Modelling the interactions of near-inertial waves and vortical motion in the ocean

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with Eric Danioux & Jin-Han Xie
Near-inertial waves

Inertia-gravity waves: fast waves with dispersion relation

\[ \omega = \pm (f^2 + N^2 k^2 / m^2)^{1/2} \quad \text{or} \quad \omega = \pm f (1 + r_d^2 k^2)^{1/2} \]

with \( r_d \) radius of deformation (= \( NH/(nf\pi) \)).

Oceanic inertia-gravity waves important for:

- vertical motion \( \Rightarrow \) biology,
- vertical shear, instability, turbulence \( \Rightarrow \) diapycnal mixing,
- mixing \( \Rightarrow \) pollutant dispersion,
- large-scale ocean circulation, through diapycnal mixing (Munk & Wunsch 2009) and dissipation (Gertz & Straub 2009).

Sources: tides, topography, winds...
Near-inertial waves

Inertia-gravity-wave spectrum is dominated by lowest frequencies: near-inertial waves, NIWs:

$$\omega \approx f, \quad k/m \ll N/f, \quad kr_d \ll 1.$$
Near-inertial waves

About 50% of wave energy in NIWs:
- generated by winds (low frequency) affecting the mixed layer ($k/m \ll 1$),
- $f$ lowest frequency available for resonant interactions,
- subharmonic instability of $M_2$ tide.

Alford 2003

‘Despite their ubiquity, energy, and many years of study, much about the behavior of inertial waves remains obscure.’ (Ferrari & Wunsch 2009)
Near-inertial waves

Main issues:

- NIW propagation into ocean interior (weak dispersion),
- role of mean flow in this propagation,
- generation of small vertical scales,
- impact of NIWs on mean flow.

Main theoretical tools: linear wave dynamics,

- WKB approximation (Kunze 1985): takes $kL_{\text{flow}} \gg 1$, but $kL_{\text{flow}} \lesssim 1$,
- Young–Ben Jelloul model (1997): assumes $\omega \approx f$, $kL_{\text{flow}} = O(1)$ to describe slow modulation of NIWs.

In this talk:

1. a variational derivation of the YBJ model,
2. scattering of NIWs by mean flow,
3. a coupled YBJ + QG model.
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YBJ model: variational derivation

Inertial balance: \( \partial_t u - fv = 0, \quad \partial_t v + fu = 0, \quad u_x + v_y + w_z = 0. \)

Write \( u + iv = M z e^{-i ft}, \quad w = \frac{1}{2} (\partial_x + i \partial_y) Me^{-i ft} + c.c. \)

YBJ model: for \( \epsilon = (Nk)^2/(fm)^2 \ll 1, \) derive a slow evolution equation for \( M(\epsilon t, x). \)

Start from the Lagrangian for hydrostatic–Boussinesq,

\[
\mathcal{L}[x, p] = \int \left( \frac{1}{2} (\dot{x}^2 + \dot{y}^2) - \left( fy + \frac{\beta y^2}{2} \right) \dot{x} + bz + p \left( \frac{\partial x}{\partial a} - 1 \right) \right) \, da
\]

Assume \( X(a, t) \) is the flow map of the (given) mean flow, and let

\[
x(a, t) = X(a, t) + \xi(X(a, t), t),
\]

with \( \xi \) a GLM-style perturbation.
YBJ model: variational derivation

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**YBJ model:** variational derivation

To leading order, \( \xi \) is an inertial-wave,

\[
\xi + i\eta = M_z e^{-i\Omega t}, \quad \zeta = \frac{1}{2} (\partial_x + i\partial_y) M e^{-i\Omega t} + \text{c.c.}
\]

Into \( \mathcal{L} \), (Whitham)-average to obtain the YBJ Lagrangian,

\[
\mathcal{L}_{\text{YBJ}}[M] = \int \left( (M_z D_t M_z^* - i\beta y|M_z|^2 \\
- \frac{i}{2f} (\nabla^2 P|M_z|^2 + P_{zz}\nabla|M|^2 - \nabla P_z \cdot (\nabla M^* M_z + \nabla M M_z^*)) \right) dx.
\]

with \( D_t = \partial_t + \mathbf{U}_h \cdot \nabla + W \partial_z \) and \( P \) background pressure.

