Ericksen–Leslie theory for nematic liquid crystals

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Outline of Talk

1. Introduction
2. Continuum description of nematic liquid crystals
3. Applications to dynamics, including ‘switching phenomena’ in liquid crystals
4. Developments related to smectic and other liquid crystals phases
5. Conclusions and future work

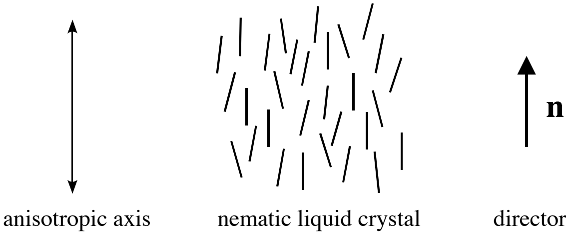
Examples

1. Flat panel visual display devices
2. Liquid crystal displays (e.g., LED displays, LCD projectors, monitors, etc.
3. different types of devices: TND (see later), IPS, STN, soliton-like switching
4. High-speed optical shuttering
5. Biological membranes (lamellar liquid crystals)
6. Cancer drug delivery mechanisms

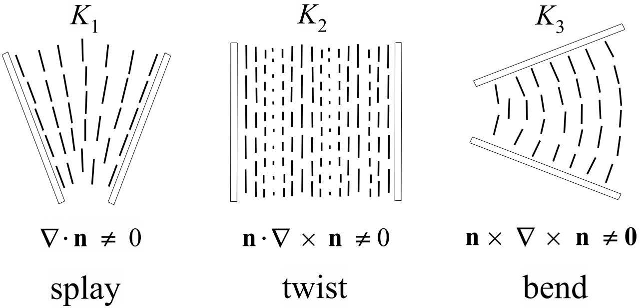
Development of the continuum theory for dynamics

- Liquid crystals can be viewed as anisotropic and non-Newtonian
- Early static theory by Ossen and Zocher in the 1920s, leading to the energy density for nematic liquid crystals established by Frank in 1958
- Early dynamic theory by Anzelius in 1931
- Many developments leading to Ericksen's approach for a theory of dynamics in 1961
- Constitutive equations for anisotropic fluids by Leslie in 1966
- Dynamic theory for nematic liquid crystals completed by Leslie in 1968, now considered as the 'Ericksen–Leslie' theory for the dynamics of nematic liquid crystals

Nematic liquid crystals



Ericksen and Leslie based their theories of dynamics on extensions to the static equilibrium theory of nematic liquid crystals. These equations require some knowledge of the energy density related to the distortions of the director.


K_1
K_2
K_3
$\nabla \cdot \mathbf{n} \neq 0$
$\mathbf{n} \cdot \nabla \times \mathbf{n} \neq 0$
$\mathbf{n} \times \nabla \times \mathbf{n} \neq \mathbf{0}$
splay
twist
bend

Nematic energy density

The energy density, W_F, for nematic liquid crystals goes back to Oseen and Zocher in the 1920s and was essentially established by Frank in 1958, now commonly called the Frank–Oseen energy. This energy density for nematics, when it is assumed that the energy is invariant to a change in the sign of $\mathbf{n}$, may be expressed as

$$w_F(\mathbf{n}, \nabla\mathbf{n}) = \frac{1}{2}K_1(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_2(\mathbf{n}\cdot\nabla\times\ \mathbf{n})^2 + \frac{1}{2}K_3(\mathbf{n}\times\nabla\times\mathbf{n})^2 + \frac{1}{2}(K_2+K_4)\nabla\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n}-(\nabla\cdot\mathbf{n})\mathbf{n}] \qquad (1)$$

where K_1 to K_4 are elastic constants.

