Operator Systems, Tsirelson's Conjectures and Quantum Chromatic Numbers

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The Graph Coloring Game and Quantum Chromatic Numbers

Given a graph $G = (V, E)$ on $n = |V|$ vertices, its **chromatic number,** $\chi(G)$ is the least integer c such that there exists a function $f : V \rightarrow \{1, ..., c\}$ such that $(v, w) \in E$ implies that $f(v) \neq f(w)$. Such a function is called a **colouring.**

In the **graph colouring game** two players, A and B try to convince a referee that they have a colouring. The referee gives A a vertex v and B a vertex w and they each respond with a colour. They win the game if whenever they are given the same vertex they respond with the same colour and whenever they are given adjacent vertices they respond with different colours.

If A and B each set in separate rooms with no communication allowed and binary machines, then the smallest c such that they can win this game with probability 1 is $\chi(G)$.

However, if quantum machines with entangled states are allowed, then it is possible to win this game with a smaller number. This is the **quantum chromatic number,** $\chi_q(G)$.We formalize this idea as follows:

Alex has POVM's $(E_{v,i})_{i=1}^{c} \subseteq M_p = L(\mathbb{C}^p)$ which represent the measurements of $|V|$ experiments with c possible outcomes, Bab has POVM's $(F_{w,j})_{j=1}^{c} \subseteq M_q = L(\mathbb{C}^q)$, and they share a state, i.e., a unit vector $\xi \in \mathbb{C}^p \otimes \mathbb{C}^q$. If A performs experiment v and B performs w, then

$\langle (E_{v,i} \otimes F_{w,j})\xi, \xi \rangle = Tr((E_{v,i} \otimes F_{w,j})|\xi >< \xi|)$

is the probability for A producing an output i and B an output j.

A and B can thus convince the referee that they have a c-colouring if the following conditions are satisfied:

$$\forall v, \forall i \neq j, \langle (E_{v,i} \otimes F_{v,j})\xi, \xi\rangle = 0,$$
$$\forall (v,w) \in E, \forall i, \langle (E_{v,i} \otimes F_{w,i})\xi, \xi\rangle = 0.$$

In this case we say that G has a quantum c-colouring. The smallest c for which G has a quantum c-colouring is called the **quantum chromatic number** and denoted $\chi_q(G)$.
Much work done on this concept, many graphs for which $\chi_q(G)$ much less than $\chi(G)$.

Quantum Correlation Matrices

The set of all $nc \times nc$ matrices,

$$(\langle E_{v,i} \otimes F_{w,j}\xi, \xi\rangle)$$

that one can obtain by letting $\{E_{v,i}\}$, $\{F_{w,j}\}$ vary over all POVM's on finite dimensional spaces and ξ vary over all unit vectors is denoted

$$Q(n, c)$$

If we allowed the POVM's to range over all HIlbert spaces, possibly infinite dimensional, then we would obtain a potentially larger set of matrices,

$$Q_{nr}(n, c)$$

These are often called the **non-relativistic** correlation matrices.

If we allowed a different QM interpretation, where POVM's act on the same Hilbert space, but A's measurements must commute with B's, i.e.,

$$E_{v,i}F_{w,j} = F_{w,j}E_{v,i}, \forall v, w, i, j$$

then we would obtain a potentially larger set of matrices,

$$(\langle E_{v,i}F_{w,j}\xi, \xi\rangle)$$

denoted

$$Q_r(n, c).$$

These are often called the **relativistic** correlation matrices. Thus, there are, potentially, 3 models for the matrices that can arise as the correlation matrices for n experiments, each with c outcomes.

Tsirelson's Problems

Tsirelson attempted to describe each of these sets and conjectured that they are all the same set.
In fact, in a short note he claimed to have shown that they were all the same. But later gaps were found in the proofs.
Tsirelson's method, in the $c = 2$ case, was to study relations among the finite set of vectors in the relativistic picture,

$$h_{v,i} = E_{v,i}\xi, k_{w,j} = F_{w,j}\xi$$

observe that the correlation matrix only depends on these vectors, i.e.,

$$(\langle E_{v,i}F_{w,j}\xi, \xi\rangle) = (\langle h_{v,i}, k_{w,j}\rangle)$$

and given a set of vectors satisfying the "proper" constraints, try to construct POVM's on a finite dimensional space yielding this matrix.
The set of matrices that can be obtained by vectors satisfying these relations we shall denote by $Q_{vect}(n, c)$.

