Fish gotta swim, Birds gotta fly, I gotta do Feynmann Graphs 'til I die:

A continuum theory of flocking

Original idea: T. Vicsek, Eötvos Institute Budapest, Hungary

Collaborators: Yu-hai Tu, IBM, Research

Everything

Markus Ulm

simulations

Inspiration: A. Hitchcock, E. Fields

Two dimensions: Ferromagnetic flock

(MMB (2

Several hundred thousand wildbeest in a moving front, grazing on the Serengeti.
.E. Sinclair (1977), reproduced by permission of the author and publisher.

photograph of wildbeest grazing on the Serengeti Plains of Africa, hundreds of
of animals form a moving front, a sharp transition in the density of the popu-
ch resembles protruding fingers, see Fig.1. A similary dense flock of flamingos
can lake demonstrates cohesiveness, a well defined margin, as well as internal
uch as directionality, see Fig.2. The roughly circular wave of density emanating
in the side of the flock may suggest some disturbance that initiated an outward

Three dimensions

MM4 3

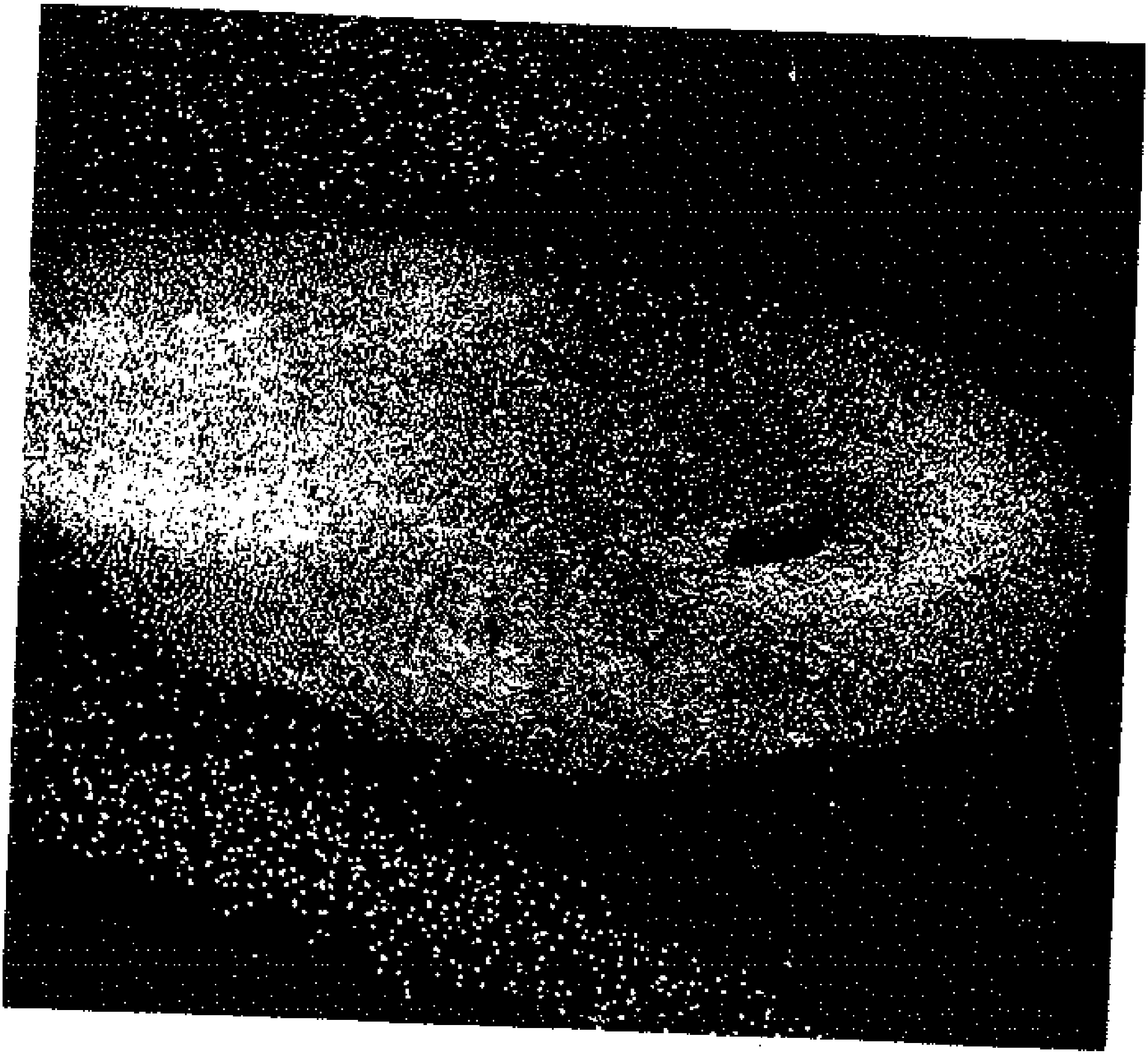


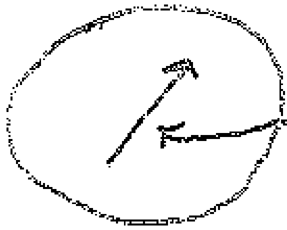
Figure 2: A flock of flamingos on an African lake. Note variations in density and in th orientations of the birds. A fairly distinct front is maintained.

which have carried the most traffic. The authors demonstrate that a model for pheromon and ant traffics accounts for experimentally observed traffic patterns.

Several contributions in this chapter further develop the theme of trail following

Bacteria:
(10^9 of 'em)

All cases Spontaneously Broken Continuous Symmetry



arbitrary direction But same for all

Short ranged interactions $\Rightarrow$ Long ranged order

Violates Mermin-Wagner Thm in $d=2$!

How? Non-Equilibrium (Motion)

$\Downarrow$

Breakdown of linearized hydrodynamics

$\Downarrow$

Stabilizes Long ranged Order

Ferromagnetic Flocks
Outline

I) Intro: What's Flocking?

II) Microscopic Models - Vicsek

Important points: Rotation Invariance (Lost sheep)
Locality:

III) Mermin-Wagner Theorem:
Are Birds smarter than Nerds?

IV) Continuum model:
- Analogy with Navier-Stokes (Fluid mechanics)
- Use symmetries
- Bury our ignorance

V) No, Birds ain't smarter:
How motion beats Mermin-Wagner

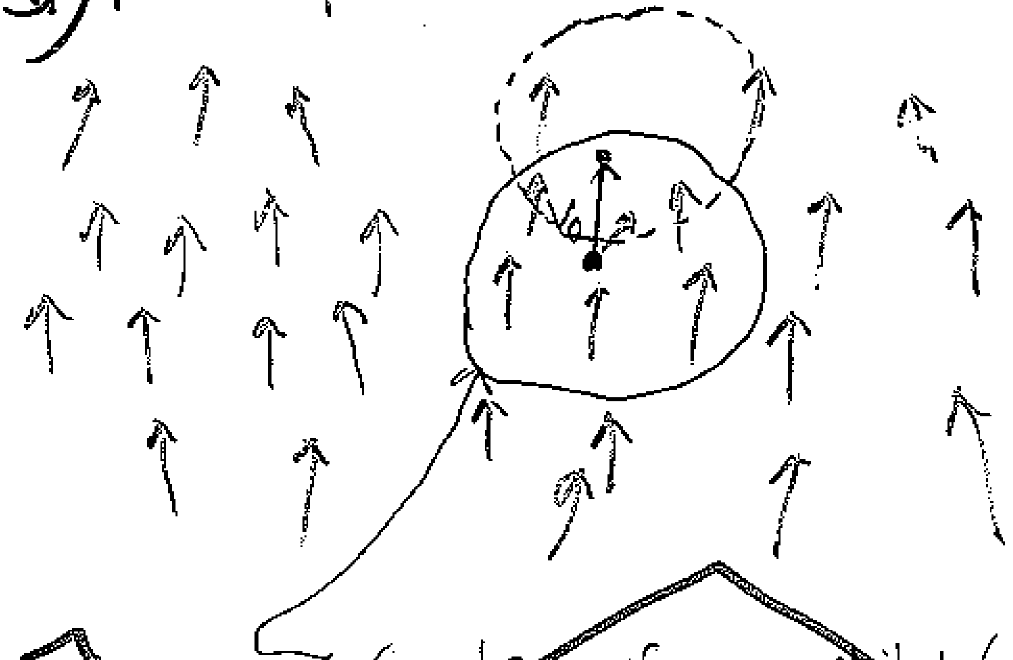
VI) Our other predictions

VII) Experiments we'd like to see.