The total elastic energy is, of course, w_F integrated over the sample volume V:

$$W = \int_V w_F(\mathbf{n}, \nabla\mathbf{n})\, dV$$

Kinematics and material frame-indifference

The Eulerian description of the instantaneous motion of a fluid with microstructure employs two independent vector fields. The first is the usual velocity $\mathbf{v}(\mathbf{x}, t)$ and the second is an axial vector $\mathbf{w}(\mathbf{x}, t)$ which, in the case of polar fluids, represents the angular velocity of the polar fluid particle at position $\mathbf{x}$ at time t. In the context of liquid crystals, $\mathbf{w}$ is interpreted as being the *local angular velocity* of the director $\mathbf{n}$. In classical continuum theory the only independent field is the velocity $\mathbf{v}$ of the fluid because the angular velocity in such theories equals one half of the curl of the velocity. We denote this particular angular velocity by $\widehat{\mathbf{w}}$ defined by

$$\widehat{\mathbf{w}} = \frac{1}{2}\nabla \times \mathbf{v}, \tag{2}$$

and refer to it as the *regional angular velocity*. It is a measure of the average rotation of the fluid over a neighbourhood of the director.

The angular velocity of the director relative to the regional angular velocity in which the director is embedded is denoted by $\boldsymbol{\omega}$ and is defined by

$$\boldsymbol{\omega} = \mathbf{w} - \widehat{\mathbf{w}} = \mathbf{w} - \tfrac{1}{2}\nabla \times \mathbf{v}. \tag{3}$$

The quantity $\boldsymbol{\omega}$ is called the *relative angular velocity* and is introduced to measure the difference between the local angular velocity $\mathbf{w}$ of the director and the regional angular velocity $\widehat{\mathbf{w}}$ of the fluid in the neighbourhood of the director.

Summary of angular velocities

- local angular velocity of $\mathbf{n}$: $\mathbf{w}$
- regional angular velocity: $\widehat{\mathbf{w}} = \tfrac{1}{2}\nabla \times \mathbf{v}$
- relative angular velocity: $\boldsymbol{\omega} = \mathbf{w} - \widehat{\mathbf{w}}$

- material time derivative (also denoted by a superposed dot)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{x}} \tag{4}$$

- rate of strain tensor A and vorticity tensor W

$$\mathsf{A} = \tfrac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right) \qquad \mathsf{W} = \tfrac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^T\right) \tag{5}$$

- co-rotational time flux of the director, defined by

$$\mathbf{N} = \boldsymbol{\omega} \times \mathbf{n}. \tag{6}$$

- By using the definition of $\boldsymbol{\omega}$, it turns out that (6) is equivalent to

$$\mathbf{N} = \dot{\mathbf{n}} - \mathsf{W}\mathbf{n}, \tag{7}$$

which is the form discussed originally by Ericksen and Leslie.

It can be shown that $\mathbf{n}$, A, $\mathbf{N}$ are material frame-indifferent and satisfy the conditions of objectivity. This is important for the construction of constitutive equations.

Now assume isothermal conditions and ignore thermal effects. As in any classically based continuum theory, conservation laws for mass, linear momentum and angular momentum must hold. The balance law for linear momentum, results in a stress tensor that is generally not symmetric.

Balance laws and the dissipation function

mass conservation:

$$\frac{D}{Dt}\int_V \rho \, dV = 0 \tag{8}$$

balance of linear momentum:

$$\frac{D}{Dt}\int_V \rho \, \mathbf{v} \, dV = \int_V \rho \mathbf{F} \, dV + \int_S \mathbf{t} \, dS \tag{9}$$

balance of angular momentum:

$$\frac{D}{Dt}\int_V \rho(\mathbf{x} \times \mathbf{v}) \, dV = \int_V \rho(\mathbf{x} \times \mathbf{F} + \mathbf{K}) \, dV + \int_S (\mathbf{x} \times \mathbf{t} + \mathbf{l}) \, dS \tag{10}$$

- ρ denotes the density, $\mathbf{x}$ is the position vector, $\mathbf{v}$ is the velocity
- $\mathbf{F}$ is the external body force per unit mass
- $\mathbf{t}$ is the surface force per unit area
- $\mathbf{K}$ is the external body moment per unit mass
- $\mathbf{l}$ is the surface moment per unit area (the couple stress vector)

No 'director inertial term' has been incorporated to the angular mo-

Rate of work

The rate of work is taken to be

$$\int_V \rho(\mathbf{F}\cdot\mathbf{v}+\mathbf{K}\cdot\mathbf{w})dV+\int_S(\mathbf{t}\cdot\mathbf{v}+\mathbf{l}\cdot\mathbf{w})dS=\frac{D}{Dt}\int_V(\tfrac{1}{2}\rho\,\mathbf{v}\cdot\mathbf{v}+w_F)dV+\int_V\mathcal{D}\,dV \tag{11}$$

where $\mathbf{w}$ is the local angular velocity of the director introduced above, and $\mathcal{D}$ is the rate of viscous dissipation per unit volume (dissipation function), assumed non-negative.

