$ and psd rank two.

Question: Can we find a family of slack matrices with an exponential sized gap between $\text{rank}_{\text{psd}}$ and $\text{rank}_+$? Such a family would give an example where the expressive power of semidefinite programming outperforms that of linear programming.
Thank you for your attention!

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