VIII) Future theory (+ some already done)

IX

II) "Microscopic Models": Vicsek's (2d)



$|\vec{v}_i(t)| = v_0$ (all i, t)

Circle of visibility (radius $\equiv 1$) C

$$\vec{v}_i(t+1) = \langle \vec{v}(t) \rangle_C + \underbrace{\vec{\eta}_i(t)}_{\text{Mistake (random)}}$$

$\langle \vec{\eta}_i(t) \rangle = 0$ (correct ~~right~~, on average)

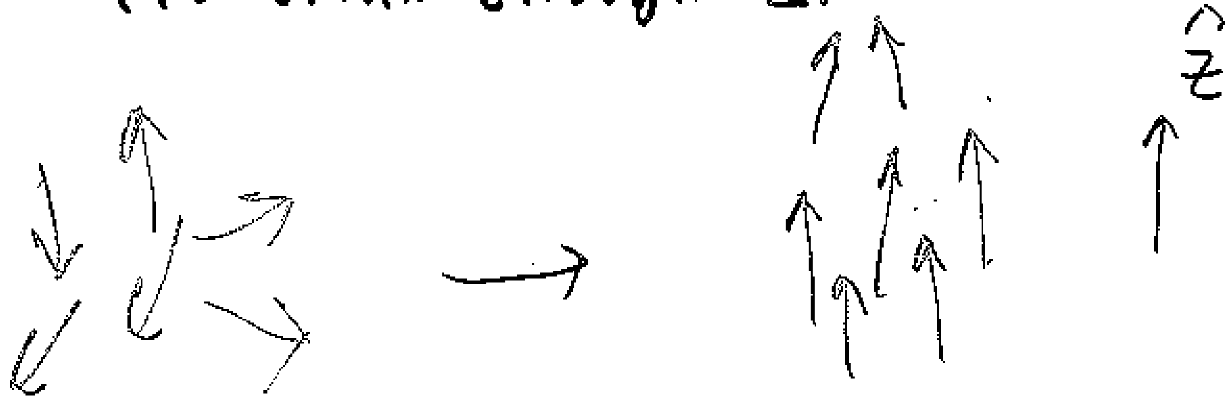
$$\langle \eta_{i\alpha}(t)\, \eta_{j\beta}(t') \rangle = \begin{cases} 0, & i \neq j \text{ (different birds' errors uncorrelated)} \\ 0, & t \neq t' \text{ (no memory)} \\ \Delta, & i=j,\ t=t',\ \alpha=\beta \text{ (noise strength)} \end{cases}$$

Note: Rotation invariant

$\Rightarrow$ "lost sheep"

Vicsek simulates this, finds:

Flock Spontaneously orders
(For small enough Δ)



$t=0$ $\qquad\qquad$ $t \gg 1$

$\langle \vec{v} \rangle = 0$ $\qquad\qquad$ $\langle \vec{v} \rangle = v_0 \hat{z} \neq 0$

Arbitrary direction

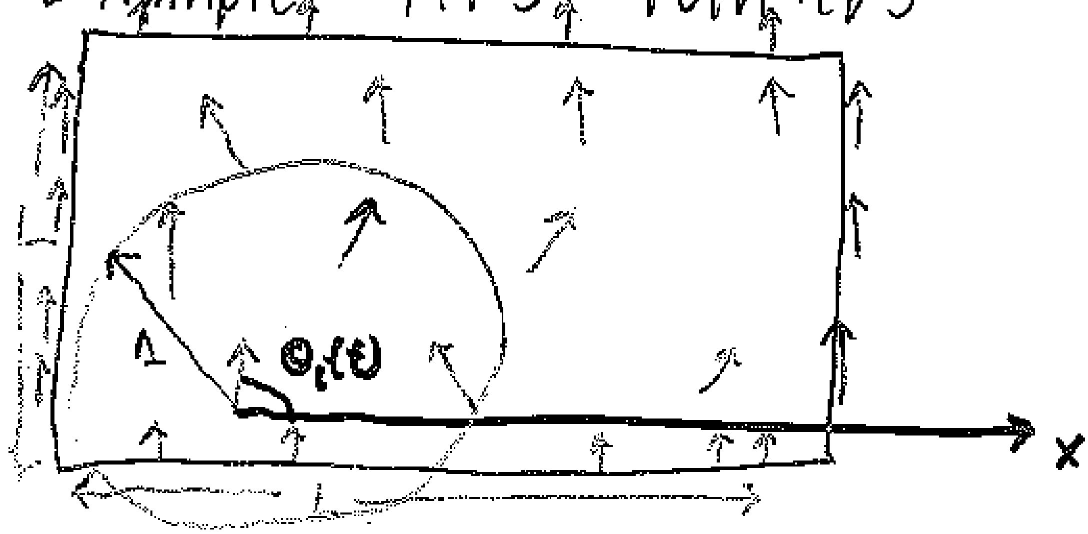
In $d=2$, this is Astonishing!

Violates Mermin-Wagner Thm!

III) Mermin-Wagner Thm: $\boxed{\text{Are birds smarter than nerds?}}$

Spontaneously Broken Continuous symmetry Impossible in $d=2$:

Example: APS Pointers



same orientational rule:

$$\theta_i(t+1) = \langle \theta_j(t) \rangle_{j \in O} + \eta_i(t)$$

$$\langle \eta_i \rangle = 0, \quad \langle \eta_i(t)\, \eta_j(t') \rangle = \Delta\, \delta_{ij}\, \delta_{tt'}$$

No motion $\Rightarrow$ 2d XY model
Equilibrium

$\Rightarrow \langle \theta_i^2 \rangle \propto \ln L \to \infty$

$\Rightarrow$ No LRO

Why? Monte Carlo for Equilibrium

2d XY model $\Rightarrow$ Mermin-Wagner Theorem

Physics: In $d \leq 2$, errors generated faster than dissipated

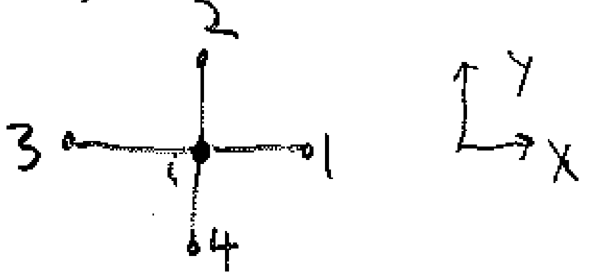
Rewrite evolution rule: subtract $\theta_i(t)$ from both sides

$$\underbrace{\theta_i(t+1) - \theta_i(t)}_{\downarrow} = \underbrace{\langle \theta_j(t) \rangle - \theta_i(t)}_{\downarrow} + \eta_i$$

$\longleftarrow$ continuum limit $\longrightarrow$

$$\partial_t \theta = D(\partial_x^2 \theta + \partial_y^2 \theta) + \eta$$

To see this, consider lattice:



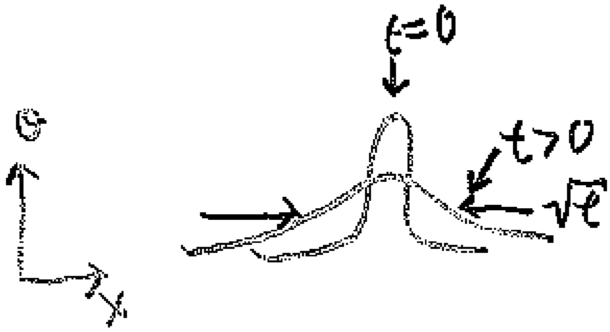
$$\langle \theta_j \rangle - \theta_i = \frac{1}{4}(\theta_1 + \theta_2 + \theta_3 + \theta_4) - \theta_i$$

$$= \frac{1}{4}\Big[\overbrace{\underbrace{(\theta_1 - \theta_i)}_{\partial_x \theta(x,y)} - \underbrace{(\theta_i - \theta_3)}_{\partial_x \theta(x-1,y)}}^{\partial_x(\partial_x \theta)} + \overbrace{\underbrace{(\theta_2 - \theta_i)}_{\partial_y \theta(x,y)} - \underbrace{(\theta_i - \theta_4)}_{\partial_y \theta(x,y-1)}}^{\partial_y(\partial_y \theta)} \Big]$$

$$\partial_t \Theta = D \nabla^2 \Theta + \eta$$

$\boxed{\text{Diffusion}}$ equation: Like Ink in water

Properties: (1) Slow



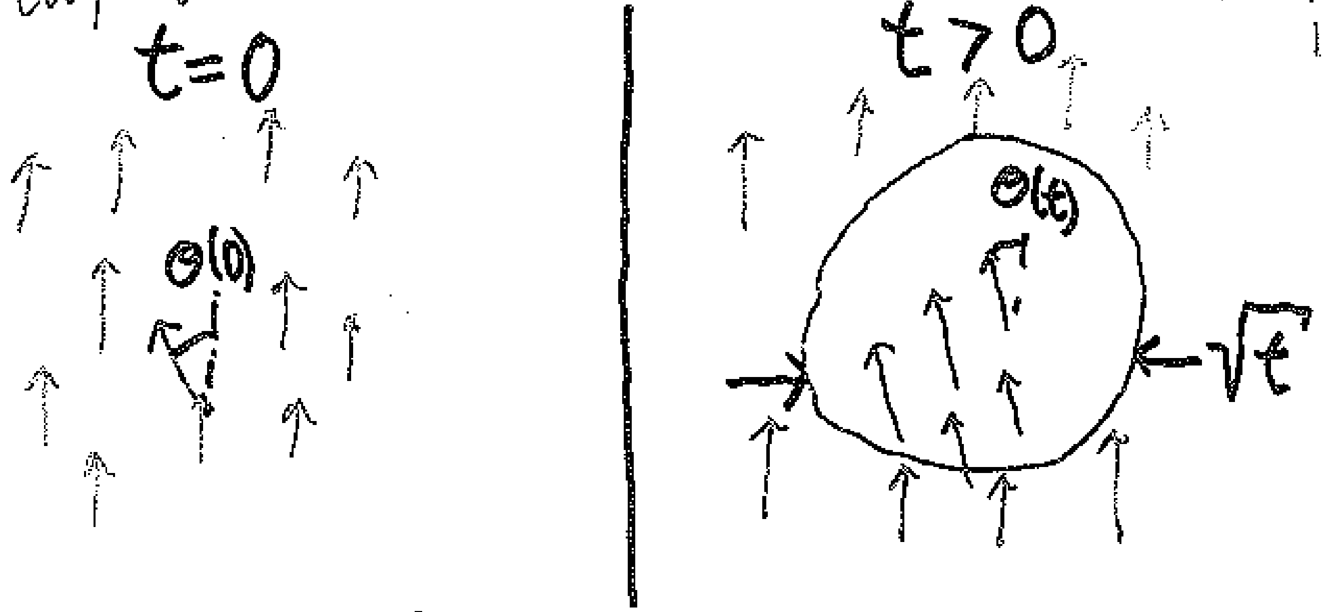
$$\frac{\Theta}{t} \sim \frac{\Theta}{x^2} \Rightarrow x^2 \propto t \Rightarrow x \propto \sqrt{t}$$

2) Conserves:

$$\partial_t \int \Theta \, d^d r = D \int \nabla^2 \Theta \, d^d r = D \int \vec{\nabla} \Theta \cdot dS$$

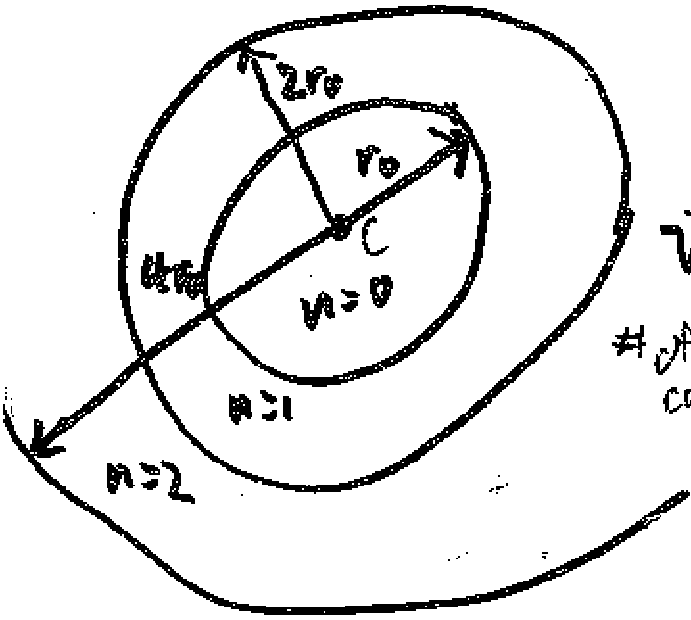
$\int \vec{\nabla}\Theta \cdot \vec{dS} = 0$ if flock size $\gg \sqrt{t}$

Decay of One error: $d\uparrow \Rightarrow$ slower decay



$$\text{conserved} \Rightarrow \theta(t) \approx \frac{\theta_0}{N} \approx \frac{\theta_0}{t^{d/2}} \quad (d \text{ dimensions})$$

Meanwhile: More errors:



Total error at center:

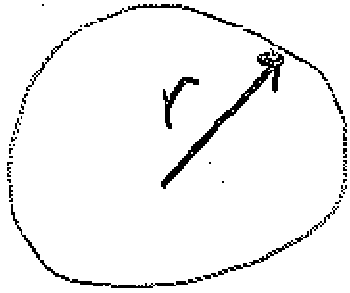
$$\sqrt{\delta\theta_0^2} = \sqrt{\underbrace{r_0^d}_{\#\text{ of error committers}}\ \underbrace{t_0}_{\#\text{ of errors each makes}}} \Big/ r_0^d$$

$$t_0 \sim r_0^2 \Rightarrow \delta\theta_0^2 \sim r_0^{2-d}$$

$$\delta\theta_1^2 \sim (2r_0)^{2-d} = \delta\theta_0^2\, 2^{2-d}$$

$$\vdots$$

Meanwhile: More errors



Diffusion time $t_D(r) \propto r^2$

$\#$ of errors/spin $\propto t_D \propto r^2$

$\#$ of spins $N \propto r^d$

$\Rightarrow$ $\#$ of errors $\propto N t_D \propto r^{d+2}$

$\Rightarrow$ RMS error/spin $\delta\theta \propto \dfrac{\sqrt{r^{d+2}}}{r^d}$

$$\Rightarrow \delta\theta \propto r^{1-\frac{d}{2}} \xrightarrow[r\to\infty]{} \begin{cases} 0, & d>2 \Rightarrow \text{LRO} \\ \infty, & d<2 \Rightarrow \text{No LRO} \\ \ln L \to \infty, & d=2 \Rightarrow \text{No LRO} \end{cases}$$

How do birds beat this?