The usual tetrahedron argument shows that the surface force t_i and surface moment l_i are expressible in terms of the stress tensor t_{ij} and couple stress tensor l_{ij}, respectively, via the relations

$$t_i = t_{ij}\nu_j, \qquad l_i = l_{ij}\nu_j. \tag{12}$$

The point forms of the balance laws can be exploited in the rate of work (11). If we assume incompressibility, the non-negativity of $\mathcal{D}$ then leads to the conclusion that (cf. Leslie 1992)

$$t_{ij} = -p\,\delta_{ij} - \frac{\partial w_F}{\partial n_{p,j}}\, n_{p,i} + \tilde{t}_{ij} \tag{13}$$

$$l_{ij} = \epsilon_{ipq} n_p \frac{\partial w_F}{\partial n_{q,j}} + \tilde{l}_{ij} \tag{14}$$

where p is an arbitrary pressure arising from incompressibility and $\tilde{t}_{ij}$ and $\tilde{l}_{ij}$ denote possible dynamic contributions.

Under the assumption that $\tilde{l}_{ij}$ does not depend on the gradients of $\mathbf{w}$, the dissipation inequality leads to

$$\tilde{l}_{ij} = 0 \tag{15}$$

In the terminology of Leslie, $\tilde{t}_{ij}$ is called the *viscous stress*.

The rate of work eventually reduces to

$$\tilde{t}_{ij} v_{i,j} - w_i \epsilon_{ijk} \tilde{t}_{kj} = \mathcal{D} \geq 0 \tag{16}$$

which leads to restrictions to the viscous stress contributions $\tilde{t}_{ij}$.

Constitutive equations

It is supposed that

$$\tilde{t}_{ij} \qquad \text{is a function of} \qquad n_i, w_i \text{ and } v_{i,j} \tag{17}$$

Under various assumptions and the properties of **w**, this is equivalent to saying

$$\tilde{t}_{ij} \qquad \text{is a function of} \qquad n_i, N_i \text{ and } A_{ij} \tag{18}$$

Material frame-indifference means that $\tilde{t}_{ij}$ is required to be a hemitropic function of the above named variables; further symmetries force it to be an isotropic function of the above variables.

The experiments of Miesowicz (1936) and Zwetkoff (1939) suggested that $\tilde{t}_{ij}$ has a linear dependence upon N_i and A_{ij} so that a constitutive assumption takes the form

$$\tilde{t}_{ij} = \mathcal{A}_{ij} + \mathcal{B}_{ijk} N_k + \mathcal{C}_{ijkp} A_{kp} \tag{19}$$

where the coefficients $\mathcal{A}_{ij}$, $\mathcal{B}_{ijk}$ and $\mathcal{C}_{ijkp}$ are functions of n_i. From the work of Smith and Rivlin (1957) tensors of such forms can be expressed explicitly. After invoking nematic symmetries, and the dissipation inequality, the final result was given by Leslie (1966)

$$\begin{aligned} \tilde{t}_{ij} &= \alpha_1 n_k A_{kp} n_p n_i n_j + \alpha_2 N_i n_j + \alpha_3 n_i N_j + \alpha_4 A_{ij} \\ &\quad + \alpha_5 n_j A_{ik} n_k + \alpha_6 n_i A_{jk} n_k \end{aligned} \tag{20}$$

The coefficients α_1, α_2,..., α_6, are called the Leslie viscosity coefficients, or simply the Leslie viscosities.

The Parodi relation

Parodi (1970) used Onsager relations to arrive at the result

$$\alpha_6 - \alpha_5 = \alpha_2 + \alpha_3 \tag{21}$$

which reduces the number of independent viscosities to five rather than six and leads to simplifications in the theory. It is a widely accepted relation in the theory.

