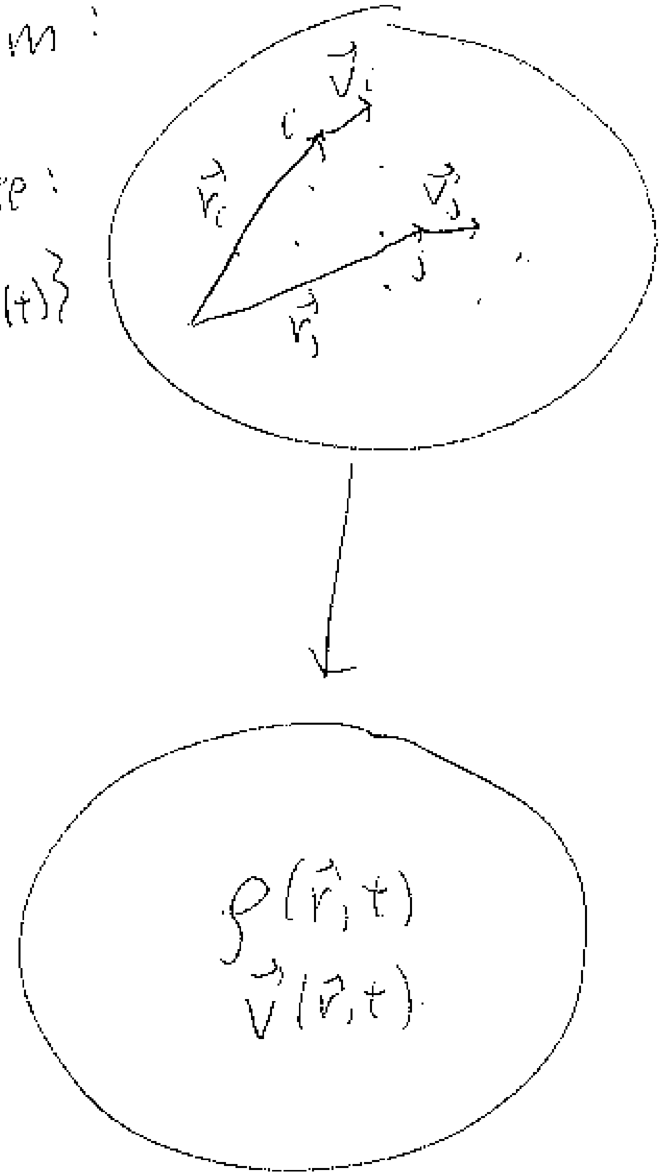
Deriving Continuum EOM:

II A1.

Hydrodynamic approach: (e.g., simple flu

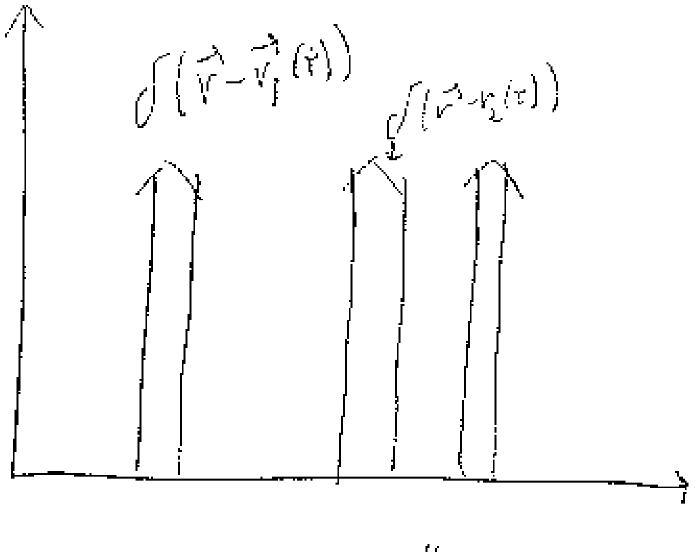
Go from:

Discrete:
$\{\vec{r}_i(t), \vec{v}_i(t)\}$



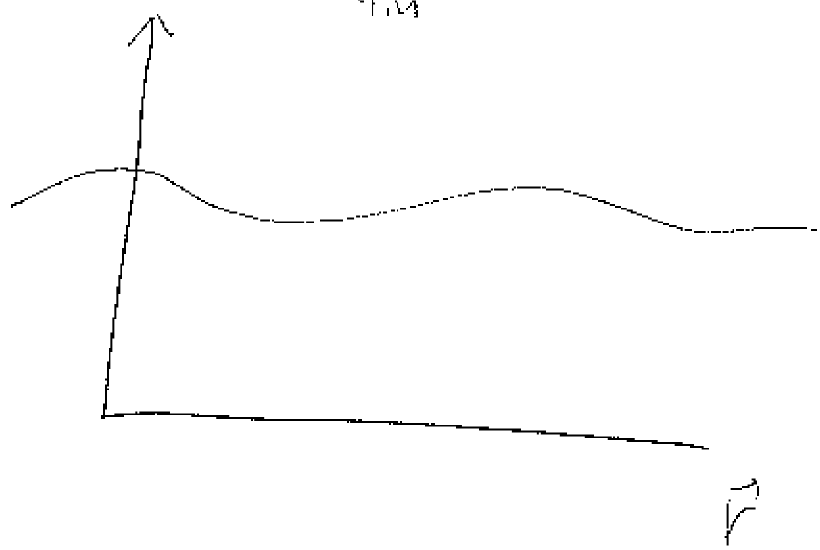
By "course-graining"
(either space, time, or both)

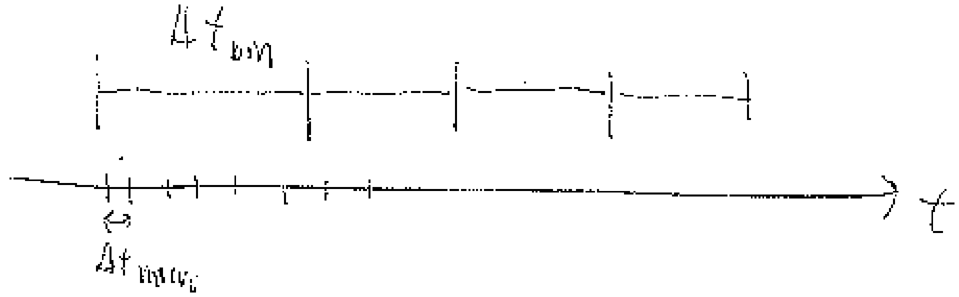
$\rho(\vec{r}, t)$ instantaneous:



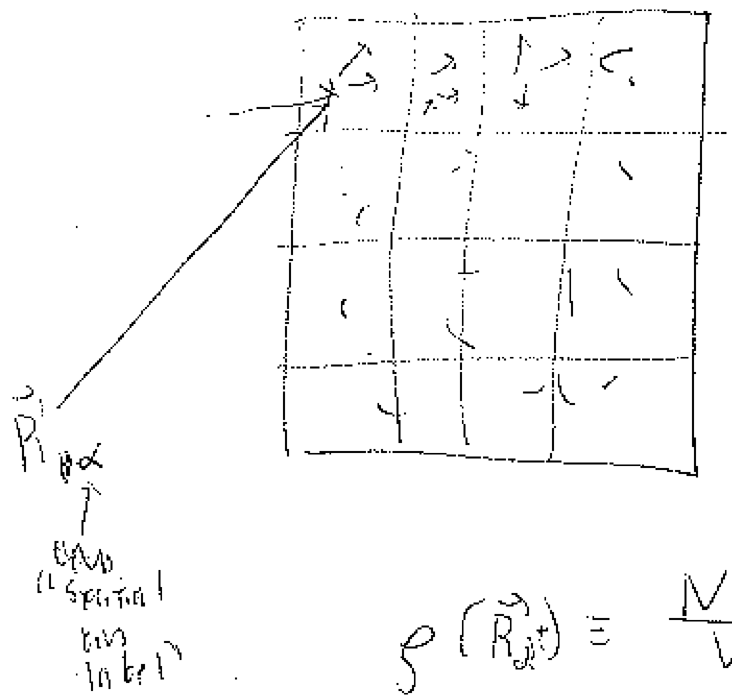
time average: over "bins" $\Delta t_{bin} \gg \Delta t_{step}$

$\langle \rho_{ins} \rangle_{\Delta t_{bin}}$





space average:



$$\rho(\vec{R}_\beta, t) \equiv \left.\frac{N \in \text{bin}}{V_{\text{bin}}}\right|_{\text{time } t}$$

$$\downarrow$$

$$\rho(\vec{r}, t)$$

Same with $\vec{v}$.

Try to "guess" EOM's from symmetry principles and conservation laws

II A

This "hydrodynamic" approach intrinsically long wavelength, long time (indeed, long ~~time~~ time at long wavelength)

$L \gg a \leftarrow$ "microscopic" length (e.g., inter-bird distance)

$t \gg \Delta t \leftarrow$ " time (e.g., ~~update~~ direction update time "ticking watch")

We hope (and invariably find) universality at these length and time scales

(e.g.: All ~~[illegible]~~ simple fluids obey Navier-Stokes equation

Problems: How do you know $\rho(\vec{r},t)$, $\vec{v}(\vec{r},t)$ are only fields you need?

What about other variables $X_\alpha(\vec{r},t)$? (E.g., $X_1(\vec{R}_\alpha,t) =$ # of birds moving "backwards" ($\theta < 90°$ to mean direction of neighbors) in bin α).

Answer: Almost all variables are "fast" (i.e., relax rapidly to values determined by local values of "slow" variables (Here, ρ and $\vec{v}$).

(See discussion in D. Forster, "Hydrodynamic ~~~~ Fluctuations, Broken Symmetry, and Correlation Functions")

Consider some "fast variable" x. What is it's EOM?

$$\cancel{\partial_t x = -\frac{1}{\tau_x}\left(x - \langle x(\text{slow})\rangle\right)}$$

$$\partial_t x = -\frac{x}{\tau_x} + f(\text{slow variables})$$

relaxation time for x.

If we're considering slow modes (i.e., modes for which (frequency) $\omega \to 0$ as $L \to \infty$), LHS $(\partial_t x) \ll \frac{x}{\tau} \Rightarrow$ drop it

$$\Rightarrow \boxed{x = \tau\, f(\text{slow variables})} \ast$$

$\Rightarrow$ If "slow variables" ~~cont~~ EOM's contain x:

$$\partial_t(\text{slow}) = g(x) + \ldots$$

Can use * to rewrite

as $\partial_t(\text{slow}) = f(\text{slow})$

So: We <u>only</u> need to consider <u>slow</u> variables

Strategy: <u>Guess</u> form of $f(\text{slow})$

(Actually, write down most general form allowed by <u>symmetries + conservation laws</u>)

What <u>are</u> slow variables?

Variables for which $\tau \to \infty$ as wavelength $L \to \infty$.

~~Why are the~~

(DA

Why are there <u>any</u> slow variables?

("slow" variables = "hydrodynamic variab

Ans: Symmetries and conservation law

Conservation laws

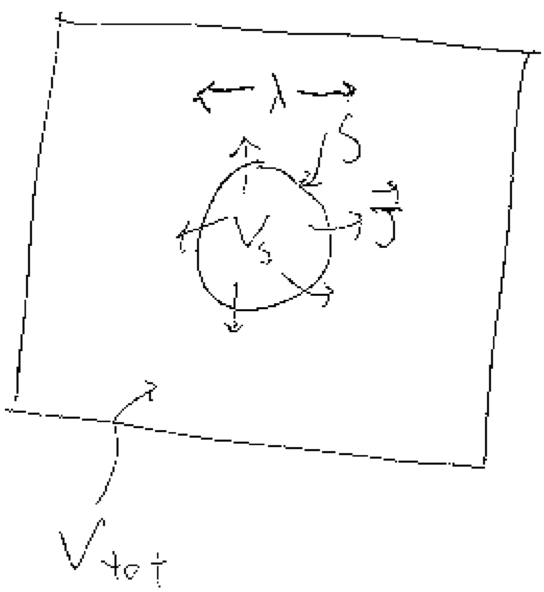
Example: Number density $\rho(\vec{r})$

(for <u>either</u> fluid or flock):

$$\int_{V_{tot}} \rho(\vec{r})\, d^d r = N = \text{constant}$$

$$\frac{d}{dt}\int_{V_S} \rho(r)\, d^d r = -\oint_S \vec{J}_n \cdot d\vec{S}$$

$\vec{J}_n$: number current



$\Rightarrow$ Divergence Thm

$$\Rightarrow \boxed{\partial_t \rho + \vec{\nabla} \cdot \vec{J} = 0}$$

$\Rightarrow$ If $\vec{J}(\vec{r},t)$ varies on length scale λ, $\vec{\nabla} \cdot \vec{J} \sim \frac{J}{\lambda}$

$$\partial_t \rho \sim \frac{J}{\lambda} \to 0 \text{ as } \lambda \to \infty$$

$\Rightarrow \rho$ hydrodynamic (slow)

Insert

Are there any ~~oth~~ other conserved quantities?