IV) Continuum theory of birds:

- Hard (impossible) to solve microscopic model with $\sim 10^6$ birds
- Harder to figure out what happens if you change model

Historical analog: Fluid mechanics
Navier-Stokes equation
@ 1840

- No theory of atoms, molecules, etc.
- No statistical Physics
- No 37" TV
- No computers

IV) Back to birds: Continuum model: Phenomenological

How do you do this?

Arbitrary spatial dimension d

Fields: $\vec{v}(\vec{r},t)$: Coarse grained velocity

$\rho(\vec{r},t)$: " " number density

Length scales: $L \gg$ Interbird distance

time scales $t \gg$ microscopic time scale

- Expand in powers of spacetime gradients
- Expand " " " fields
- keep all terms consistent with symmetries

: Rotation invariance: $\vec{v} \to \underset{\sim}{R}\,\vec{v}$

$\vec{r} \to \underset{\sim}{R}\,\vec{r}$

$\Rightarrow \vec{\nabla} \to \underset{\sim}{R}\,\vec{\nabla}$

⇒ Equations of Motion: for Birds

$F = ma$

$$\partial_t \vec{v} + \underbrace{\lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v}}_{\text{Convection}} + \underbrace{\lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v})}_{\text{crunch anxiety}} + \underbrace{\lambda_3 \vec{\nabla}(|\vec{v}|^2)}_{\substack{\text{speed kills } (\lambda_3>0) \\ \text{speed thrills } (\lambda_3<0)}}$$

$$= \underbrace{\alpha \vec{v} - \beta |\vec{v}|^2 \vec{v}}_{\text{keep moving: } v_0 = \sqrt{\alpha/\beta} = |\vec{v}|} - \vec{\nabla} P(\rho) \rightarrow \text{Pressure Forces}$$

$$+ \underbrace{D_B \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) + D_1 \nabla^2 \vec{v} + D_2 (\vec{v}\cdot\vec{\nabla})^2 \vec{v}}_{\text{Follow your neighbors}} + \underbrace{\vec{f}}_{\text{mistakes}}$$

Bird conservation:

$$\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0$$

$$P(\rho) \equiv \sum_{n=1}^{\infty} \alpha_n (\rho - \langle\rho\rangle)^n$$: Only $n = 1, 2$ important

$\vec{f}$ Gaussian, white

$$\langle \vec{f} \rangle = 0$$

$$\langle f_i(\vec{r}, t)\, f_j(\vec{r}\,', t') \rangle = \Delta\, \delta_{ij}\, \delta(\vec{r} - \vec{r}\,')\, \delta(t - t')$$

Solve using fluctuating hydrodynamics techniques (Dynamical P.T., RG, etc)

Results: Negative Feedback stabilizes order:

Fluctuations $\uparrow \Rightarrow$ Diffusion $\uparrow \Rightarrow$ Fluctuations $\downarrow$

(anomalous diffusion: $D_\perp(L) \propto L^{4/5}$)

seen in simulations by Yu-hai Tu

(like "breakdown of hydrodynamics" in $d=2$ fluid, but much stronger)

$\Downarrow$

LRO $\underline{\underline{\text{is}}}$ possible in $d=2$ ~~Mermin-Wagner~~

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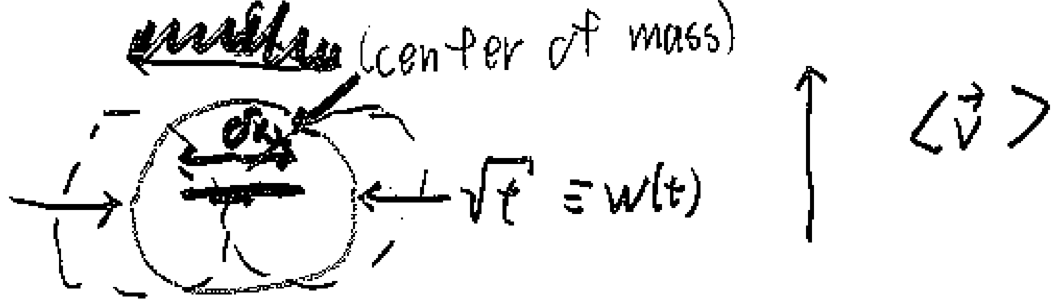
V) How motion beats Mermin-Wagner

Hand waving:



How far $\delta x_\perp$ does blob move $\perp \langle \vec{v} \rangle$?

$$\delta x_\perp \sim \sqrt{v_\perp^2}\, t \sim v_0 \sqrt{\theta^2}\, t$$

remember: $\sqrt{\theta^2} \propto r^{1-\frac{d}{2}} \propto t^{\frac{1}{2}-\frac{d}{4}}$

$$\Rightarrow \boxed{\delta x_\perp \propto t^{\frac{3}{2}-\frac{d}{4}}}$$

Compare with width of blob:

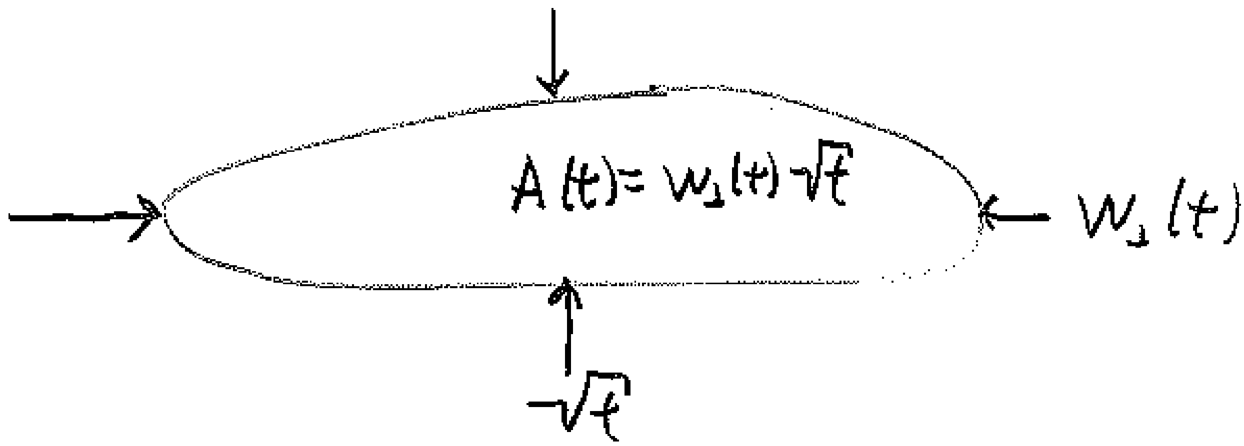
$$\frac{\delta x_\perp}{w(t)} \propto \frac{t^{\frac{3}{2}-\frac{d}{4}}}{t^{1/2}} \propto t^{1-\frac{d}{4}} \xrightarrow[t\to\infty]{} \begin{cases} 0, & d>4 \\ \infty, & d<4 \end{cases}$$

$\Rightarrow$ Bird motion dominates diffusion for $d<4$

$\Rightarrow$ Physics changes in $d<4$

("Breakdown of linearized hydrodynamics")

So what <u>does</u> happen in $d < 4$?
Blobs spread anisotropically



$\#$ of errors/spin $\propto t$

$\#$ of spins $\propto A(t) \propto W_\perp(t)\sqrt{t}$

$\Rightarrow \#$ of errors $\propto A(t)\,t \propto W_\perp(t)\,t^{3/2}$

$\Rightarrow$ RMS error/spin $\delta\theta \propto \dfrac{\sqrt{W_\perp(t)\,t^{3/2}}}{W_\perp(t)\sqrt{t}} \propto \dfrac{t^{1/4}}{W_\perp^{1/2}}$