Variations \( \delta M \) give the YBJ equation,

\[
(D_t M_z)_z + i\beta y M_{zz} + \frac{i}{2f} (\nabla^2 P M_{zz} + P_{zz} \nabla^2 M - 2\nabla P_z \cdot \nabla M_z) = 0.
\]
YBJ model

Mean flow does not need to be QG, balanced, or horizontal.

Symmetries $M \mapsto M e^{i\phi}$ and $t \mapsto t + \phi$ yield:
- action conservation (NIW kinetic energy)
  $$\mathcal{A} = \int |M_z|^2 \, dx,$$
- energy conservation
  $$\mathcal{H} = \frac{1}{2f} \int (i f M_z^* U \cdot \nabla M_z$$
  $$+ M^* \nabla^2 (PM_{zz}) + 2P|\nabla M_z|^2 + M_{zz}^* P \nabla^2 M) \, dx.$$  

Hamiltonian system: in terms of $L = M_z$, with symplectic form $idN \wedge dN^*$. 
YBJ model

For $\mathbf{U}$ QG, barotropic, $P_{zz} \approx N^2$, YBJ reduces to

$$
\partial_t M_{zz} + \mathbf{U} \cdot \nabla M_{zz} + i \beta y M_{zz} + i \frac{N^2}{2f} \nabla^2 M + i \frac{\Omega}{2} M_{zz} = 0.
$$

- **advection**: by the background flow, cf. passive scalar,
- **dispersion**: approximates frequency as

\[
(f^2 + N^2 k^2 / m^2)^{1/2} \approx f + N^2 k^2 / (2fm^2),
\]

or

\[
f (1 + r_d^2 k^2)^{1/2} \approx f + fr_d^2 k^2 / 2
\]

- **refraction**: $\Omega = \nabla \times \mathbf{U}$, frequency shift (Kunze 1985).

YBJ model used to demonstrate the effect of background flow on NIW structure and vertical propagation (Balmforth et al 1998, Balmforth & Young 1999, Klein et al 2004...).
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YBJ model

Asymptotic limits:
- $kL_{\text{flow}} \gg 1$: WKB, refraction $\ll$ advection,
- $kL_{\text{flow}} \ll 1$: homogenisation, advection $\ll$ refraction.

Homogenisation + strong dispersion $N^2k^2/(2m^2f) \gg \Omega$ shows small-scale flow increases dispersion and vertical propagation (Young–Ben Jelloul 1997).

Scattering limit: take $kL_{\text{flow}} = O(1)$ and $N^2k^2/(2m^2f) \gg \Omega$: refraction $\sim$ advection, both causing NIW scattering.
NIW scattering

Let $\epsilon \ll 1$ be the scale separation between $k^{-1} \sim L_{\text{flow}}$ and envelope scale. 

Describe NIW field using the Wigner function,

$$W(t, x, k) = \frac{1}{4\pi^2} \int e^{ik \cdot y} M_z(t, x - \frac{\epsilon y}{2}) M_z^*(t, x + \frac{\epsilon y}{2}) \, dy,$$

Assume $\Omega = O(\epsilon^{1/2}) \times fr_d^2 k^2$: strong dispersion. Take $\psi = \nabla^{-2} \Omega$ steady, Gaussian random process with $\langle \psi(x) \psi(y) \rangle = R(x - y)$.