Insert II A

For fluid moving in vacuum, yes: momentum density $\vec{g} \equiv m\rho \vec{v}$ ← particle mass

For fluid (or flock) moving on substrate: No! Friction between substrate and fluid (or flock)

$\Rightarrow$ As discussed earlier, can eliminate $\vec{g}$.

In fact, we all know:

$$\vec{j} = \rho \vec{v} \equiv \vec{g}/m$$

number current

$$\Rightarrow \quad \boxed{\partial_t \rho + \vec{\nabla} \cdot \left(\frac{\vec{g}}{m} \right) = 0}$$

continuity equation

$$\Rightarrow \boxed{\partial_t \vec{g} = -\mu \vec{g} - \vec{\nabla} P(\rho)}$$

linear friction coefficient

$\Rightarrow$ for slow modes, $|\partial_t \vec{g}| \ll |\mu \vec{g}|$ (def'n of hydrodyn limit)

$$\Rightarrow \quad 0 \simeq -\mu \vec{g} - \vec{\nabla} P(\rho)$$

$$\Rightarrow \quad \vec{g} = -\frac{1}{\mu} \vec{\nabla} P(\rho)$$

Put in continuity equation

$$\Rightarrow \boxed{\partial_t \rho = \frac{1}{\mu m} \vec{\nabla}^2 P(\rho)} \quad (1)$$

Now, if we assume $P(\rho) = \rho \chi^{-1}$

inverse compressibility

$$\Rightarrow \boxed{\partial_t \rho = \left(\frac{1}{\mu m \chi}\right) \nabla^2 \rho \equiv D \nabla^2 \rho}$$

Diffusion equation.

Note, in any case, whatever $P(\rho)$ is, (1) involves only ρ. (I.e., only hydrodynamic variables appear in hydrodynamic eq'n)

I A.

(See, e.g., Ramaswamy + Mazenko, 198(?)) for discussion of beyond hydrodynamic description of this)

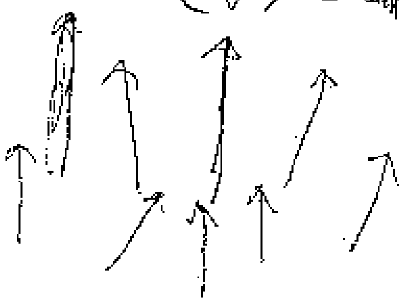
Now: 2nd class of hydrodynamic variables: Goldstone modes

Spontaneously broken continuous symmetry

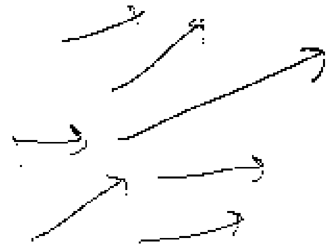
Example: Flock

a priori,

$\langle \vec{v} \rangle = v_0 \hat{n}_1$



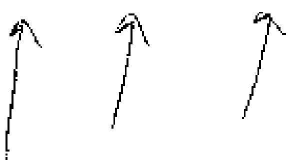
and



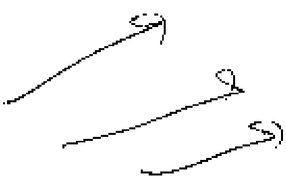
$\langle \vec{v} \rangle = v_0 \hat{n}_2$

are equally likely. (For rotationally invariant rules)

If a state with all $\vec{v}(\vec{r}) = v_0 \hat{n}_1$ is a steady state



So is



$v(\vec{r}) = v_0 \hat{n}_2$

VI A1

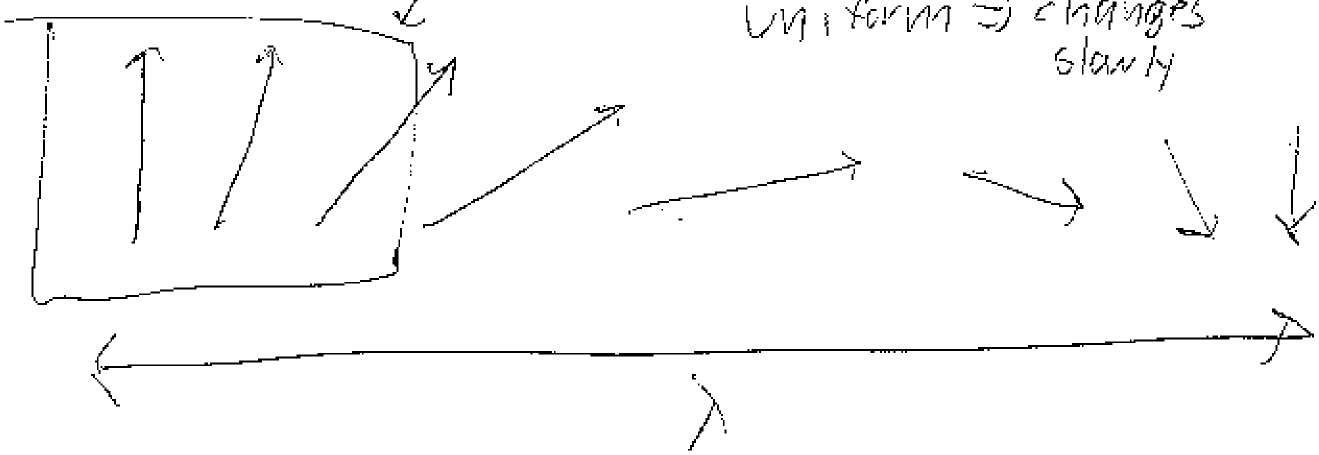
But: flock can only be in one of this continuum of possible states ("ferromagnetic")

$\Rightarrow$ By establishing flocking state, $(\langle \vec{v} \rangle \neq \vec{0})$, spontaneously breaks continuous symmetry (in this case, rotation invariance)

But $\Rightarrow$ by continuity, if $\partial_t \vec{v}|_{\text{uniform state}} = \vec{0}$

$$\partial_t \vec{v}|_{\text{slowly varying state}} \to 0 \quad \text{as } \lambda \to \infty$$

locally, looks very close to uniform $\Rightarrow$ changes slowly



$\Rightarrow$ ~~although~~

$\vec{v}$ is still a hydrodynamic variable (or, at least, some components are) for flock on substrate, in broken symmetry state (i.e., when flock actually moves). $\Rightarrow$ $\vec{v}$ is a Goldstone (Nambu) mode

Other examples from condensed matter:

~~magnetization~~

system	broken symmetry	Goldstone mode ~~variable~~
ferromagnet	rotation invariance	magnetization $\vec{M}(\vec{r},t)$
flock	"	velocity $\vec{v}(\vec{r})$
nematic (equilibrium or active)	"	$\underset{\sim}{Q}$ (nematic order parameter)
crystal (active or equilibrium)	translational invariance	$\vec{u}(\vec{r})$ displacement field

Start 3rd lecture

II A2 | II A

Formulating hydrodynamic equations.

~~We~~ ~~h~~

Gradient expansion

We know hydrodynamic variables $\rho(\vec{r},t)$, $\vec{v}(\vec{r},t)$

How do we write down EOM's?

$$\partial_t \rho = \text{stuff}(\rho, \vec{v})$$

$$\partial_t \vec{v} = \overrightarrow{\text{stuff}}(\rho, \vec{v})$$

Hard way: derive "stuff" from microscopic.

Easy way: Write down all terms in stuff allowed by symmetries (~~...~~ in flocks, rot. inv.) + conservation laws (in flocks, # conservation ...

(II A2.3

Still a lot of terms.

Further reduction: Gradient expansion

Since we're interested in wavelength $\lambda \to \infty$, and $\vec{\nabla} \sim \frac{1}{\lambda}$;

keep only terms with fewest gradients. (small

Eg: in $\vec{v}$ eqn., ~~terms~~

$\overrightarrow{stuff_1} \equiv \eta \nabla^2 \vec{v}$ and

$\overrightarrow{stuff_2} \equiv \eta_4 \nabla^4 \vec{v}$

are both allowed by ~~symmetr~~ rotation invariance

But $stuff_1 \sim \frac{V}{\lambda^2} \underset{\lambda \to \infty}{>>} stuff_2 \sim \frac{V}{\lambda^4}$

$\Rightarrow$ drop $\overrightarrow{stuff_2}$

Instantly eliminates ∞'ly many terms: $\eta_n \nabla^{2n} \vec{v} \equiv \overrightarrow{\text{stuff}_n}$ terms can all be dropped.

~~By~~ Reduces eqns to finite set of terms.

We "bury our ignorance" of microscopic details in phenomenolog. parameters like η

Different flocks (or fluids) will have different η, but <u>all</u> flocks (or fluids) will have $\eta \nabla^2 \vec{v}$ term in $\vec{v}$ EOM.

OK, so let's go:

$\partial_t \rho = ?$

Conservation of #

$\Rightarrow \partial_t \rho = -\vec{\nabla}\cdot\vec{J}$

We know that $\vec{J} = \frac{\vec{g}}{m} = \rho\vec{v}$

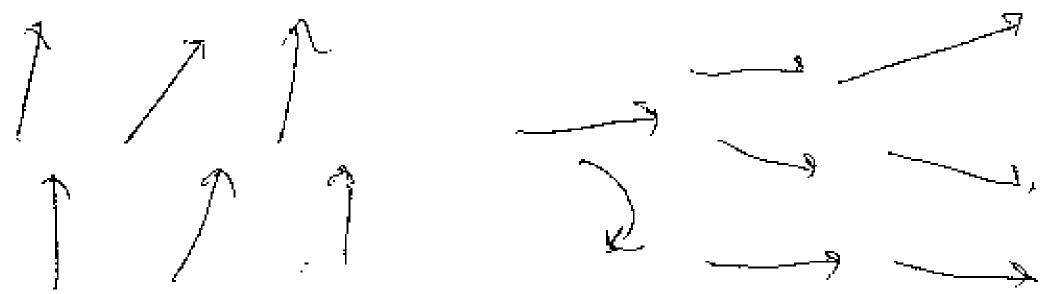
So $\boxed{\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$ 1 down, 1 to go.

What about $\vec{v}$ equation?

Constrained by rotation invariance

$\Rightarrow$ $\overrightarrow{\text{stuff}}$ can <u>not</u> contain any a priori fixed vectors $\vec{n}$ (these pick out special direction)

If we rotate whole picture:



EOM for $\vec{v}$ should not change

$\Rightarrow$ If $\vec{v} \to \underset{\sim}{R}\, \vec{v}$

$\vec{r} \to \underset{\sim}{R}\, \vec{r}$

$\vec{\nabla} \to \underset{\sim}{R}\, \vec{\nabla}$

EOM should not change.

So, what sort of terms does that allow for $\overrightarrow{\text{stuff}}$?

$$\partial_t \vec{v} \to \underset{\sim}{R}\, \partial_t \vec{v}$$

$\overrightarrow{\text{stuff}_1} = \alpha \vec{V}$ is allowed

also,

$\overrightarrow{\text{stuff}_2} = f(|\vec{V}|)\,\vec{V}$ " "

any f (will discuss f's that ~~allow~~ have ferromagnetic flock state later)

What else?

$\vec{\nabla} P(\rho) \to \underset{\sim}{\beta}\,\vec{\nabla} P(\rho)$ is allowed (any $P(\rho)$)

$P(\rho) \equiv$ "Pressure"

What else?

~~Any combination of ∂_i, V_j~~

$\partial_t V_i =$ Any comb of (∂_j, V_k, ρ) with one free index i

(IIA

Any vector $\vec{V}$ made out of $\vec{\partial}, \vec{v}, \rho$.

But: because gradient expansion, need only keep ~~a~~ terms with small # of ~~gra~~ $\vec{\nabla}$'s.