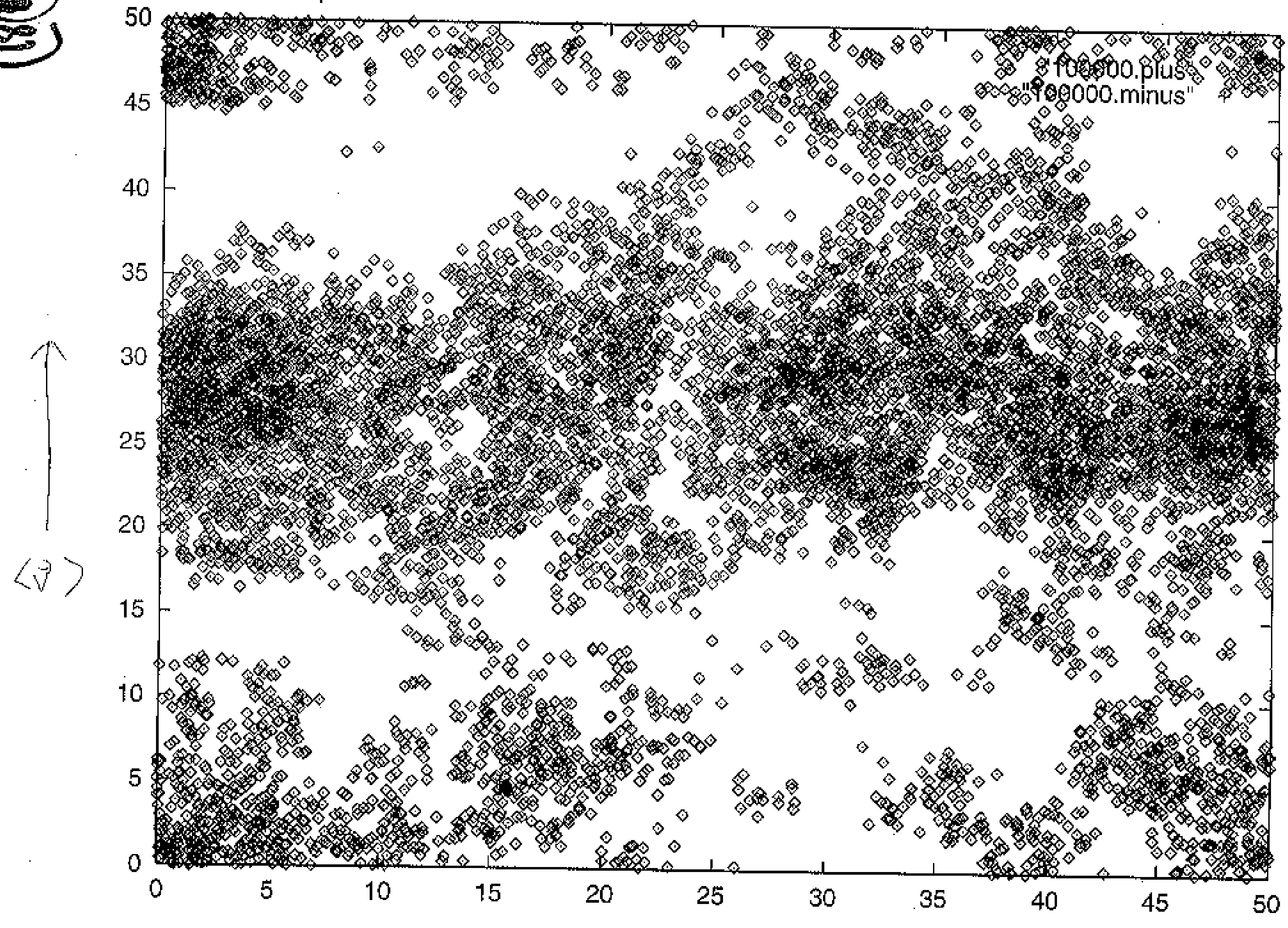
self-consistency:

$$W_\perp(t) \sim v_0\,\delta\theta\,t \propto \frac{t^{5/4}}{W_\perp^{1/2}} \Rightarrow W_\perp^{3/2} \propto t^{5/4} \Rightarrow W_\perp \propto t^{5/6} \gg t^{1/2}$$

$$\Rightarrow \delta\theta \propto \frac{t^{1/4}}{t^{5/12}} \propto t^{-1/6} \to 0 \text{ as } t \to \infty$$