What we do/don't know:

- $Q(2,2) = Q_{nr}(2,2) = Q_r(2,2)$.
- $Q_r(n,c)$ always closed
- $Q(n,c)^- = Q_{nr}(n,c)^-$
- Unknown if $Q(n,c)$ or $Q_{nr}(n,c)$ closed, except for $n = c = 2$.
- Unknown if $Q(n,c) = Q_{nr}(n,c)$
- Unknown if $Q(n,c)^- = Q_r(n,c)$,
- In the notation of Navascue/Pirinio/Acin, our $Q_{vect}(n,c) = Q^1(n,c)$ and theyknow $Q^1(n,c) \neq Q(n,c)$ so known that Tsirelson's basic construction fails. They also give numerical evidence for the conjecture that $Q(n,c)^- = Q_r(n,c)$

- (Werner/Scholz) If Connes' embedding conjecture true, then $Q(n,c)^- = Q_r(n,c)$
- (JNPPSW) Connes' embedding conjecture true iff $Q^N(n,c)^- = Q_r^N(n,c), \forall N, n, c,$
- (FKPT) Connes' embedding true iff $Q^N(3,2)^- = Q_r^N(3,2), \forall N$
- (Ozawa) $Q(n,c)^- = Q_r(n,c)$ $\forall n, c$ iff Connes' embedding conjecture is true.

Connes Embedding Conjecture: Every II_1 factor embeds in an ultrapower of the hyperfinite II_1 factor.
Kirchberg: Connes' embedding conjecture true iff $C^*(G_1) \otimes_{min} C^*(G_2) = C^*(G_1) \otimes_{max} C^*(G_2)$ for all pairs of groups which contain the free group on two generators and have the lifting property iff there exists a pair of such groups with the tensor products equal.

“New” Quantum Chromatic Numbers

Motivated by the potential differences of these correlation sets, we introduce “new” quantum chromatic numbers that all collapse to $\chi_q(G)$ if Tsirelson's conjectures are true. Hope is that these coarser integer invariants might be computable and that the combinatorics of graphs might give insight into behaviour for larger n and c than is computationally feasible. However, equality of chromatic numbers won't show correlation sets are same only that certain “patterns” of 0's are the same.
I'll explain a few of these definitions, rest are pretty self-evident.
In all cases $\{E_{v,i}\}_{1 \le i \le c}$ and $\{F_{w,j}\}_{1 \le i \le c}$ are families of POVM's.

The **relativistic quantum chromatic number,** $\chi_{qr}(G)$ is the least c such that there exists a HIlbert space, POVM's with $E_{v,i}F_{w,j} = F_{w,j}E_{v,i}$ and a unit vector ξ such that

$$\forall v, \forall i \neq j, \langle (E_{v,i}F_{v,j})\xi, \xi \rangle = 0,$$

$$\forall (v,w) \in E, \forall i, \langle (E_{v,i}F_{w,i})\xi, \xi \rangle = 0.$$

The **approximate quantum chromatic number,** $\chi_{qa}(G)$ is the least number c such that $\forall \epsilon > 0$, there exists POVM's(depending on ϵ) on finite dimensional spaces(whose dimension can depend on ϵ) with

$$\forall v, \forall i \neq j, \langle (E_{v,i} \otimes F_{v,j})\xi, \xi \rangle < \epsilon, \\ \forall (v,w) \in E, \forall i, \langle (E_{v,i} \otimes F_{w,i})\xi, \xi \rangle < \epsilon.$$

This corresponds to there being a c such that $\forall \epsilon > 0$ you are able to convince a referee with probability $1 - \epsilon$ that you have a quantum c-coloring.
The integers $\chi_{qs}(G)$ and $\chi_{qsa}(G)$ are defined similarly, by allowing the Hilbert spaces to be infinite dimensional, but still requiring a tensor product of operators.

The **vectorial colouring number,** $\chi_{vect}(G)$ is defined using Tsirelson's vector relations. There are several others defined using state spaces on tensor products of operator systems(more on this later).Here's a partial summary of the relationships:

$$\chi_{q,max}(G) \le \chi_{vect}(G) \le \chi_{q,c}(G) = \chi_{qr}(G) \le \chi_{q,min}(G) = \chi_{qsa}(G) = \chi_{qa}(G) \le \chi_q(G),$$

If all of Tsirelson's claims are true, then $\forall G, \chi_{vect}(G) = \chi_q(G)$.
If Connes true, then $\forall G, \chi_{qr}(G) = \chi_q(G)$.
If correlation sets are closed, then $\forall G, \chi_{qa}(G) = \chi_q(G)$ and $\chi_{qsa}(G) = \chi_{qs}(G)$.

Roberson-Stahlke: $\chi_{vect}(C_5 \boxtimes K_3) = 7 < 8 \leq \chi_q(C_5 \boxtimes K_3)$.
In particular, $Q_{vect}(15, 7) \neq Q(15, 7)$ since "patterns" of 0 sets are different.
In fact, there is a 105×105 correlation matrix in $Q_{vect}(15, 7)$ that has 0's in a set of

$$|V|c(c-1) + 2c|E| = 910$$

entries and no correlation matrix in $Q(15, 7)$ has 0's in all of those sites.
This means that Tsirelson's approach to proving equality of quantum correlation sets can not succeed.
Their proof uses the theory of graph homomorphisms developed in [CMRSSW] and Roberson's projective rank $\xi_f(G)$.
Open: Can their 105 vectors be attained by commuting POVM's on a, necessarily, infinite dimensional Hilbert space or approximated by POVM's on finite dimensional spaces?