Possibilities with 1 gradient:

$$(\partial_t \vec{v}_0)_{\lambda_1} = \lambda_1 (\vec{v} \cdot \vec{\partial}) \vec{v}$$

(or: ~~in theu~~

$$(\partial_t v_i)_{\lambda_1} \equiv \lambda_1 v_j \partial_j v_i$$

(like "convective derivative" in fluid mechanics:

$$\partial_t \vec{v}_{fluid} + (\vec{v}_f \cdot \vec{\nabla}) \vec{v}_f = - - - - -$$

$$\Rightarrow \partial_t \vec{v}_f = -(\vec{v}_f \cdot \vec{\nabla}) \vec{v}_f + \cdots$$

$$\Rightarrow \boxed{\lambda_1^{fluid} = -1}$$

II A

~~Digression: Why does~~

Digression: Why does $\lambda_1 = 1$ in fluid?

No substrate $\Rightarrow$ Galilean invariance (extra symmetry)

$$\Rightarrow \quad \vec{v}(\vec{r}, t) \rightarrow \vec{v}(\vec{r} + \vec{v}_0 t, t) + \vec{v}_0$$

speed of moving frame

(Galilean transformation (go to moving frame))

$$\Rightarrow \quad \partial_t \vec{v} \rightarrow \partial_t \vec{v} + \vec{v}_0 \cdot \vec{\nabla} \vec{v}$$

$$(\vec{v} \cdot \vec{\nabla}) \vec{v} \rightarrow \vec{v} \cdot \vec{\nabla} \vec{v} + \vec{v}_0 \cdot \vec{\nabla} \vec{v}$$

$$\cancel{\Rightarrow \partial_t \vec{v} = \lambda_1 \vec{v}}$$

$$\partial_t \vec{v} = \lambda_1 \vec{v} \cdot \vec{\nabla} \vec{v} \rightarrow \partial_t \vec{v} + \vec{v}_0 \cdot \vec{\nabla} \vec{v} = \lambda_1 \vec{v} \cdot \vec{\nabla} \vec{v} + \lambda_1 \vec{v}_0 \cdot \vec{\nabla} \vec{v}$$

must balance $\Rightarrow \lambda_1 = 1$

But for flock on substrate,

No Galilean invariance

(substrate $\Rightarrow$ special frame)

$\Rightarrow$ ~~λ_1 need not~~ $\lambda_1 \neq 1$ in general

Note: $f(|\vec{v}|)\vec{v}$ term also violates Galilean invariance $\Rightarrow$ forbidden in fluid mechanics (no substrate) allowed in presence of substrate.

For fluid, no other ~~terms of~~ $v \nabla v$ terms allowed (Galilean invarian

For flock, 2 more:

$\lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v})$

$$\partial_t v_i = \lambda_2 v_{0i}(\partial_j v_j)$$

forbidden by Galilean invariance

$$\lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) \to \lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) + \lambda_2 \vec{v}_0(\vec{\nabla}\cdot\vec{v})$$

Nothing to cancel $\Rightarrow \lambda_2 = 0$ if Gal. inv. holds

Can also have

$$\lambda_3 \vec{\nabla} |\vec{v}|^2$$

$$\partial_t v_i = \lambda_3 \partial_i (v_j v_j)$$

Anything else with $\perp \vec{\nabla}$?

Yes (thanks to SR):

$$\lambda_4 \partial_j (v_i v_j P_2(\rho))$$

absorb in $\lambda_1(\rho), \lambda_2(\rho)$

$$= \lambda_4 P_2(\rho) [v_j(\partial_j v_i) + v_i(\partial_j v_j)]$$

$$+ \underbrace{\lambda_4 [v_i v_j \partial_j P_2(\rho)}_{\text{New piece}}$$

(potentially relevant;
doesn't change linear results;
might change non-linear results;
hard to treat;
will drop for now)

~~Neme~~

3 $\vec{\nabla}$ terms:

$D_4 \nabla^2 \vec{v}$, $D_3 \vec{\nabla}(\vec{\nabla}\cdot\vec{v})$

$D_2 (\vec{v}\cdot\vec{\nabla})^2 \vec{v}$

(more powers of $\vec{v}$, but $\langle \vec{v} \rangle \neq \vec{0}$

⇒ not necessarily smaller than $n_{1,2}$ terms ⇒ keep.

~~n_1, n_2 could potentially~~

$D_{T,B}(\rho, |\vec{v}|)$, $D_2(\rho, |\vec{v}|)$: shear, bulk, and funny viscosities

---

~~2 $\vec{\nabla}$'s: Ignore (gradient expansion)~~

$\vec{v}\, \nabla^2 \rho$? $\ll$ $\rho(\vec{v}\cdot\vec{\nabla})\vec{v}$ term ⇒ drop.

> 2 $\vec{\nabla}$'s: Drop ($\vec{\nabla}$ expansion)

Final term: Langevin noise $\vec{f}(\vec{r},t)$
(mistakes) Gaussian, white noise,
short spatio-temporal correlatio[n]

Comment: Crucial difference between flock with substrate
($\vec{v}$ slow because of symmetry, not momentum conservation)

and
fluid without substrate
($\vec{v}$ slow because $\vec{g}$ conserved)

In no-substrate case, $\vec{g}$ conserved
$\Rightarrow \vec{\nabla}\cdot\vec{f} = 0 \Rightarrow$ Effects of $\vec{f}$ small at
long wavelengths λ

Why? Spatial Fourier transform:

$$\vec{q}\cdot\vec{f}(\vec{q},t) = 0$$

$$\Rightarrow \boxed{\langle f_i(\vec{q},t) f_j(-\vec{q},t')\rangle = \Delta(\vec{q})\, P_{ij}(\vec{q})\, \delta(t-t')}$$

$$P_{ij}(\vec{q}) \equiv \delta_{ij} - \frac{q_i q_j}{q^2}$$: Transverse projection operator

But: $\langle ff\rangle$ must be analytic function of $\vec{q}$ (otherwize, long ranged correlations)

$$\Rightarrow \Delta(\vec{q}\to 0) \to C q^2 \to 0 \text{ as } |\vec{q}| \to 0$$

~~Elec~~ Flock with substrate: IIA

$\vec{v}$ slow because of rotation invariance

$\Rightarrow \vec{\nabla}\cdot\vec{f}$ ~~needed~~ $\neq 0$

$\Rightarrow$ $\langle ff \rangle$ does <u>not</u> vanish as $\vec{q} \to \vec{0}$.

$$\langle f_i(\vec{q},t)\, f_j(-\vec{q},t') \rangle = \underset{\substack{\uparrow \\ \text{constant} \\ \text{as } |\vec{q}| \to 0}}{\Delta}\, \delta(t-t')$$

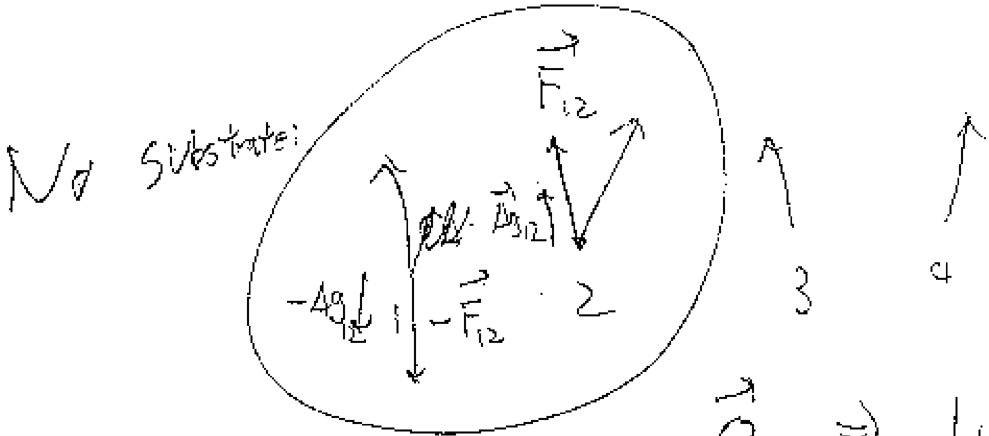
$\Downarrow$

$$\langle f_i(\vec{r},t)\, f_j(\vec{r}',t') \rangle = \Delta \delta_{ij} \delta^d(\vec{r}-\vec{r}')\,\delta(t-t')$$

No sub

II A2

Physics: If $\vec{g}$ conserved, random forces only <u>transfer</u> momentum from particle to particle:



$\Delta\vec{p}_{total} = \vec{0} \Rightarrow$ little long distance effect

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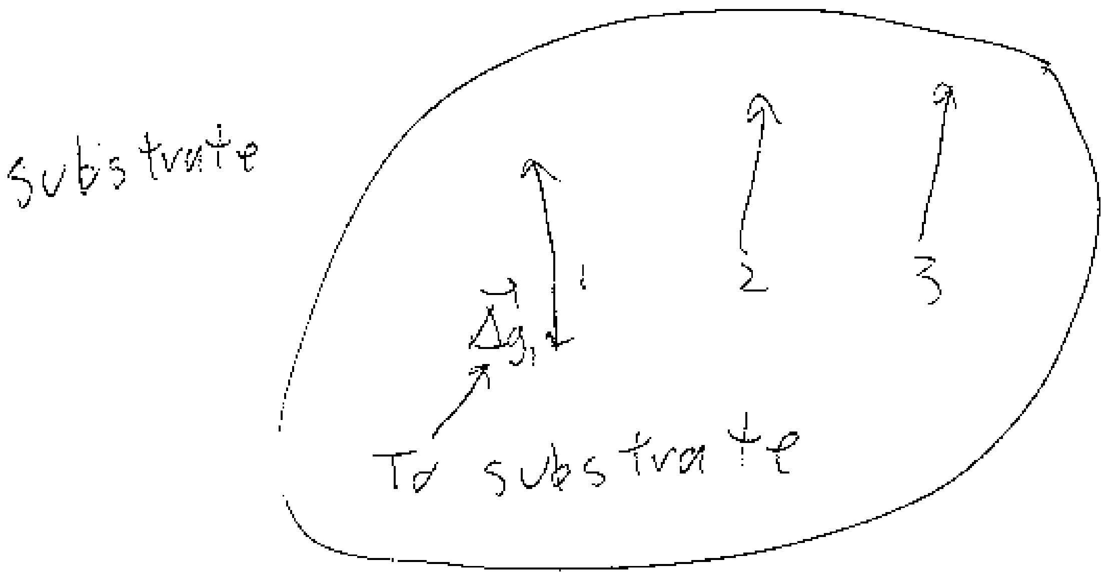


$\vec{\Delta g}_{net} = \vec{\Delta g}_1 \neq 0$

For flock, you get to have your kick and speed up, too

So, full EOM:

$$\boxed{\begin{aligned}\partial_t \vec{v} = &-\lambda_1 (\vec{v}\cdot\vec{\nabla})\vec{v} - \lambda_2 \vec{v}(\vec{\nabla}\cdot\vec{v}) - \lambda_3 \vec{\nabla}(|\vec{v}|^2) \\ &+ f(|\vec{v}|)\vec{v} - \vec{\nabla} P(\rho) + D_B \vec{\nabla}(\vec{\nabla}\cdot\vec{v}) \\ &+ D_T \nabla^2 \vec{v} + D_2 (\vec{v}\cdot\vec{\nabla})^2 \vec{v} + \vec{f}\end{aligned}}$$

$$\langle f_i(\vec{r},t)\, f_j(\vec{r}\,',t')\rangle = \Delta\, \delta_{ij}\, \delta^d(\vec{r}-\vec{r}\,')\,\delta(t-t')$$

+ Continuity eqn:

$$\boxed{\partial_t \rho + \vec{\nabla}\cdot(\rho\vec{v}) = 0}$$

We've "buried our ignorance" (D. Forster, "Hydrodynamic fluctuations, Broken Symmetries, and Correlation Functions") in small number of parameters Δ, $\lambda_{1,2,3}$, $f(|\vec{v}|,\rho)$, $P(\rho)$, $D_{T,B,2}(|\vec{v}|,\rho)$

We'll now reduce unknown functions to unknown #'s.