When the Parodi relation (21) holds, the viscous dissipation inequality given by (16) reduces to

$$\begin{aligned} \mathcal{D} &= \alpha_1(n_iA_{ij}n_j)^2 + 2\gamma_2 N_iA_{ij}n_j + \alpha_4 A_{ij}A_{ij} \\ &\qquad + (\alpha_5 + \alpha_6)n_iA_{ij}A_{jk}n_k + \gamma_1 N_iN_i \;\geq\; 0 \end{aligned} \tag{22}$$

where, for convenience, we set

$$\gamma_1 = \alpha_3 - \alpha_2, \qquad \gamma_2 = \alpha_6 - \alpha_5$$

The dissipation inequality leads to restrictions on the viscosity coefficients:

$$
\begin{aligned}
\gamma_1 \equiv \alpha_3 - \alpha_2 &\geq 0 \\
\alpha_4 &\geq 0 \\
2\alpha_4 + \alpha_5 + \alpha_6 &\geq 0 \\
2\alpha_1 + 3\alpha_4 + 2\alpha_5 + 2\alpha_6 &\geq 0 \\
4\gamma_1(2\alpha_4 + \alpha_5 + \alpha_6) &\geq (\alpha_2 + \alpha_3 + \gamma_2)^2
\end{aligned}
$$

We are now in a position to state the Ericksen–Leslie dynamic equations for nematics in the in the incompressible isothermal case.

Ericksen–Leslie equations

Director constraint:

$$n_i n_i = 1$$

Incompressibility:

$$v_{i,i} = 0$$

Balance of linear momentum:

$$\rho \dot{v}_i = \rho F_i - (p + w_F)_{,i} + \tilde{g}_j n_{j,i} + G_j n_{j,i} + \tilde{t}_{ij,j}$$

Balance of angular momentum:

$$\left(\frac{\partial w_F}{\partial n_{i,j}} \right)_{,j} - \frac{\partial w_F}{\partial n_i} + \tilde{g}_i + G_i = \lambda n_i$$

G_i is the generalised body force, related to the external body moment K_i per unit mass through relation $\rho K_i = \epsilon_{ipq} n_p G_q$. λ is a Lagrange multiplier due the director constraint. $\tilde{g}_i$ is defined by

$$\tilde{g}_i = -\gamma_1 N_i - \gamma_2 A_{ip} n_p$$

There are eight equations in eight unknowns: the three components of $\mathbf{v}$, the three components of $\mathbf{n}$, the pressure p and the Lagrange multiplier λ.

Simplifications

Following Ericksen (1962), we can often incorporate the effects of gravity and magnetic fields by setting

$$\rho F_i = \frac{\partial \Psi}{\partial x_i}, \quad G_i = \frac{\partial \Psi}{\partial n_i}, \qquad \Psi = -\rho \Psi_g + \Psi_m, \tag{23}$$

when ρ is constant and Ψ_g is the usual gravitational potential and Ψ_m is the associated magnetic potential (or electric potential). It can be shown that

$$\rho F_i + G_j n_{j,i} = \frac{\partial \Psi}{\partial x_i} + \frac{\partial \Psi}{\partial n_j} n_{j,i} = \Psi_{,i} \tag{24}$$

which simplifies the balance of linear momentum to

$$\rho \dot{v}_i = \tilde{g}_j n_{j,i} - \tilde{p}_{,i} + \tilde{t}_{ij,j} \tag{25}$$

where

$$\tilde{p} = p + w_F - \Psi \tag{26}$$

For example, as used below, for a magnetic field $\mathbf{H}$ with $H = |\mathbf{H}|$, Ψ_m may be expressed as (other versions differ only in the contribution of constants that do not affect the alignment)

$$\Psi_m = \tfrac{1}{2}\chi_a\left((\mathbf{n}\cdot\mathbf{H})^2 - H^2\right)$$

where χ_a is the magnetic anisotropy of the nematic (which is generally positive, but there are examples when it is negative). There is an analogous expression for electric fields.