Derive the transport equation (Ryzhik et al. 1996),

$$\partial_t W + fr_d^2 k \cdot \nabla W = 4\pi \int \left( (k \times p)^2 + \frac{|p - k|^4}{4} \right) \hat{R}(p - k) \delta(k^2 - p^2) \times [W(t, x, p) - W(t, x, k)] \, dp.$$
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NIW scattering

Equation describes:

- transport of NIW energy density with \( c_g = f r_d^2 k \),
- scattering by mean flow, restricted to constant frequency shell \( |\mathbf{p}| = |\mathbf{k}| \).

Predicts:

- Isotropisation of NIW energy in \( k \)-space,
- Slow down of scale cascade by dispersion.
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Numerical experiment: barotropic YBJ equation, with isotropic random velocity field.
NIW scattering

$R_d = 10 \text{ km}$
$L_{\text{wave}} = 85 \text{ km}$
$\zeta_{\text{rms}}/f = 0.07$
Coupled model

Impact of NIWs on mean flow:

- non-dissipative framework,
- evolution of NIW amplitude $M_z$ and mean flow on the same time scale,
- an averaged model that respects dynamical constraints (momentum, energy conservation, circulation...).

Derive a new model using

- variational formulation of Generalised Lagrangian Mean (Salmon 2013),

Start with hydrostatic Boussineq $\mathcal{L}[x, p]$, introduce

$$x(a, t) = X(a, t) + \xi(X(a, t), t), \quad \xi + i\eta = M_z e^{-i ft},$$

and average to obtain $\bar{\mathcal{L}}[X, M, P]$. 
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and average to obtain $\bar{\mathcal{L}}[X, M, P]$. 
**Coupled model**

**Variations:**

- $\delta X^{-1}, \delta P$: mean-flow dynamics,
- $\delta M$: YBJ equations, as before.

For QG mean flow, we obtain the coupled QG-YBJ model (for $\beta = 0$):

\[
\partial_t q + \partial(\psi, q) = 0, \quad \text{with} \quad \left( \nabla^2 + \partial_z \left( \frac{f^2}{N^2} \partial_z \right) \right) \psi = q + F(M^*, M),
\]

\[
(D_t M_z)_z + \frac{i}{2} \left( \nabla^2 \psi M_{zz} + \left( \frac{N^2}{f} + \psi_{zz} \right) \nabla^2 M - 2 \nabla \psi_z \cdot \nabla M_z \right) = 0,
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with $F(M^*, M)$ bilinear operator.

The model conserves action and energy:

\[
\mathcal{A} = \int |M_z|^2 \, dx,
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Coupled model

\[ \mathcal{H} = \frac{1}{2} \int \left( |\nabla \psi|^2 + \frac{f^2}{N^2} (\partial_z \psi)^2 + \frac{N^2}{2} |\nabla M|^2 \right) \, dx. \]

- evolution governed by PV \( q \) and NIW amplitude \( M \),
- advecting velocity \( \nabla \perp \psi \) depends on both \( q \) and \( M \),
- energy \( \mathcal{H} \) is simple in terms of \( \psi \), complicated in terms of \( q \),
- ‘velocity split’ (for the class of 1996 only),
- simple Hamiltonian structure.

Physics of NIWs–mean flow interactions constrained by conservation laws:

- \( A = \text{const} \): no spontaneous NIW generation,
- \( \mathcal{H} = \text{const} \): mean-flow energy decays as \( |\nabla M| \) increases.
Coupled model

\[ \mathcal{H} = \frac{1}{2} \int \left( \left| \nabla \psi \right|^2 + \frac{f^2}{N^2} \left( \partial_z \psi \right)^2 + \frac{N^2}{2} \left| \nabla M \right|^2 \right) \, dx. \]

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Conclusion

- Near inertial waves: everywhere in the ocean, significant energy;
- Length scales comparable to mean-flow scales – high frequencies: subgrid scale in time;
- YBJ model: relevant model for NIW dynamics, variational structure;
- Dispersion constrains NIW scale cascade;
- Coupled YBJ-QG model: a (Hamiltonian) subgrid scale model (cf. Gjaja & Holm 1996);
- Coming soon: numerical integrations of the coupled model.