If asked:

Why not

$$\langle f_i f_j \rangle = v_i v_j \Delta \cdots$$

Ans: long. piece (along $\langle \vec{v} \rangle \neq 0$) irrelevant (that comp. of $\vec{v}$ <u>not</u> slow)

$\vec{v} = \langle \vec{v} \rangle + \delta\vec{v}$

$\langle \vec{v} \rangle$: mean; $\delta\vec{v}$: small fluctuations

transverse piece $|\delta\vec{v}| \ll 1$

$\Rightarrow$ these terms $\ll$ those already kept

II B.1

The broken symmetry state

Ignore noise: What are steady state solutions? Depend on form of $f(|\vec{v}|)$:

$\boxed{\rho = \rho_0}$, $\boxed{\vec{v}(\vec{r},t) = \vec{v}_0 = \text{consta}}$

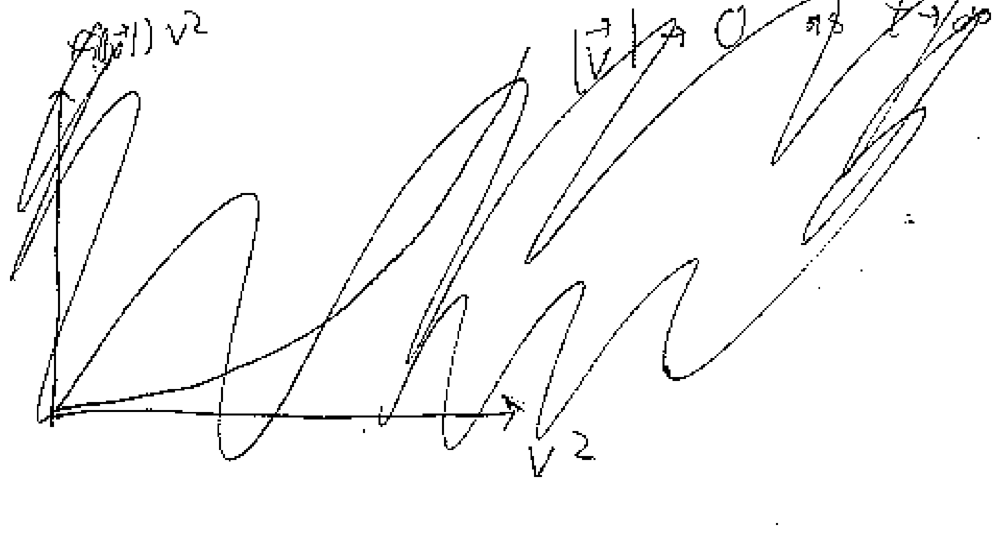
Spatially uniform state: $\vec{\nabla}\rho, \vec{\nabla}\vec{v} = 0$

$\Rightarrow \vec{v}\cdot[\partial_t \vec{v} = f(|\vec{v}|)\vec{v}]$

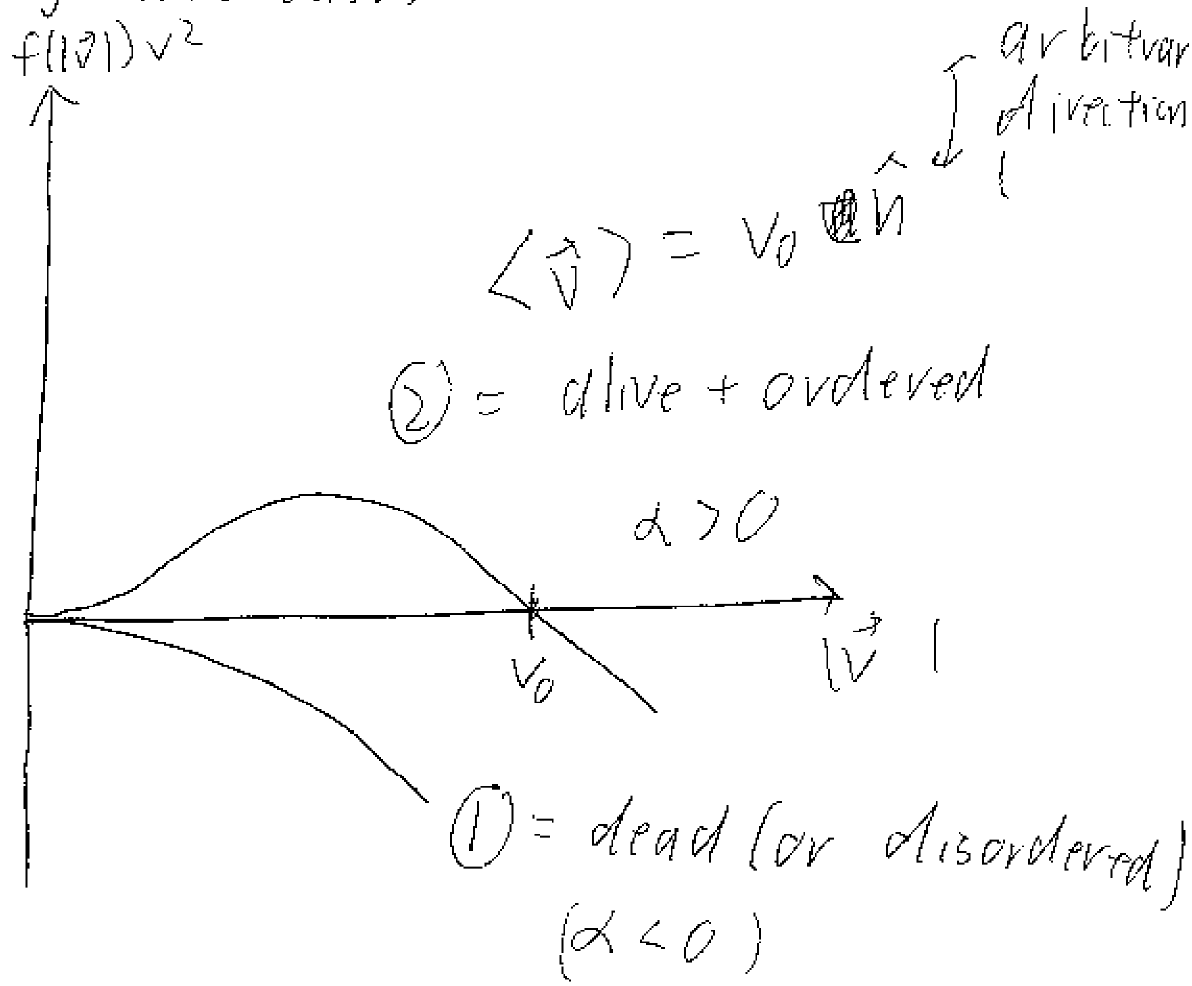
$\Rightarrow$ $\vec{v}$ always in same direction

Speed: $|\vec{v}|^2$:

$\frac{d}{dt}|\vec{v}|^2 = f(|\vec{v}|)|\vec{v}|^2$



Two generic cases:



~~Most of~~

Simplest example:

$$f(|\vec{v}|) = \alpha - B|\vec{v}|^2$$

$\alpha < 0 \Rightarrow$ ~~case (1)~~ dead or disordered

$\alpha = 0 \Rightarrow$ transition

$\alpha > 0$: ordered $\left(v_0 = \sqrt{\frac{\alpha}{B}}\right.$

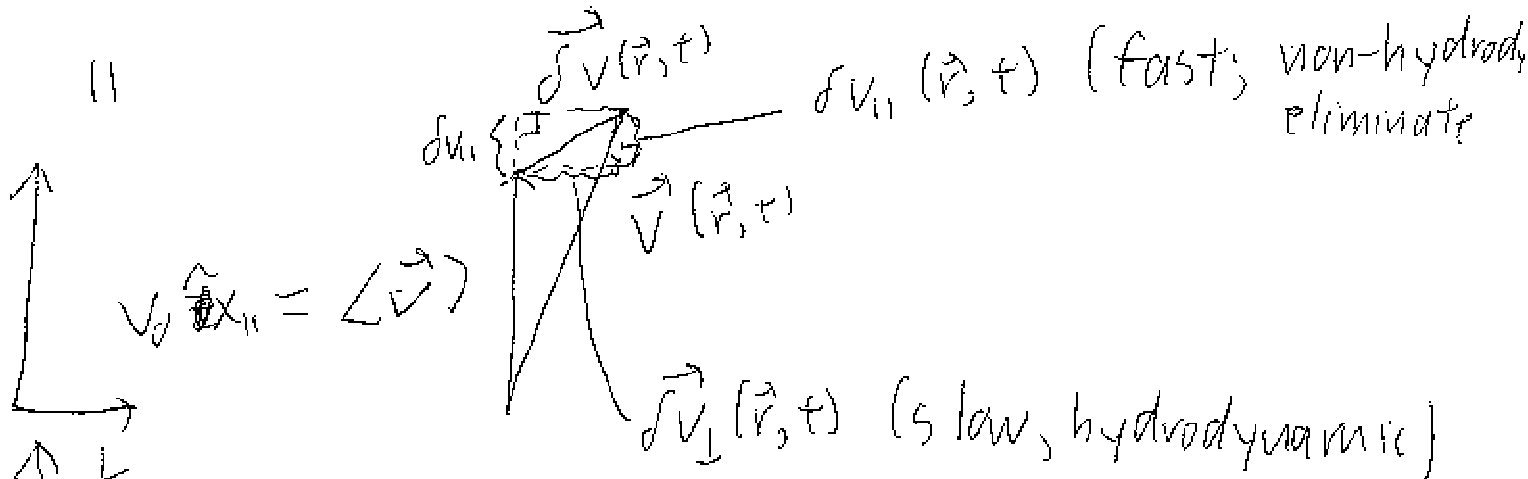
~~Lin~~

Fluctuations around ordered state:

~~Linear~~

~~M~~ Assume:

$$\vec{v}(\vec{r},t) = \underset{\text{mean}}{\langle \vec{v} \rangle} + \underset{\substack{\text{small}\\ \text{fluctuations}}}{\overrightarrow{\delta v}(\vec{r},t)} \equiv v_0 \hat{x}_{\parallel} + \overrightarrow{\delta v}(\vec{r},t)$$



$$|\vec{v}(\vec{r},t)| = v_0 + \delta v_{\parallel} + O(|\overrightarrow{\delta v}_{\perp}|^2)$$

$\Rightarrow \vec{v}_{\perp}$ does <u>not</u> change $f(|\vec{v}|) \Rightarrow f(|\vec{v}|)$ terms drop out of $\vec{v}_{\perp}$ EOM $\Rightarrow \vec{v}_{\perp}$ is slo

Note: Broken symmetry state is <u>anisotropic</u> (∞'ly so)

$$P \equiv \sigma_1 \, \delta\rho + P_0$$

II C-2

$$\cancel{\delta v_{\parallel}}$$

$$\partial_t \delta v_{\parallel} = -\sigma_1 \partial_{\parallel} \delta\rho + \underbrace{\delta v_{\parallel} \left(\frac{\partial f}{\partial |\vec{v}|}\right)_{\rho}}_{\equiv \Gamma_1} + \underbrace{\delta\rho \left(\frac{\partial f}{\partial \rho}\right)_{|\vec{v}|}}_{\equiv \Gamma_2} \Bigg|_{\substack{\rho=\rho_0 \\ |\vec{v}|=v_0}}$$

neglect $\partial_t \ll \Gamma_1$

$$\Rightarrow \boxed{\delta v_{\parallel} = -\frac{\sigma_1}{\Gamma_1}\partial_{\parallel}\delta\rho - \frac{\Gamma_2}{\Gamma_1}\delta\rho}$$

$$\cancel{\Rightarrow \partial_t \vec{v}_{\perp} + \gamma \partial_{\parallel} \vec{v}_{\perp}}$$

$$(\vec{v}_\phi \cdot \vec{\nabla})\vec{v}_{\perp} = v_{\parallel} \partial_{\parallel} \vec{v}_{\perp} + (\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp})\vec{v}_{\perp}$$

$$\boxed{\begin{aligned}\partial_t \vec{v}_{\perp} + \gamma \partial_{\parallel} \vec{v}_{\perp} + \lambda_1 (\vec{v}_{\perp} \cdot \vec{\nabla}_{\perp})\vec{v}_{\perp} + \lambda_2 (\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp})\vec{v}_{\perp} \\ = -\vec{\nabla}_{\perp} P + D_B \vec{\nabla}_{\perp}(\vec{\nabla}_{\perp} \cdot \vec{v}_{\perp}) + D_T \nabla_{\perp}^2 \vec{v}_{\perp} \\ + D_{\parallel} \partial_{\parallel}^2 \vec{v}_{\perp} + \vec{f}_{\perp}\end{aligned}}$$

$$\boxed{\partial_t \delta\rho + \rho_0 \vec{\nabla}_{\perp} \cdot \vec{v}_{\perp} + \vec{\nabla}_{\perp} \cdot (\vec{v}_{\perp} \delta\rho) + v_2 \partial_{\parallel} \delta\rho = D_\rho \partial_{\parallel}^2 \delta\rho}$$

$$v_2 \equiv v_0 - \frac{\rho_0 \Gamma_2}{\Gamma_1}$$

(III C.4

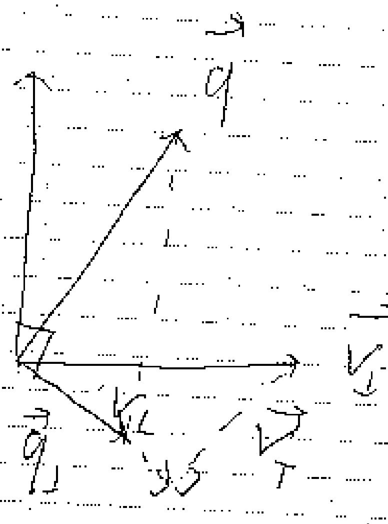
~~$i(\omega - \gamma q_{\parallel})$~~ +

$$\boxed{\begin{aligned} &-i(\omega - \gamma q_{\parallel})\vec{v}_{\perp} + D_B \vec{q}_{\perp}(\vec{q}_{\perp}\cdot\vec{v}_{\perp}) + D_T q_{\perp}^2 \vec{v}_{\perp} \\ &+ D_{\parallel} q_{\parallel}^2 \vec{v}_{\perp} + i\vec{q}_{\perp} \sigma_1 \delta\rho = \vec{f}_{\perp} \end{aligned}}$$

$$\boxed{\left[-i(\omega - v_2 q_{\parallel}) + D_\rho q_{\parallel}^2\right]\delta\rho + i\rho_0 \vec{q}_{\perp}\cdot\vec{v}_{\perp} = 0}$$