Another simplification often used is the following one-constant approximation for w_F (there are other versions):

$$K \equiv K_1 = K_2 = K_3, \quad K_4 = 0$$

which reduces w_F to

$$w_F = \tfrac{1}{2}K\|\nabla\mathbf{n}\|^2 = \tfrac{1}{2}Kn_{i,j}n_{i,j}$$

Simplified form of Ericksen–Leslie equations

Under the above simplifications, the Ericksen–Leslie equations reduce to:

$$n_i n_i = 1$$

$$v_{i,i} = 0$$

$$\rho \dot{v}_i = -\tilde{p}_{,i} + \tilde{g}_j n_{j,i} + \tilde{t}_{ij,j}$$

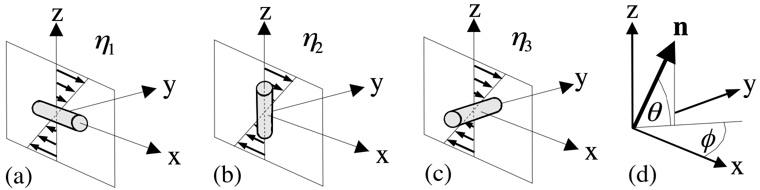
$$K n_{i,jj} + \tilde{g}_i + G_i = \lambda n_i$$

where

$$\tilde{p} = p + w_F - \Psi$$

These equations will be used below for examining the dynamics of the Freedericksz transition.

Physical interpretation of the viscosities



The three basic flow geometries considered by Miesowicz in the 1930s (cf. his review in 1983) allow the measurement of:

(a) η_1 when **n** is parallel to **v**,
(b) η_2 when **n** is parallel to ∇**v**,
(c) η_3 when **n** is orthogonal to both **v** and ∇**v**.

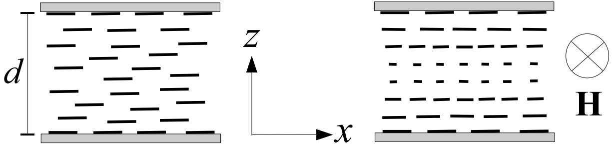
Miesowicz viscosities	$\eta_1 = \frac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6)$
	$\eta_2 = \frac{1}{2}(-\alpha_2 + \alpha_4 + \alpha_5)$
	$\eta_3 = \frac{1}{2}\alpha_4$
	$\eta_{12} = \alpha_1$
rotational viscosity	$\gamma_1 = \alpha_3 - \alpha_2$
torsion coefficient	$\gamma_2 = \alpha_2 + \alpha_3 = \eta_1 - \eta_2$
splay viscosity	$\eta_{splay} = \gamma_1 - \alpha_3^2/\eta_1$
twist viscosity	$\eta_{twist} = \gamma_1 = \alpha_3 - \alpha_2$
bend viscosity	$\eta_{bend} = \gamma_1 - \alpha_2^2/\eta_2$

Table: The principal viscosities for nematics in terms of the Leslie viscosities. The Miesowicz viscosities and the rotational viscosity make up a canonical set of five independent viscosities.

Miscellaneous comments

- $\gamma_1 = \alpha_3 - \alpha_2$ is called the *rotational viscosity* or *twist viscosity*. It generally determines the rate of relaxation of the director.
- $\gamma_2 = \eta_1 - \eta_2$ characterises the contribution to the torque and is called the *torsion coefficient*. It often reflects a coupling between the orientation of the director and shear flow.
- η_3 corresponds to the usual dynamic viscosity that arises in standard isotropic Newtonian fluids.
- The viscosities γ_1 and γ_2 have no counterpart in isotropic fluids.
- Liquid crystals exhibit non-Newtonian behaviour because the first and second normal stress differences $\sigma_1 = t_{11} - t_{22}$ and $\sigma_2 = t_{22} - t_{33}$ are not generally zero.
- The exceptional case of a pure shear in the geometry of Figure (c) above gives rise to Newtonian flow that involves η_3.