$\Rightarrow$ LRO!! ~~Mermin-Wagner~~

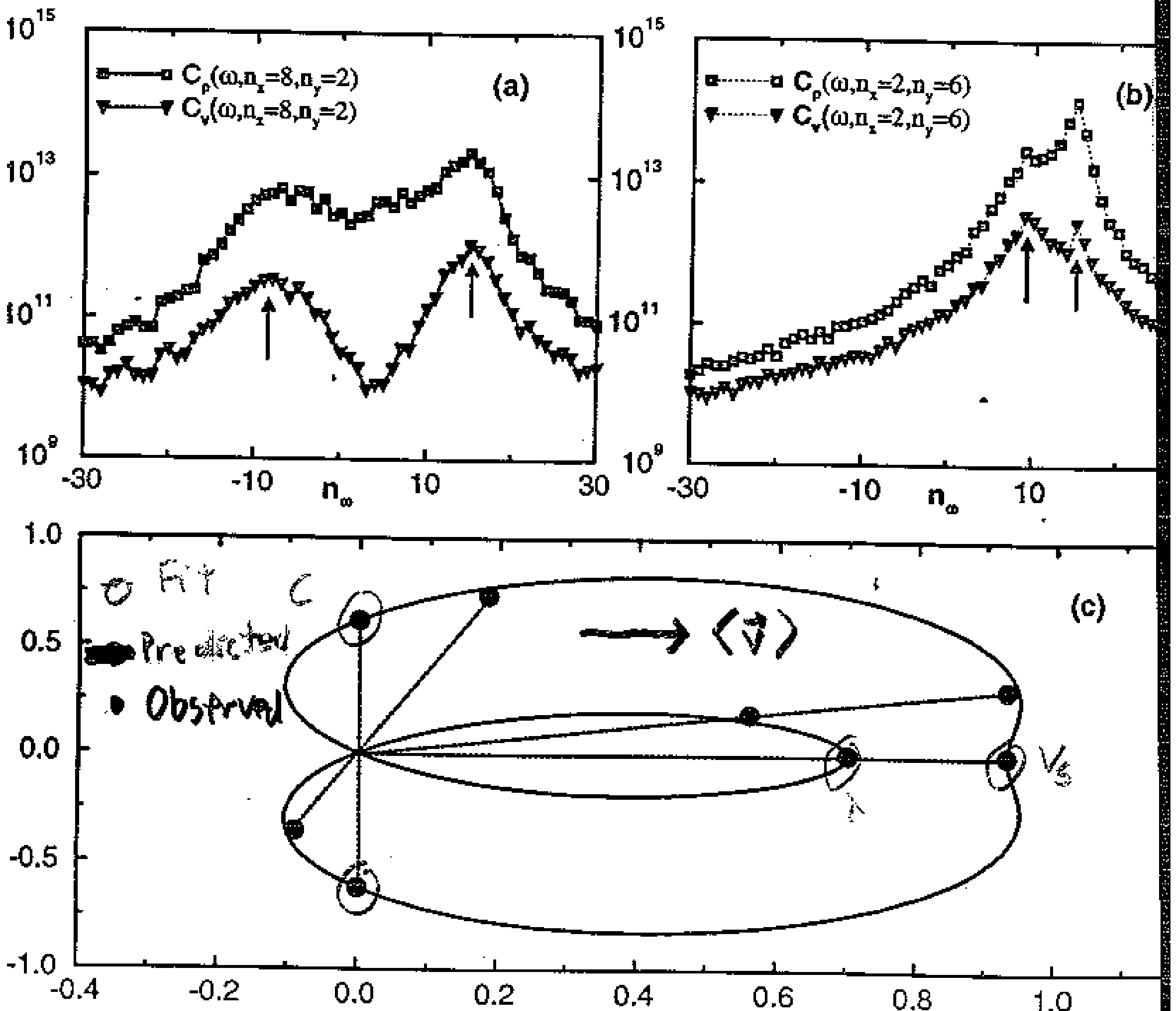
Simulated by Markus Ulm



1) 0 Our (other) Predictions: - Anisotropic sound speeds

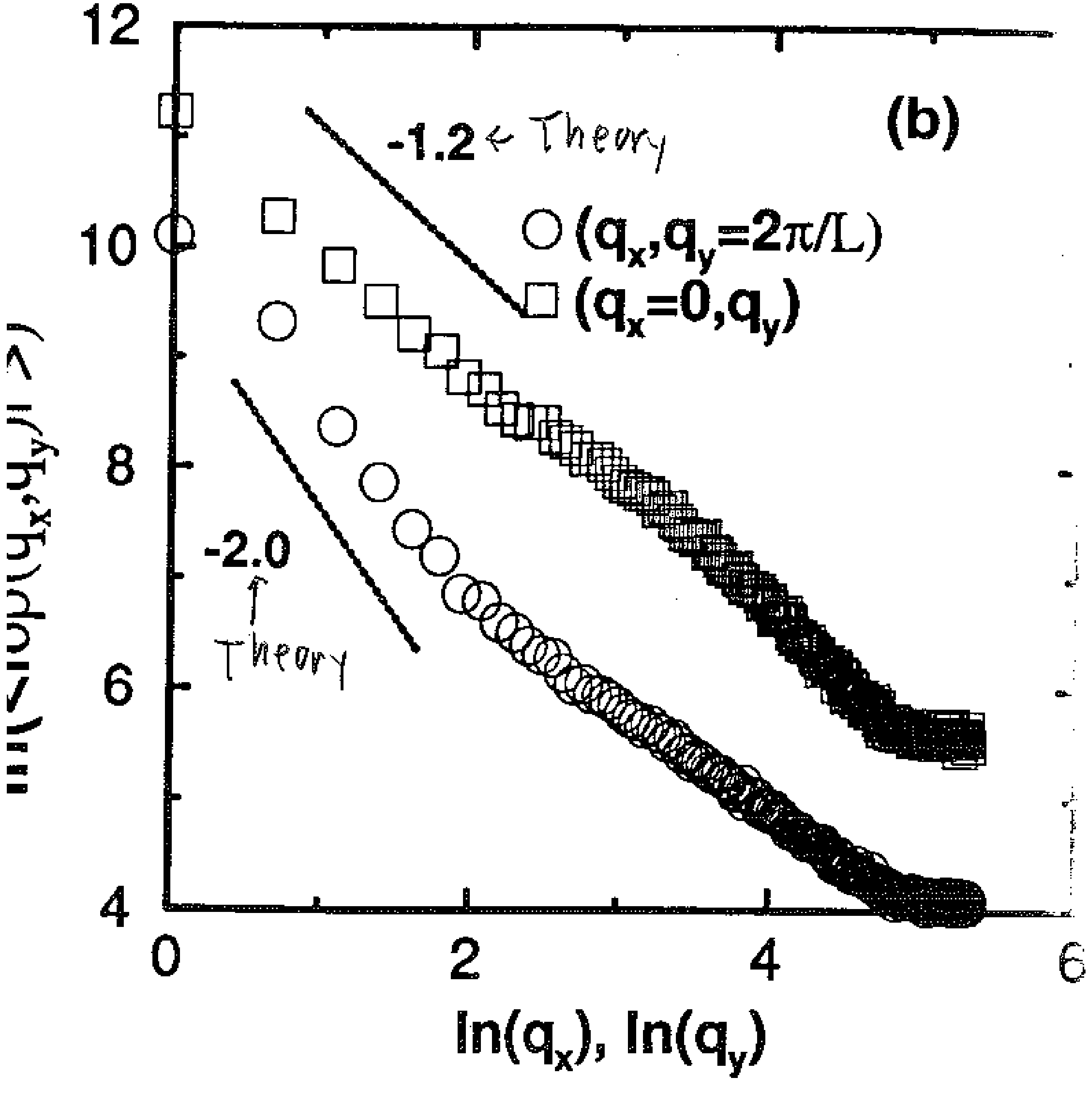
Simple fluid:  - 2 sound modes

Verify in simulations:

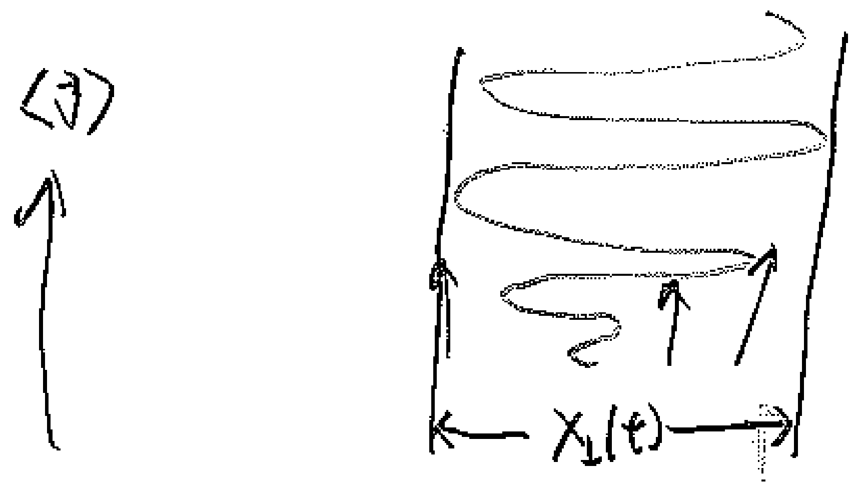


Big, Anisotropic Density Fluctuations:

Simulation results:

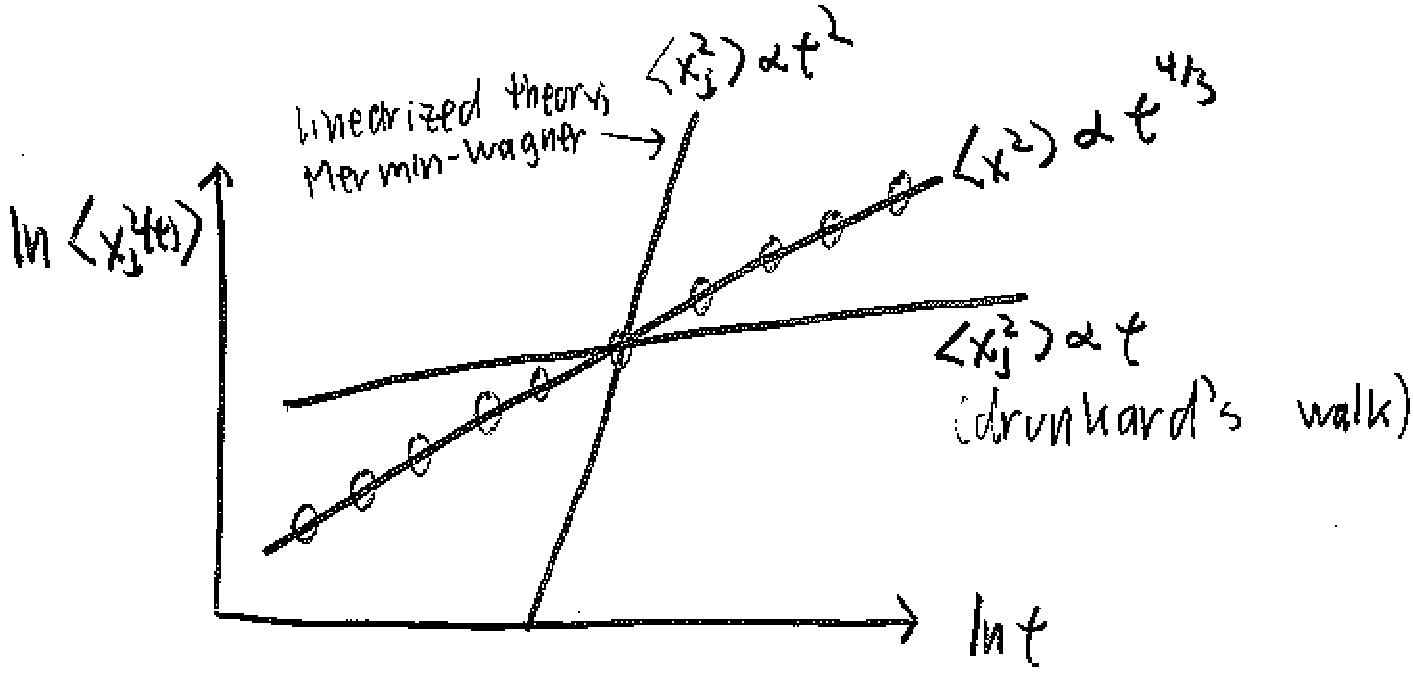


Motion of individual bird in flock:



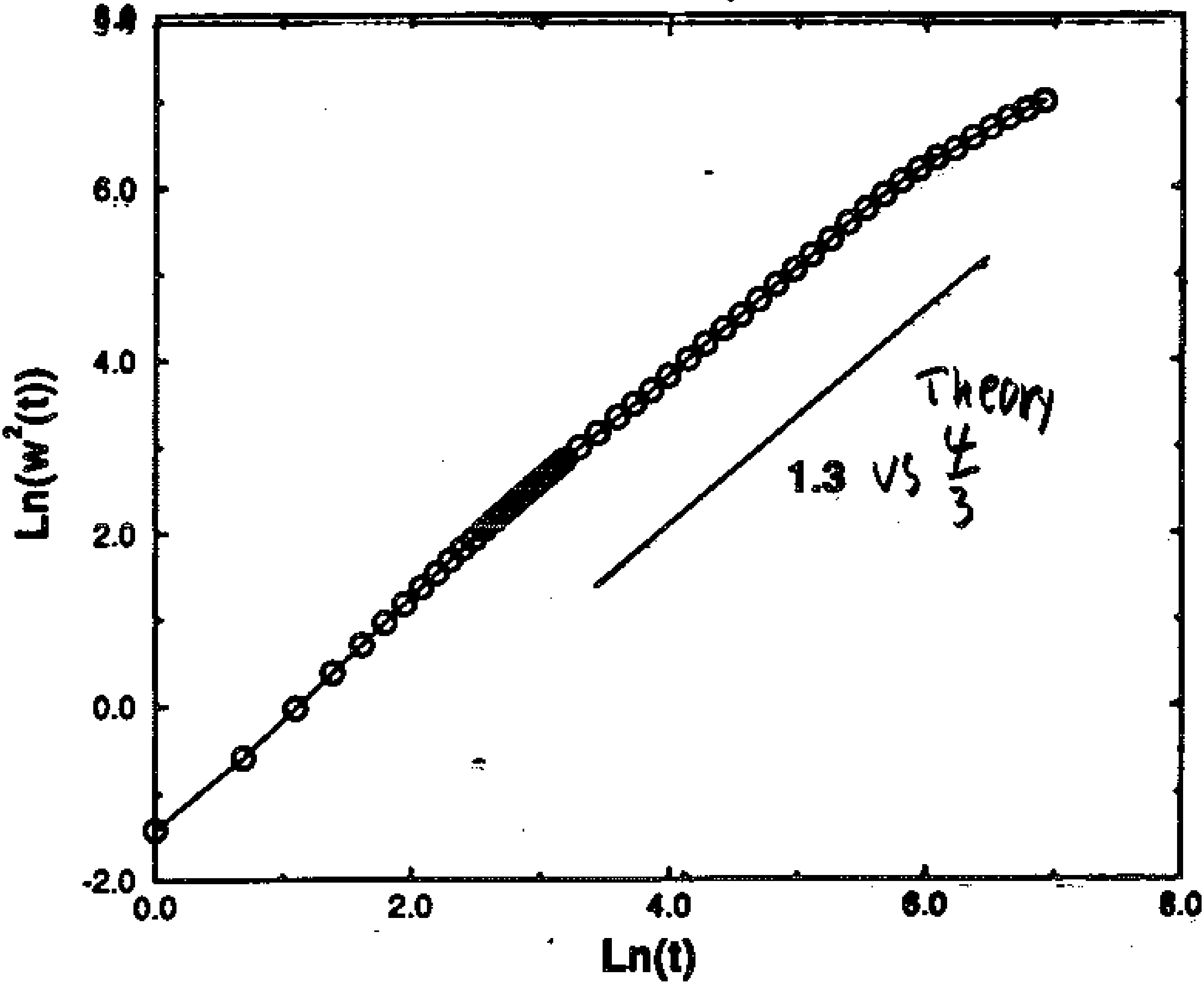
We predict: $\langle x_\perp^2(t)\rangle \propto t^{4/3}$

Simulations by Ulm: Viczek algorithm



Same plot, different
"microscopic rules" (avoid collisions, etc)
(Oil vs. water)

Universality:



VII) The future:

What we have so far:

1) Powerful analytic theory
2) $\boxed{\text{Numerical}}$ experiments that confirm analytic theory

So what's missing?

Real Experiments.

What sort of experiments?
" " " data?
" kind of analysis?

Basically, repeat our numerical experiments with real critters

E.g.: Video images (microscopy, film, etc.

Data? trajectories $\vec{r}_i(t)$
labels birds

$\Rightarrow$ Fourier transformed density

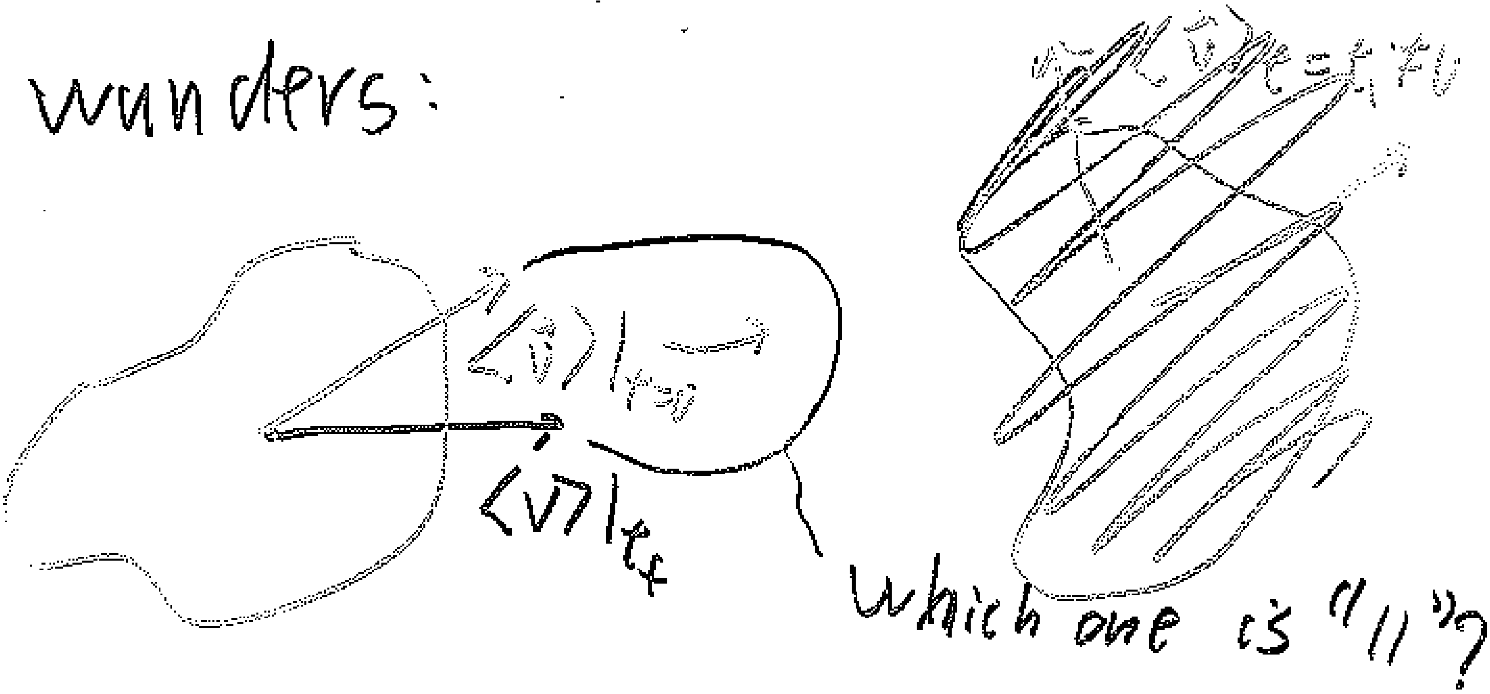
$$\rho(\vec{q},t) = \sum_i e^{i\vec{q}\cdot\vec{r}_i(t)}$$