Simplify further: project along and $\perp$ to $\vec{q}_{\perp}$:

$\langle\vec{v}\rangle = v_0 \hat{x}_{\parallel}$



Note: No transverse component $\vec{v}_T$ in $d = 2$!

$$\Rightarrow \Big[-i(\omega - \gamma q_\parallel) + \underbrace{\big[(\overbrace{D_B + D_T}^{D_L}) q_\perp^2 + D_{\parallel B}\, q_\parallel^2\big]}_{\Gamma_L(\vec{q})} \Big] v_L + i\sigma_1 q_\perp \delta\rho = f_L$$

$$\Big[-i(\omega - \gamma q_\parallel) + \underbrace{\big(D_T q_\perp^2 + D_{\parallel} q_\parallel^2\big)}_{\equiv \Gamma_T(\vec{q})} \Big] \vec{v}_T = \vec{f}_T$$

$$\Big[-i(\omega - v_2 q_\parallel) + \underbrace{D_\rho q_\parallel^2}_{\Gamma_\rho(\vec{q})} \Big] \delta\rho + i\rho_0 q_\perp v_L = 0$$

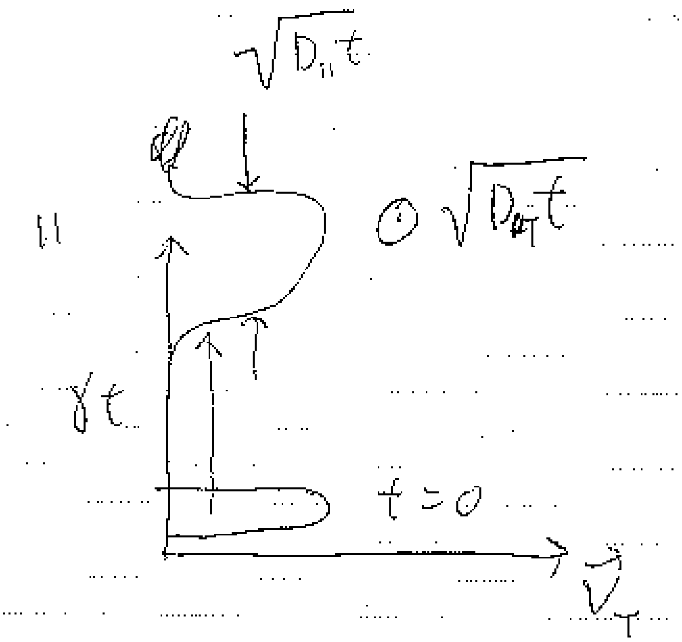
Note: $\vec{V}_T$ decouples.

~~But~~ Finding eigenfrequencies and eigenmodes: $\vec{f} = \vec{0}$

$\vec{V}_T$ obvious eigenmode:

$$\omega = \gamma q_\parallel - i\overbrace{(D_T q_\perp^2 + D_\parallel q_\parallel^2)}^{\text{anisotropic diffusion}}$$

γ: "convecting speed"



Coupled v_L, $\delta\rho$ equations:

Guess $\omega = c(\hat{q})\, q$ ← direction of q

Work to leading order in q as $q \to 0$

$$-i(\omega - \gamma q_{\parallel}) v_L + i\sigma_1 q_\perp \delta\rho = 0$$

$$-i(\omega - v_2 q_{\parallel}) \delta\rho + i\rho_0 q_\perp v_L = 0$$

$\Rightarrow$

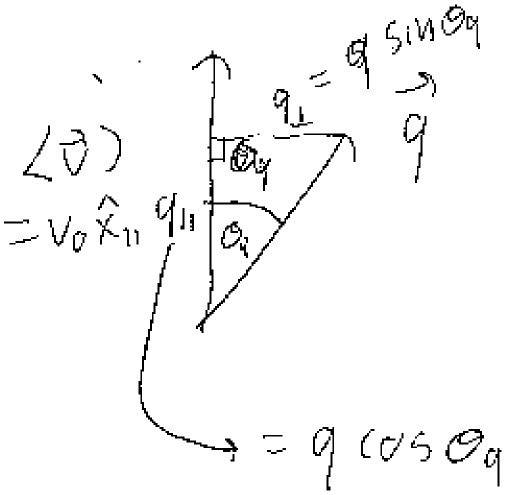
II C

$$(\omega-\gamma q_\parallel)(\omega - v_2 q_\parallel) + \sigma_1 \rho_0 q_\perp^2 = 0$$

$$\Rightarrow -\omega^2 + (\gamma + v_2) q_\parallel \omega - (\gamma v_2 q_\parallel^2 + \sigma_1 \rho_0 q_\perp^2) = 0$$

$$\cancel{\Rightarrow \omega = - \gamma + v}$$

$$\omega(\vec{q}) \equiv c(\theta_{\vec{q}})\, q$$



$$\Rightarrow c^2 - (\gamma + v_2)\cos\theta_q\, c + \gamma v_2 \cos^2\theta_q - \sigma_1 \rho_0 \sin^2\theta_q = 0$$

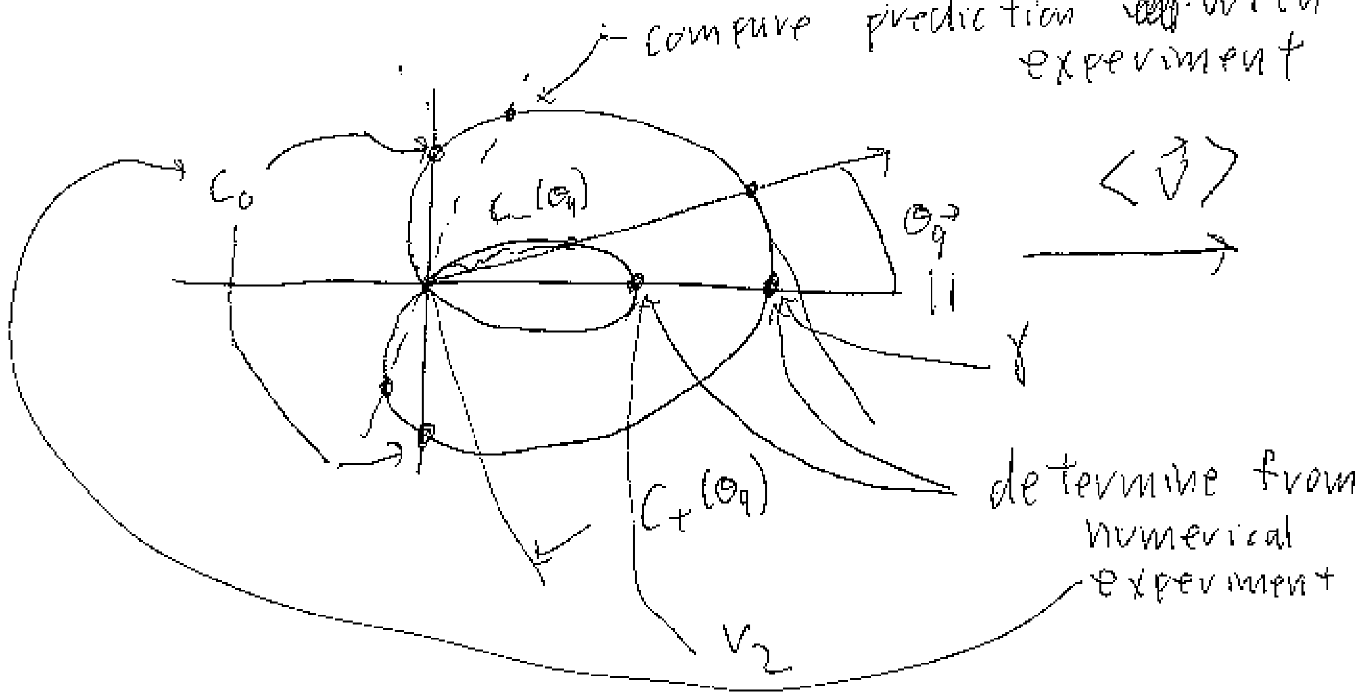
$$\Rightarrow c_\pm(\theta_q) = \left(\frac{\gamma + v_2}{2}\right)\cos\theta_q \pm \frac{1}{2}\sqrt{(\gamma+v_2)^2\cos^2\theta_q + 4\sigma_1\rho_0 \sin^2\theta_q - 4\gamma v_2 \cos^2\theta_q}$$

$$= \left(\frac{\gamma + v_2}{2}\right)\cos\theta_q \pm \sqrt{\frac{(\gamma - v_2)^2}{4}\cos^2\theta_q + \sigma \rho_0 \sin^2\theta_q}$$

3 parameters: $\gamma,\ \nu_2,\ \underbrace{c_0 \equiv \sqrt{\sigma_1 \rho_0}}$

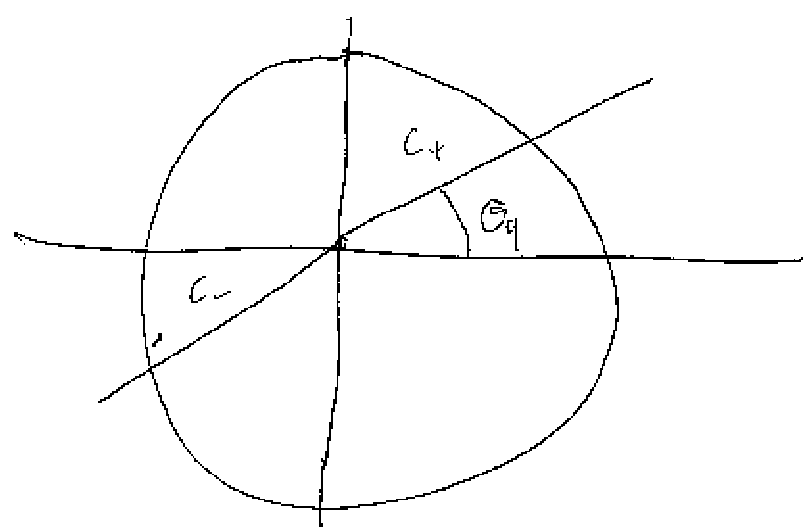
like ordinary, compression wave in simple fluid

Note: Rotated 90^0



Cf: same plot for simple fluid:

$\omega = \pm c_0 q$



(II C

Simple directions:

$\theta_q = 0$ (along flock motion)

$$c_\pm = \frac{\gamma + v_2}{2} \pm \frac{\gamma - v_2}{2} = \begin{cases} \gamma & + \\ v_2 & - \end{cases}$$

$\theta_q = \pm\frac{\pi}{2}$ ($\perp$ to flock motion)

$$c_\pm = \pm\sqrt{\sigma_1 \rho_0} \equiv \pm c_0$$

Damping (important for fluctuations, (finite height of resonance)

assume $\omega = \pm c_\pm(\theta_{\vec{q}})\, q + \delta_\pm(\vec{q})$

$$\delta(\vec{q}) \ll cq$$

Solve to <u>next</u> order in q

(DC

find

$$d_\pm(\vec{q}) = -i\Big[\big(\underbrace{D_{\theta_{11}}}_{\text{Dissipative}} q_{\parallel}^2 + D_\perp q_\perp^2\big) f_1(\theta_q) + D_s q_{\parallel}^2 f_2(\theta_{\vec{q}})\Big]$$

$$f_1(\theta_q) \equiv \frac{v_\pm(\theta_q)}{2\, c_2(\theta_q)}$$

$$v_\pm(\theta_q) \equiv \pm\left(\frac{\gamma - v_2}{2}\right)\cos\theta_q + c_2(\theta_q)$$

$$c_2(\theta_q) \equiv \sqrt{}$$

Summary of longitudinal modes

$$\omega(\vec{q}) = \underbrace{c_\pm(\theta_{\vec{q}})\, q}_{\text{propagating}} \underbrace{- i q^2 F(\theta_{\vec{q}})}_{\text{damping}}$$

Note: Damping $F q^2 \ll$ real part cq as $q \to 0$. $\Rightarrow$ <u>very</u> sharp resonances <u>very</u> high q

Aside: $z = 1$
or
$z = 2$?