The Freedericksz transition in the ‘twist geometry’



(a) $H < H_c$ (b) $H > H_c$

$$\mathbf{n} = (\cos\phi(z,t), \sin\phi(z,t), 0)$$

$$\mathbf{H} = H(0,1,0)\,, \quad H = |\mathbf{H}|\,, \quad H_c = \frac{\pi}{d}\sqrt{\frac{K_2}{\chi_a}}$$

Strong anchoring boundary conditions applied, namely,

$$\phi(0,t) = \phi(d,t) = 0\,, \quad t \geq 0$$

In simplest case, ignore flow. Then w_F and the dissipation $\mathcal{D}$ are

$$\widehat{w}_F = \tfrac{1}{2}K_2\left(\frac{\partial\phi}{\partial z}\right)^2 \quad \text{and} \quad \mathcal{D} = \gamma_1 N_i N_i = \gamma_1 \frac{\partial n_i}{\partial t}\frac{\partial n_i}{\partial t} = \gamma_1\left(\frac{\partial\phi}{\partial t}\right)^2$$

The Ericksen–Leslie equations reduce to identifying the pressure p and the Lagrange multiplier λ explicitly. These can be determined – details omitted here for brevity. The only remaining equation arises from the angular momentum equations and is given by

$$\gamma_1 \frac{\partial \phi}{\partial t} = K_2 \frac{\partial^2 \phi}{\partial z^2} + \chi_a H^2 \sin\phi \cos\phi$$

where $\gamma_1 > 0$.

In the initial stages of the 'switch-on' process, ϕ is small and additionally satisfies some non-zero initial conditions:

$$\phi(z, 0) = \phi_0(z), \quad |\phi_0(z)| \ll 1, \quad \phi_0(0) = \phi_0(d) = 0$$

The above dynamic equation can be linearised around the zero state to give the perturbation equation for $\phi(z, t)$ as

$$\gamma_1 \frac{\partial \phi}{\partial t} = K_2 \frac{\partial^2 \phi}{\partial z^2} + \chi_a H^2 \phi$$

This can be solved exactly for strong anchoring boundary conditions.

Switch-on equation:

$$\gamma_1 \frac{\partial \phi}{\partial t} = K_2 \frac{\partial^2 \phi}{\partial z^2} + \chi_a H^2 \phi$$

Solution:

$$\phi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi z}{d}\right) \exp\left(-\frac{t}{\tau_n}\right)$$

where

$$A_n = \frac{2}{d}\int_0^d \phi_0(z) \sin\left(\frac{n\pi z}{d}\right) dz \qquad \text{and} \qquad \tau_n = \frac{\gamma_1}{\chi_a(n^2 H_c^2 - H^2)}$$

The switch-on time τ_{on} is defined, for $H > H_c$, by $\tau_{on} = -\tau_1$. Thus

$$\tau_{on} = \frac{\gamma_1}{\chi_a(H^2 - H_c^2)}$$

An analogous result holds for electric fields with H replaced by E, χ_a replaced by the appropriate dielectric anisotropy.

This switch-on time is also called the reaction time or rise time and is one measure of the time taken for the reorientation of the director **n** to take place.

A similar procedure for a switched cell leads to the 'switch-off' time τ_{off} when the magnetic field is suddenly removed. It is given by

$$\tau_{off} = \frac{\gamma_1}{K_2}\left(\frac{d}{\pi}\right)^2$$

The switch-off time is also called the decay time or relaxation time.

The incorporation of flow into other geometries not discussed here (such as the 'splay geometry' or 'bend geometry') leads to phenomena called *backflow* and *kickback* – beyond the scope of this basic introduction. There is much experimental confirmation of these theoretical phenomena.

Other applications

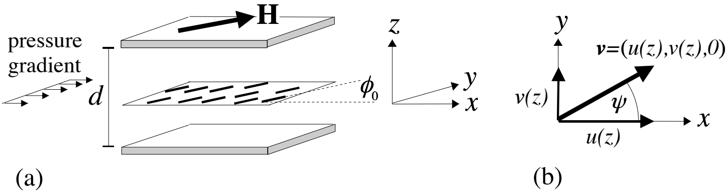
A key verification of the Ericksen–Leslie dynamic theory was provided by the experimental observations of Fisher and Frederickson (1969) of an unusual scaling law for Poiseuille flow which confirmed a theoretical prediction made by Atkin (1970); this result helped to establish the Ericksen–Leslie theory as the generally accepted dynamic theory for nematic liquid crystals. (Note: Despite the timeline in publication, Fisher and Frederickson had corresponded with Atkin and revised the analysis of their data in the light of his work before he published his theoretical predictions – now known as the Atkin scaling laws.)