What is an operator system, why should I care?

Short Answer: Operator systems are what is needed to define what is meant by a map being completely positive. They are the natural domains and ranges of quantum channels.

Longer Answer: There is an $(n-1)c$-dimensional operator system

$$\mathcal{S}(n,c) = span\{e_{v,i} : v \in V, 1 \le i \le c\}$$

such that if $\{E_{v,i}\} \subseteq B(H)$ is a family of POVM's corresponding to n experiments with c outcomes each on a Hilbert space H then there is a unital, completely positive map(UCP),

$$\phi : \mathcal{S}(n,c) \to B(H), \text{ with } \phi(e_{v,i}) = \frac{1}{n}E_{v,i}$$

and conversely.

Thus, studying such families of POVM's is equivalent to studying UCP maps on $\mathcal{S}(n,c)$.

A bivariate set of correlations $\langle E_{v,i} \otimes E_{w,j}\xi, \xi \rangle$ can be thought of as the image of $e_{v,i} \otimes e_{w,j}$ under a state functional, $s : \mathcal{S}(n,c) \otimes \mathcal{S}(n,c) \to \mathbb{C}$.

Thus, defining a set of correlation matrices, determines a set of "allowed" states on the tensor product $\mathcal{S}(n,c) \otimes \mathcal{S}(n,c)$, which in turn determines the positive elements of the tensor product.

Dually, each time we define a particular operator system structure on the tensor product, then the image of the state space under the map

$$s \to \big(s(e_{v,i} \otimes e_{w,j})\big)$$

defines a set of "correlation matrices".

Hence, "correlation matrices" and "positive cones in $\mathcal{S}(n,c) \otimes \mathcal{S}(n,c)$" are dual concepts.

We have shown that the families of correlation matrices obtained in this fashion are closed and lie between $Q(n,c)$ and the set of non-signalling boxes.

However, an operator system structure on $\mathcal{S}(n,c) \otimes \mathcal{S}(n,c)$ means that for each N we also have a positive cone on

$$M_N(\mathcal{S}(n,c) \otimes \mathcal{S}(n,c)) \equiv M_N \otimes \mathcal{S}(n,c) \otimes \mathcal{S}(n,c)$$

these are what is needed to discuss the completely positive maps on $\mathcal{S}(n,c) \otimes \mathcal{S}(n,c)$.
To understand this additional structure we need to consider "matricial correlation matrices". For each "matrix state", i.e., UCP map $\phi : \mathcal{S}(n,c) \otimes \mathcal{S}(n,c) \to M_N$ consider the block matrix

$$\big(\phi(e_{v,i} \otimes e_{w,j})\big)$$

These explain the sets $Q^N(n,c)$ occurring earlier.

In the language of operator systems what [FKPT] prove is that the spacial and commuting models of POVM's lead to two operator system structures on $\mathcal{S}(n,c) \otimes \mathcal{S}(m,d)$ denoted $\mathcal{S}(n,c) \otimes_{min} \mathcal{S}(m,d)$ and $\mathcal{S}(n,c) \otimes_{c} \mathcal{S}(m,d)$.
We prove that:

$$\mathcal{S}(3,2) \otimes_{min} \mathcal{S}(3,2) = \mathcal{S}(3,2) \otimes_{c} \mathcal{S}(3,2) \implies \mathcal{S}(n,c) \otimes_{min} \mathcal{S}(m,d) = \mathcal{S}(n,c) \otimes_{c} \mathcal{S}(m,d) \forall m,n,c,d$$

This is why studying 3 experiments with 2 outcomes, but correlations whose entries are matrices is sufficient to solve Tsirelson/Connes/Kirchberg.

Each operator system structure α consistently defined on all tensor products leads to a "colouring" of $G = (V, E)$ by demanding that there exists a state $s : \mathcal{S}(n, c) \otimes_\alpha \mathcal{S}(n, c) \to \mathbb{C}$ such that

$$\forall v, \forall i \neq j, s(e_{v,i} \otimes e_{v,j}) = 0, \qquad \forall (v, w) \in E, \forall i, s(e_{v,i} \otimes e_{w,i}) = 0.$$

The least c for which such a colouring exists can be used to define a α-chromatic number, $\chi_\alpha(G)$.
There are many such tensor products that have been studied for operator systems which yield many, potentially different, chromatic numbers. For many of these chromatic numbers we still have no "operational" definitions.But these chromatic numbers give us a tool to help distinguish some differences in the geometry of the state spaces of these tensor products.

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