I'll take $z=2$, because <u>damping</u> controls fluctuations

(II C

Look ahead: Non-linearities do <u>not</u> change $c_I(\theta_q)\, q$, but <u>do</u> change damping

$$q^2 \to \begin{cases} q_\perp^{z}, \quad q_\perp \sim q_\parallel^{z} \quad z < 2 \qquad (q_\perp^{z} \gg q^2) \\ q_\parallel^2, \quad q_\perp \ll q_\parallel \end{cases} \qquad d=2: \quad z = \frac{6}{5}$$

Calculating fluctuations:

Put forces back in, solve linear eqns for $\vec{v}_\perp(\vec{q},\omega)$, $\delta\rho(\vec{q}_\perp,\omega)$ i.t.o. $\vec{f}_\perp(\vec{q},\omega)$

~~Get:~~ $\cancel{G_{\rho\rho} = \Delta}$

Get:

$$\boxed{\delta\rho(\vec{q},\omega) = G^{(\vec{q},\omega)}_{\rho i_L} f_L(\vec{q},\omega)} + G_{\rho\rho} f_\rho($$

$$\boxed{v_{i_L}(\vec{q},\omega) = G_{i_L i_L}(\vec{q},\omega)\, f_{i_L}(\vec{q},\omega) +} \; G_{L\rho} f_\rho$$

(II C.

~~$G_{\vec q}$'s are complicated,~~

$$\vec{v}_T(\vec q, \omega) = G_{TT}(\vec q, \omega)\, \vec{f}_T(\vec q, \omega)$$

Response functions

$$G_{TT}^{(\vec q, \omega)} = \frac{1}{-i(\omega - \gamma q_{\parallel}) + \Gamma_T(\vec q)}$$

$$G_{LL}^{(\vec q, \omega)} = \frac{i(\omega - v_{02} q) - \Gamma_\rho(\vec q)}{\underbrace{(\omega - c_+(\theta_q) q)(\omega - c_-(\theta_q) q) + i\omega[\Gamma_L(\vec q) + \Gamma_\rho(\vec q)] - q_{\parallel}[v_0 \Gamma_L(\vec q) + \gamma \Gamma_\rho(\vec q)]}_{\mathrm{Den}(\vec q, \omega)}}$$

$$G_{L\rho}^{(\vec q, \omega)} = \frac{i \sigma_1 q_\perp}{(\omega - c_+ q)(\omega - c_- q) + i\omega[\Gamma + \cdots]}$$

$$G_{\rho L}(\vec q, \omega) = \frac{i \rho_0 q_\perp}{\mathrm{Den}(\vec q, \omega)}$$

$$G_{\rho\rho}(\vec q, \omega) = \frac{i(\omega - \gamma q_{\parallel}) - \Gamma_L(\vec q)}{\mathrm{Den}(\vec q, \omega)}$$

Note resonance structure:

at $\omega = c_{\pm}(\theta_q)\, q$, $\gamma q_{\parallel}$

For Long. modes — for transverse modes,

all $G_{\alpha\beta}$'s peak

$\omega = c_{\pm} q$, Den $\sim \omega\, \Gamma_L \sim (cq)(Dq^2)$

$$G_{LL} \sim \frac{(c-v)q}{(cq)(Dq^2)} \sim \frac{1}{q^2} \sim G_{L\rho}, G_{\rho L}$$

peak height $\to \infty$ as $q \to 0$

$\Downarrow$

Long length scales $(q \to 0)$ have largest fluctuations ("stirred by noise")

Calculate correlations:

$$\langle v_L^{\alpha}(\vec{q},\omega)\, v_L^{\alpha}(-\vec{q},\nu-\omega)\rangle$$

$$= |G_{LL}(\vec{q},\omega)|^2 \langle f_L^{\alpha}(\vec{q},\omega)\, f_L^{\beta}(-\vec{q},-\omega)\rangle$$

What are $\vec{f}(\vec{q},\omega)$ correlations?

$$\langle f_i(\vec{q},\omega)\, f_j(-\vec{q},-\omega)\rangle = \int e^{i[\vec{q}\cdot(\vec{r}-\vec{r}\,') - \omega(t-t')]}$$

$$\times \frac{\langle f_i(\vec{r},t)\, f_j(\vec{r}\,',t')\rangle\, d^dr\, d^dr'\, dt\, dt'}{\Omega\, T}$$

Ω ← volume

We took Gaussian, white noise: $\langle f_i f_j\rangle = \Delta\, \delta^d(\vec{r}-\vec{r}\,')\,\delta(t-t')\,\delta_{ij}$

$$\Rightarrow \langle f_i(\vec{q},\omega)\, f_j(-\vec{q},-\omega)\rangle = \Delta$$

(constant, indep. of $\vec{q},\omega$ "white")

Aside: What would have happened if we assumed finite range spatio-temporal correlations of $\vec{f}$?

$$\langle f_i(\vec{r},t)\, f_j(\vec{r}\,',t')\rangle = \delta_{ij}\, g(\vec{r}-\vec{r}\,', t-t')$$

$$\Rightarrow \langle f_i(\vec{q},w)\, f_j(-\vec{q},-w)\rangle = \delta_{ij}\, \tilde{g}(\vec{q},w)$$

$$g(\vec{q},w) \equiv \int d^d\Delta r\, dt\;\; g(\vec{\Delta r},\Delta t)\, e^{i(\vec{q}\cdot\vec{r}-wt)}$$

If $g(\vec{\Delta r},\Delta t)$ decays sufficiently rapidly as $\vec{\Delta r}, \Delta t \to \infty$,

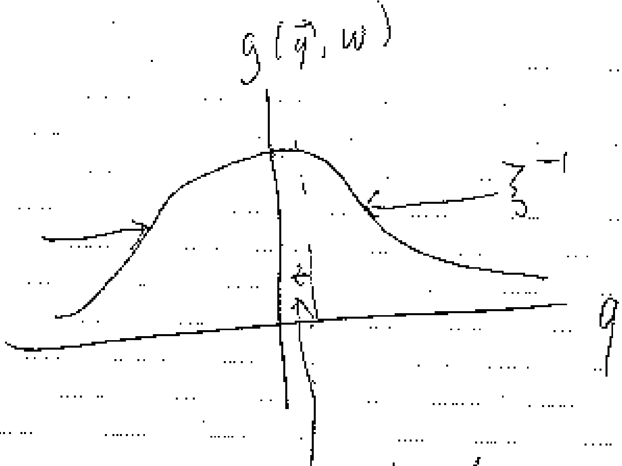
$$g(\vec{q}\to\vec{0}, w\to 0) \to \int d^d\Delta r\, dt\; g(\vec{\Delta r},\Delta t)$$

finite $\equiv \Delta$

$$\Rightarrow \langle f_i(\vec{q},w)\, f_j(-\vec{q},-w)\rangle \underset{q,w\to 0}{\to} \Delta$$

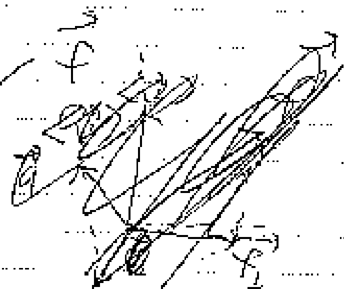
$\Rightarrow$ small $\vec{q}, w$ behavior exactly the same

E.G: $g \propto e^{-\left(\frac{\Delta r}{\xi}\right)} \Rightarrow g(\vec{q},\omega) = \frac{C}{q^2+\xi^{-2}} \rightarrow C\xi^2 \equiv \Delta$



hydrodynamic limit
(Guaranteed for short ranged correlations)

Getting $\vec{f}^{\#}_{\perp}$ from $\vec{f}$



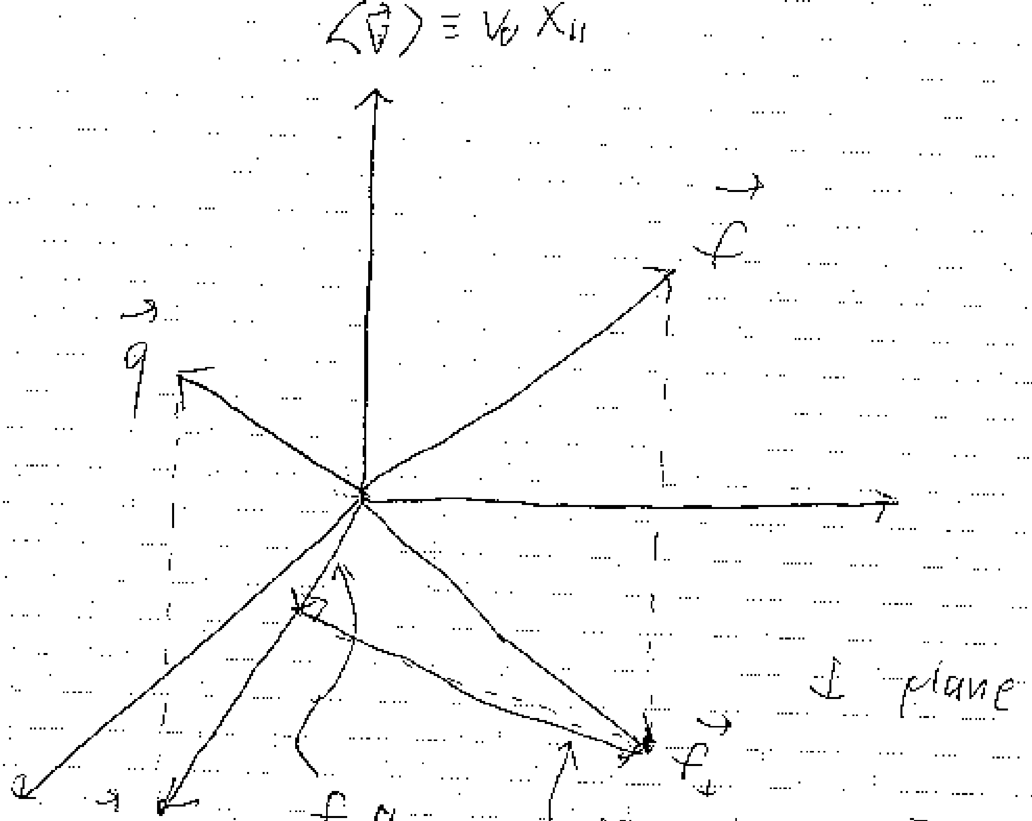
2 projections:

1) Project $\perp$ to $\langle \vec{v} \rangle$

2) " $\perp$ to $\vec{q}_{\perp}$ in $\perp$ plane.