Many other applications:

- transverse flow phenomena
- non-Newtonian shear flow effects near boundaries
- LCD flat panel display technologies
- standard types of experiments: Couette flow, Poiseuille flow, etc
- light scattering

Transverse flow

Rather than neglect flow in a special case, the director can be fixed in space by a very strong magnetic field, effectively imposing a constant orientation. The experimental results, and theoretical results, presented here are due to Pieranski and Guyon (1974).

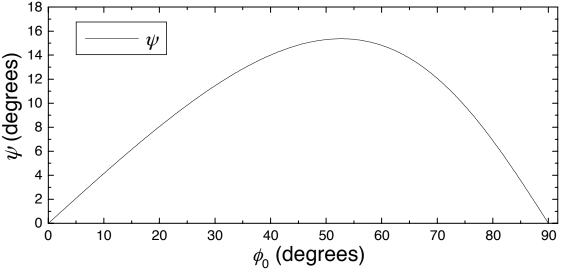


$$u(z) = -\frac{\left[\eta_1 \sin^2\phi_0 + \eta_3 \cos^2\phi_0\right]}{2\eta_1\eta_3}\, az(d-z)$$

$$v(z) = -\frac{(\eta_3 - \eta_1)}{2\eta_1\eta_3} \sin\phi_0 \cos\phi_0\, az(d-z)$$

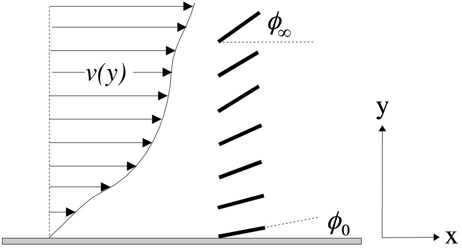
The theoretical model is astonishingly accurate when the director is held fixed by a strong magnetic field at the angle ϕ_0, selected by the experimentalist.

$$\tan\psi = \frac{v}{u} = -\frac{(\alpha_3+\alpha_6)\sin\phi_0\cos\phi_0}{\alpha_4+(\alpha_3+\alpha_6)\sin^2\phi_0} = \frac{(\eta_3-\eta_1)\sin\phi_0\cos\phi_0}{\eta_1\sin^2\phi_0+\eta_3\cos^2\phi_0}$$



The flow is diverted by the angle ψ depicted in Figure Fig. (b) above. This angle ψ has been calculated here for the material parameters of MBBA near 25°C.

Shear Flow Near a Boundary



$$y = \int_{\phi_0}^{\phi} \left[-\frac{c\gamma_2}{f(\zeta)} \int_{\zeta}^{\phi_\infty} \frac{\cos(2\psi) - \cos(2\phi_\infty)}{g(\psi)} \, d\psi \right]^{\frac{1}{2}} d\zeta. \qquad (27)$$

This provides y as a function of ϕ, that is, the solution ϕ is implicitly given as a function of y in (27); ϕ increases monotonically with y. The constant c The constant c can be determined by the shear stress, as commented by Müller (1985). f depends on the elastic constants, g depends on the viscosities; both depend on the orientation of $\mathbf{n}$.

The velocity profile is given by

$$v(y) = \int_0^y \frac{c}{g(\phi(\zeta))}\, d\zeta. \tag{28}$$

It can be calculated using the solution for ϕ in the previous equation.
A 'boundary layer' type of behaviour for the above solution has also been discussed by Leslie (1968). For example, if we set

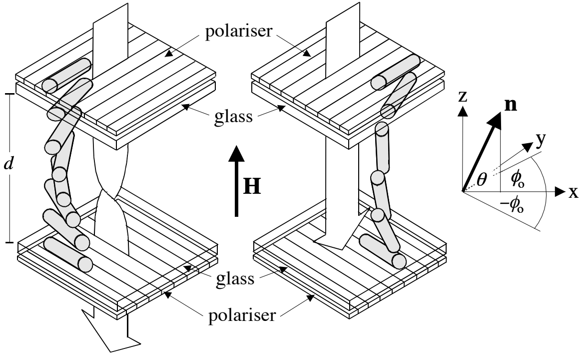
$$\xi = (|K_1| + |K_3|)\, |c|^{-1}, \tag{29}$$

then, by taking basic inequalities, it can be shown that

$$\lim_{\xi\to 0} y \le \lim_{\xi\to 0} \sqrt{\xi} \int_{\phi_0}^{\phi} \left[-\gamma_2 \int_{\zeta}^{\phi_\infty} \frac{\cos(2\psi) - \cos(2\phi_\infty)}{g(\psi)}\, d\psi \right]^{\frac{1}{2}} d\zeta = 0, \tag{30}$$

provided $\phi \neq \phi_\infty$. This demonstrates the existence of a boundary layer phenomenon when the length $\sqrt{\xi}$ is sufficiently small.
The procedure adopted by Leslie can be applied to other boundary layer phenomena in liquid crystals.