$$\Downarrow$$

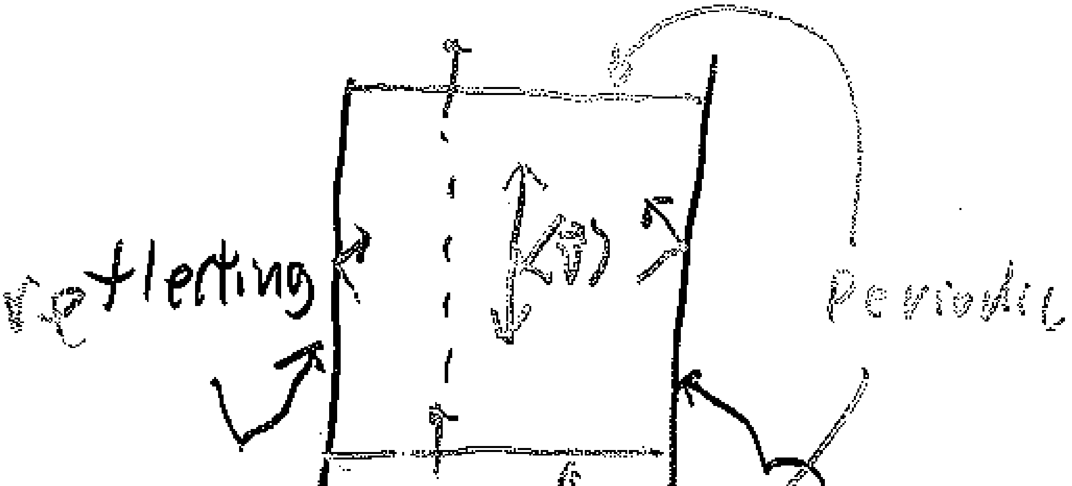
$$\underbrace{\langle |\rho(\vec{q},t)|^2 \rangle}_{\text{time average}} \propto \begin{cases} \frac{1}{q_\parallel^2}, & \vec{q} \text{ along flock motion} \\ \frac{1}{q_\perp^{6/5}}, & \vec{q} \perp \text{flock motion} \end{cases}$$

Caveat: Very important to know which way flock moves.

In free flock, this direction wanders:

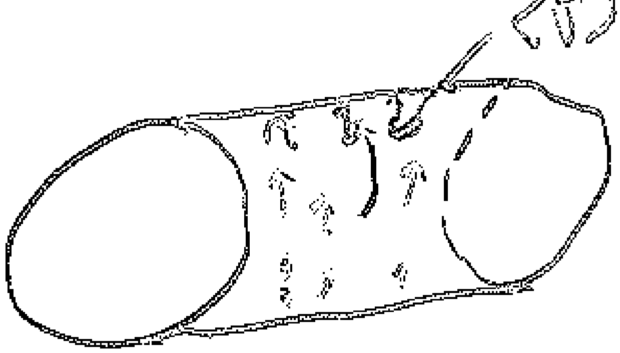


In numerical experiment, we fixed $\parallel$ with boundary conditions:

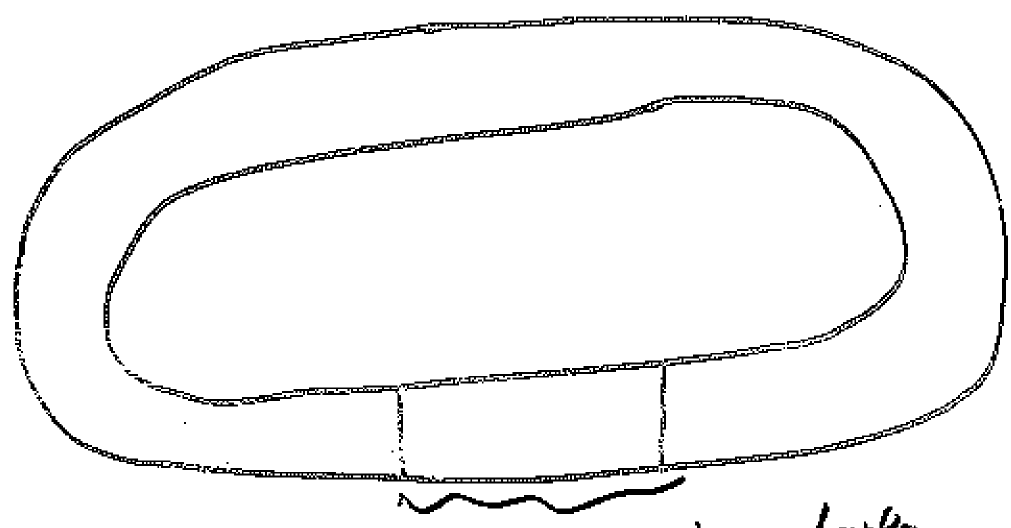


Doing this in real experiment:

1) "Tin can" (cylinder):



2) "Straightaway":



only take data here.

Other caveats:

1) Need huge herds:

hydrodynamics valid for $L \gg a$

$\Rightarrow N \sim \left(\frac{L}{a}\right)^d \gg 1$

---

2) Respect symmetries!

No compasses!

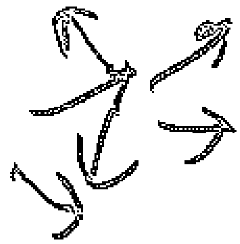
Creatures must spontaneously pick their direction of motion.

Future theoretical directions:

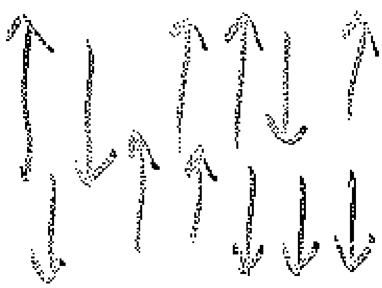
I "Ferromagnetic flocks" are only one possible "phase" of flocks.

Many others possible:

Disordered phase

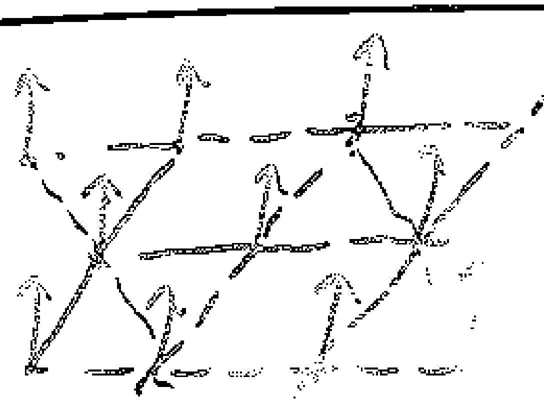


nematic phase
S. Ramaswamy, JT
Simha



Found experimentally in melanocytes (skin pigment)

"Flying crystal"



Hydrodynamics of background fluid: (Simha + Ramaswamy):

Instability at long wavelengths, low Reynolds number.

May have been seen in experiments

Phase transitions between the phases

Boundaries of flocks: Surface tension? Shapes?