$f_{L,T}$ correlations:

f_L = comp. of $\vec{f}$ along $\vec{q}_\perp$



$\vec{f}_T = \underset{\sim}{P}^\perp \vec{f}$ $\left(f^T_i = P^\perp_{ij} f_j \right)$

$$f_L = \hat{q}_\perp \cdot \vec{f} = \frac{q^\perp_i f_i}{q_\perp}$$

$$\vec{f}_T = \vec{f}_\perp - \hat{q}_\perp f_L = \vec{f}_\perp - \hat{q}_\perp (\hat{q}_\perp \cdot \vec{f})$$

$$\Rightarrow f^T_i = f^\perp_i - \frac{q^\perp_i q^\perp_k f_k}{q_\perp^2}$$

$$\Rightarrow \quad \langle f_L(\vec{q},\omega)\, f_L(-\vec{q},-\omega)\rangle = \frac{q_i^\perp q_j^\perp}{q_\perp^2} \langle f_i(\vec{q},\omega)\, f_j(-\vec{q},-\omega)\rangle$$

$$= \Delta \frac{q_i^\perp q_j^\perp}{q_\perp^2}\delta_{ij} = \Delta \frac{q_\perp^2}{q_\perp^2} = \Delta$$

$$\langle f_i^T f_j^T\rangle = \langle f_i^\perp f_j^\perp\rangle - 2\frac{q_i^\perp q_k^\perp}{q_\perp^2}\langle f_i^\perp f_k^\perp\rangle$$

$$+ \frac{q_i^\perp q_j^\perp q_k^\perp q_l^\perp}{q_\perp^4}\langle f_k^\perp f_l^\perp\rangle$$

$$\delta_{ij}^\perp = \begin{array}{c|cc} & \perp & \parallel \\ \hline \perp & \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ \parallel & & \end{array}$$

$$= \Delta\left[\delta_{ij}^\perp - 2\frac{q_i^\perp q_j^\perp}{q_\perp^2} + \frac{q_i^\perp q_j^\perp q_k^\perp q_l^\perp}{q_\perp^4}\delta_{kl}^\perp\right]$$

$$= \Delta\left[\delta_{ij}^\perp - \frac{q_i^\perp q_j^\perp}{q_\perp^2}\right] \equiv \Delta P_{ij}^\perp(\hat{q}_\perp)$$

$P_{ij}^\perp$: "transverse projection operator in $\perp$ plane"

$$P_{ij}^\perp A_j = A_i^T$$

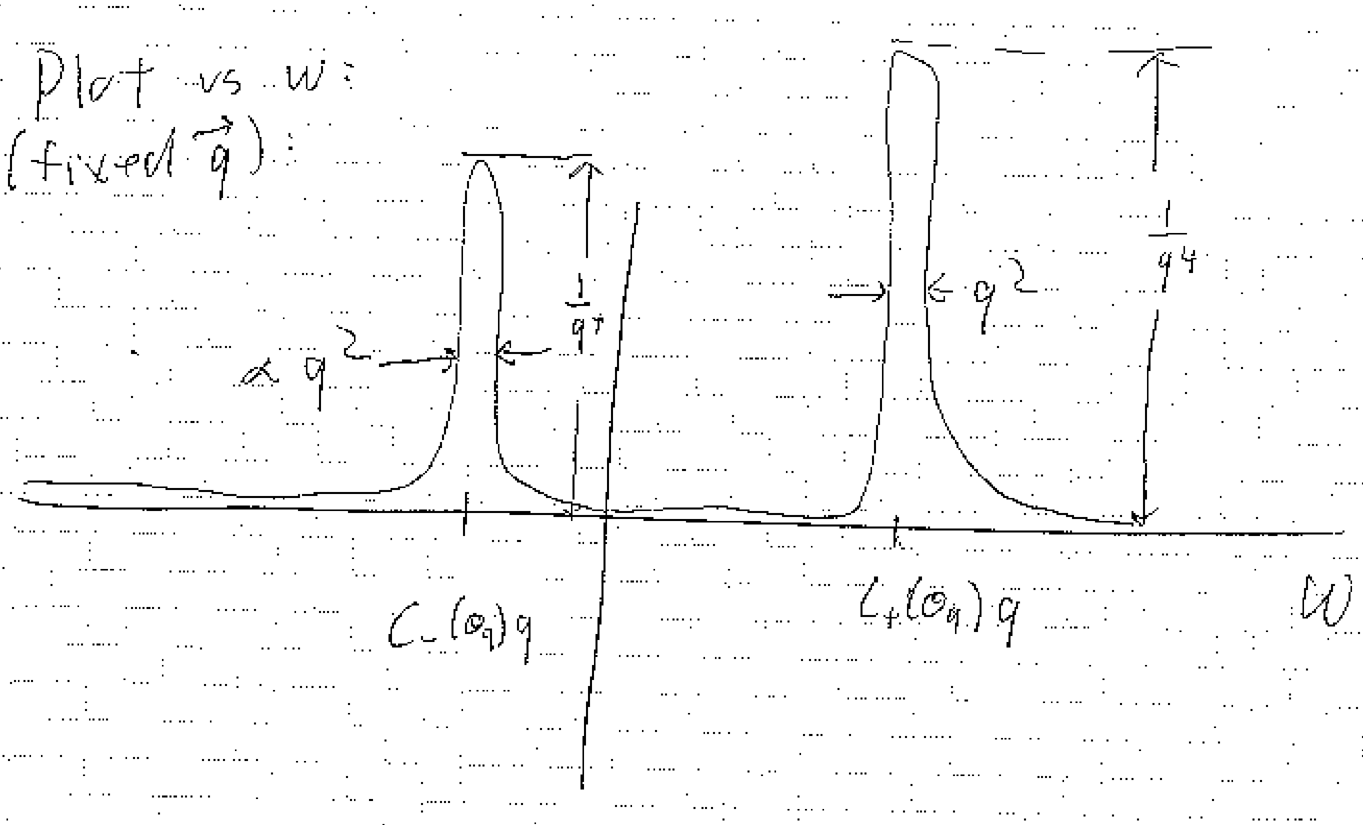
any vector

So,

$$\langle |v_L(\vec{q},\omega)|^2 \rangle \equiv C_{LL}(\vec{q},\omega) = |G_{LL}(\vec{q},\omega)|^2 \Delta$$

$$= \frac{\Delta\left((\omega - v_2 q_{\parallel})^2 + \Gamma_\rho^2\right)}{(\omega - c_+(\theta_q)q)^2(\omega - c_-(\theta_q)q)^2 + \left(\omega(\Gamma_L + \Gamma_\rho) - q_{\parallel}(v_2\Gamma_L + \gamma\Gamma_\rho)\right)^2}$$

Plot vs ω:
(fixed $\vec{q}$):



as $q \to 0$, width << distance between peaks

$\Rightarrow$ each peak nearly Lorentzian

What is $\int_{-\infty}^{\infty} C_{LL}(\vec{q},\omega)\, d\omega$?

$$= \overset{\text{equal time}}{C_{LL}(\vec{q})} = \langle |V_L(\vec{q},t)|^2 \rangle \ ?$$

~~real~~ roughly (peak height $\sim \frac{1}{q^4}$

$\times$ peak width $\sim q^2$) $\sim \frac{1}{q^2} \xrightarrow[q\to 0]{} \infty$

$\Rightarrow$ Huge ~~small q~~ Long wavelength fluctuations!

In fact, if we go to real space:

$$\langle |V_L(\vec{r},t)|^2 \rangle = \int d^d q \, \langle |V_L(\vec{q},t)|^2 \rangle$$

$$\sim \int \frac{d^d q}{q^2} \underset{d=2}{\overset{\uparrow}{\propto}} \ln(\Lambda L) \xrightarrow[L\to\infty]{} \infty$$

~~q~~ $q \gtrsim \frac{1}{L}$

$\Lambda > q > \frac{1}{L}$

ultra-violet cutoff; $\sim \frac{1}{a}$

$\Downarrow$

No LRO!

(Mermin-Wagner Theorem)

More precisely,

$$C_{LL}^{\text{equal time}}(\vec{q}) = \frac{\Delta \phi(\theta_{\vec{q}})}{\Gamma_L(\vec{q})}$$

$$\phi(\theta_{\vec{q}}) = \frac{1}{(c_+ - c_-)q} \left[\frac{(c_+ q - v_2 q_{\parallel})^2}{c_+ q - v_2 q_{\parallel} + (c_+ q - \gamma q_{\parallel}) \frac{\Gamma_\rho}{\Gamma_L}} + \frac{(c_- q - v_2 q_{\parallel})^2}{c_- q - v_2 q_{\parallel} + (c_- q - \gamma q_{\parallel}) \frac{\Gamma_\rho}{\Gamma_L}} \right]$$

$$\propto \frac{q^2}{q\, q} \propto q^0 \Rightarrow \text{only fn of } \theta_q.$$

$$\boxed{\text{Boring, finite, } \cancel{\text{non-zero}} > 0 \text{ in all directions}}$$

$$\int C_{LL}(\vec{q})\, d^d q = \underbrace{\left[\int d(\angle) \left(\frac{\phi(\theta_q)}{D_\perp \sin^2\theta_q + D_\parallel \cos^2\theta_q} \right) \right]}_{\text{finite, } O(1)} \times \underbrace{\left(\int \frac{q^{d-1}\, dq}{q^2} \right)}_{\propto \ln L \text{ in } d=2}$$

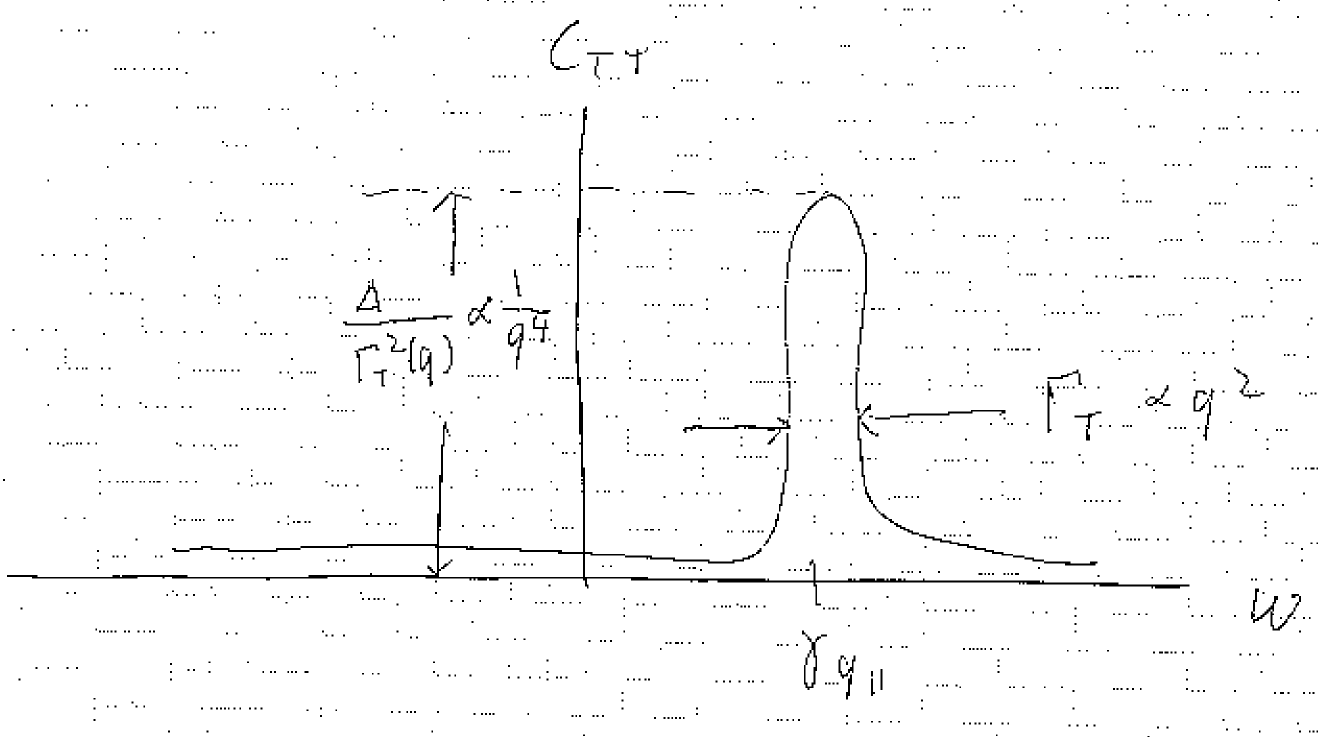
(IIC.2)

Same for C_{TT}:

$$\langle v_i^T(\vec{q},\omega)\, v_j^T(-\vec{q},-\omega)\rangle = |G_{TT}(\vec{q},\omega)|^2 \langle f_i^T(\vec{q},\omega) f_j^T(-\vec{q},-\omega)\rangle$$

$$= C_{TT}(\vec{q},\omega)\; P_{ij}^{\perp}(\hat{q}_\perp)$$

$$C_{TT}(\vec{q},\omega) = \frac{\Delta}{(\omega-\gamma q_\parallel)^2+\Gamma_T^2(\vec{q})}$$



Again,

$$\langle |\vec{v}_T(\vec{q},w)|^2 \rangle = \langle v_i^T v_i^T \rangle$$

$$= C_{TT}(\vec{q},w)\, \underbrace{P_{ii}^{\perp}}_{\underbrace{\delta_{ii}^{+}}_{=d-1} - \underbrace{\frac{q_i^{\perp} q_i^{\perp}}{q_{\perp}^2}}_{=1} = d-2}$$

(No transverse components in $d=2$

$$\Rightarrow \langle |\vec{v}_T(\vec{q},w|^2 \rangle = C_{TT}(\vec{q},w)(d-2)$$

$$\Rightarrow C_{TT}^{\text{eq. time}}(\vec{q}) \equiv \langle |\vec{v}_T(\vec{q},t)|^2 \rangle$$

$$= \int_{-\infty}^{\infty} dw\, C_{TT}(\vec{q},w)\,(d-2)$$

$$= \frac{(d-2)\Delta}{2\Gamma_T(\vec{q})} = \frac{(d-2)\Delta}{2(D_T q_\perp^2 + D_{\parallel} q_{\parallel}^2)}$$

$$\propto \frac{1}{q^2} \xrightarrow[(\vec{q}\to 0)]{} \infty$$

$\Rightarrow$ real space fluctuations:

~~Hhd~~
Why is this different from simple fluid with no substrate?