The twisted nematic device



(a) 'off state' $H < H_c$ (b) 'on state' $H > H_c$

This was studied theoretically by Leslie in 1970 and was applied by Schadt and Helfrich in the early 1970s to twisted nematic liquid crystal devices using an electric rather than a magnetic field (Figure from IWS 'The Static and Dynamic Continuum Theory of Liquid Crystals' Taylor & Francis, 2004).

H_c satisfies

$$\chi_a d^2 H_c^2 = \pi^2 K_1 + 4\phi_0^2 (K_3 - 2K_2)$$

provided

$$K_3 - 2K_2 \geq 0 \qquad \text{or} \qquad \phi_0 \leq \frac{\pi}{2} \sqrt{\frac{K_1}{2K_2 - K_3}}$$

when $K_3 - 2K_2 < 0$.

The Ericksen–Leslie theory can be used for modelling the switching dynamics.

Comments

- The Ericksen–Leslie theory, when linearised, matches many aspects of previous models, e.g., Martin, Parodi & Pershan (1972)
- It is possible to symmetrise the Ericksen stress used in the static version of the continuum theory (cf. Landau & Lifshitz). This is not possible in the dynamic theory unless further omissions or restrictions are made on viscosity coefficients.

Developments using similar continuum theory approaches

- Many people involved in continuum theories for liquid crystals have gone unmentioned in the above discussion
- Key experiments and theory were carried out by de Gennes and the Orsay Group, and many others, on nematics and smectics
- Leslie also produced continuum theories for the dynamics of choleseteric liquid crystals in 1968 & 1969; further theory for biaxial nematics by Leslie, Carlsson and Laverty in 1990s
- This led to further continuum theories by de Gennes (1974) for smectic A liquid crystals and Leslie and co-workers in the 1990s for smectic C (SmC) and ferroelectric liquid crystals (SmC*)
- More recent work on smectic A dynamics, developed from the work of Ericksen and Leslie, by E (1997), Auernhammer, Brand and Pleiner (2000s) and Stewart (2007)
- Various related microstructure approaches, e.g., Capriz, Eringen...

Conclusions and further work

Conclusion:

- Ericksen–Leslie theory has been highly successful
- Ideas have been developed to further the modelling of other liquid crystal phases

Further work:

- Use of Frank's original energy from his 1958 paper when it is 'odd' in the sign of **n**, e.g., work by May (2000) and De Vita and Stewart (2013) – important in polarisable media
- Applications of similar continuum approaches in the non-Newtonian modelling of blood plasma with red blood cells – combinations of anisotropy and coagulation processes/gelation processes – recent work by Gonzalez-Moyers, Owens and others, who use the Smoluchowski coagulation model combined with non-Newtonian flow behaviour. Shear rate related to coagulation kernel.

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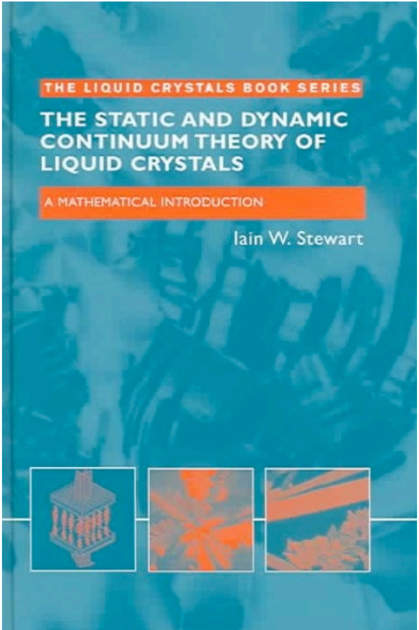
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