Undramatic: anisotropy
More important: Noise correlations

~~$\langle f f \rangle \propto FT(g$~~

$$\langle f_i(\vec{q},\omega) f_j(-\vec{q},-\omega) \rangle = \int d^d r\, dt \int d^d r'\, dt' \, e^{i[\vec{q}\cdot(\vec{r}-\vec{r}') + \omega(t-t')]} \langle f_i(\vec{r},t) f_j(\vec{r}',t') \rangle$$

(no substrate)

But for simple fluid, momentum is conserved $\Rightarrow \int d^d r \; f(\vec{r},t) = 0$

$\Rightarrow \lim_{q \to 0} \langle f f \rangle \to 0$

in fact, $\langle f f \rangle \propto q^2$

$\Rightarrow C_v^{\text{equal time}}(\vec{q}) \propto q^2 (\text{flock answer}) \propto q^2 \left(\frac{1}{q^2}\right) \xrightarrow[q \to 0]{} \text{constant}$

II 6.25

What about dead fluid on substrate?

Now, $\vec{f}$ need not conserve; but

$\vec{v}$ is not a slow variable

$$\partial_t \vec{v} = -\alpha \vec{v} + \vec{f}$$

$$\Rightarrow \langle |v(q,\omega)|^2 \rangle = \frac{1}{\alpha^2} \langle |f|^2 \rangle = \frac{D}{\alpha^2} \xrightarrow[q \to 0]{} \text{const}$$

Only for broken symmetry state (flock) can you have your kick and ~~eat it~~ speeded up too)

That is: Only for broken symmetry state (moving flock) is:

1) $\vec{v}$ slow (Goldstone mode)

2) $\vec{v}$ not conserved ($\Rightarrow$ big noise)

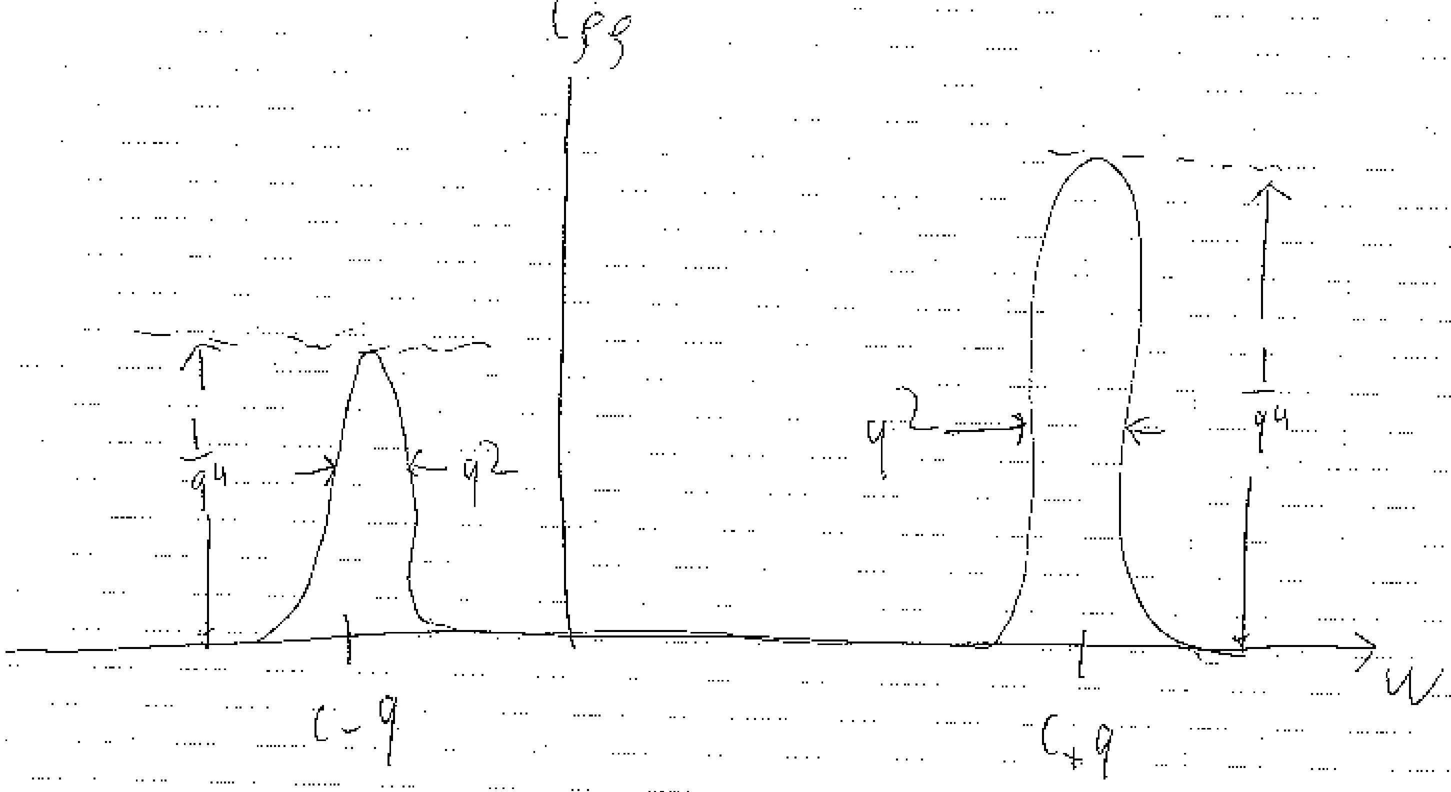
~~Final comment~~

~~Dense~~

Linearized theory also predicts:

Giant number fluctuations:

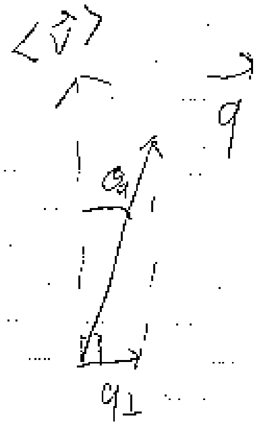
$$C_{\rho\rho}(\vec{q},\omega) = \frac{\Delta \rho_0^2 q_\perp^2}{\text{Den}}$$



again, $\int d\omega\, C_{\rho\rho}(q,\omega) = C^{\text{equal time}}_{\rho\rho}(\vec{q}) = \langle |\rho(\vec{q},t)|^2 \rangle$

$$= \frac{W(\theta_{\vec{q}})}{q^2}$$

Since num $\propto q_\perp^2$, $W(\theta_q \to 0) \to 0$



Can rewrite:

$$C_{\xi\xi}^{E.T.}(\vec{q}) = \frac{q_\perp^2}{q^4} \overbrace{W_2(\theta_q)}^{\text{finite, non-zero all } \theta_q}$$

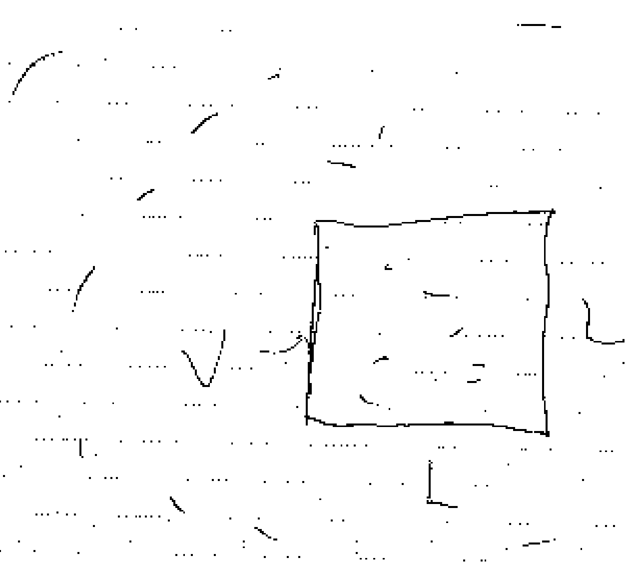
$\Rightarrow$ as $\theta_q \to 0$, $q \to q_{\parallel}$

$$C_{\xi\xi}^{ET} \propto \frac{q_\perp^2}{q_\parallel^4}$$

Back to real space

II (.25

Giant # flucs:



$$\delta N = \int_{\vec r \in V} \delta\rho(\vec r, t) \qquad \left(\int_{\vec r} \equiv \int d^d r \right)$$

$$\langle \delta N^2 \rangle = \int_{\substack{\vec r, \vec r' \\ \in \square_L}} \langle \delta\rho(\vec r, t)\, \delta\rho(\vec r', t) \rangle$$

$$\langle \delta\rho(\vec r, t)\, \delta\rho(\vec r', t) \rangle = \frac{1}{V} \sum_{q, q'} e^{i(\vec q \cdot \vec r + \vec q' \cdot \vec r')} \underbrace{\langle \delta\rho(q, t)\, \delta\rho(q', t) \rangle}_{= C^{E}_{\rho\rho}(\vec q)\, \delta_{q+q'}}$$

$$= \int \frac{d^d q}{(2\pi)^d}\, e^{i \vec q \cdot (\vec r - \vec r')}\, C^{E}_{\rho\rho}(\vec q)$$

II C.2

$$\Rightarrow \langle \delta N^2 \rangle = \int_{r,r'} \underbrace{\int_q e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')} C^{ET}_{\rho\rho}(\vec{q})}_{= C_{\rho\rho}(\vec{r}-\vec{r}\,')}$$

$$C^{ET}_{\rho\rho}(\vec{q}) = \frac{f(\theta_q)}{q^2}$$

$$\int e^{i\vec{q}\cdot(\vec{r}-\vec{r}\,')} \frac{f(\theta_q)}{q^2} d^d q \Rightarrow C^{ET}_{\rho\rho}(\vec{r}) = \frac{g(\theta_{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^{d-2}}$$

Long ranged spatial correlations of density fluctuations!

$$\vec{q} \equiv \frac{\vec{Q}}{|\vec{r}-\vec{r}\,'|}$$

$$\Rightarrow \langle \delta N^2 \rangle = \int_{r,r'} \frac{g(\theta_{\vec{r}-\vec{r}\,'})}{|\vec{r}-\vec{r}\,'|^{d-2}}$$

$$(\vec{r}, \vec{r}\,') \equiv L(\vec{R}, \vec{R}\,')$$

$$\Rightarrow \langle \delta N^2 \rangle = L^{2d-(d-2)} \int_{R,R' \in \square_1} \frac{g(\theta_{R-R'})}{|\vec{R}-\vec{R}\,'|^{d-2}}$$

$$\propto L^{d+2} \propto \langle N \rangle^{\frac{d+2}{d}} = \langle N \rangle^{1+\frac{2}{d}}$$

$$\langle N \rangle = \rho_0 L^d \Rightarrow L \propto \langle N \rangle^{\frac{1}{d}}$$

#(.30

⇒ RMS fluctuations:

$$\sqrt{\langle \delta N^2 \rangle} \propto \langle N \rangle^{\frac{1}{2}+\frac{1}{d}} \gg \sqrt{\langle N \rangle}$$: "Law of large numbers"

Ex: $d=2$, $\sqrt{\langle \delta N^2 \rangle} \propto \langle N \rangle$