The semi-geostrophic equations - a model for large-scale atmospheric flows

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Introduction - Motivation

The behaviour of the atmosphere is described by the compressible Navier-Stokes equations

$$D_t\mathbf{u} + 2\Omega \times \mathbf{u} + \nabla\Phi + \frac{1}{\rho}\nabla p = \nu\Delta\mathbf{u},$$
$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0,$$
$$D_t\theta = 0, \qquad p^{1-\kappa} = R\rho\theta.$$

(geopotential $\Phi = gx_3$.)

Notation: H typical vertical scale, L horizontal scale
U typical horizontal velocity , $T \sim \frac{U}{L}$ typical timescale

From now on, assume no viscosity ($\nu = 0$) - **compressible Euler equations**

Reductions to a model for large scale flows

(1) **Shallow atmosphere**: $\frac{H}{L} << 1$,

This implies that the Coriolis force has no effect in the vertical direction

$$2\Omega \times \mathbf{u} = (-fu_2, fu_1, 0) \Longrightarrow D_t\mathbf{u} + (fu_2, -fu_1, g) + \frac{1}{\rho}\nabla p = 0$$

f is variable

(2) **Hydrostatic balance:**

$$\frac{\partial p}{\partial x_3} = -g\rho$$

In the resulting system, the vertical velocity can be resolved if either the Rossby number $\varepsilon = \frac{U}{fL}$ or the Froude number $\eta = \frac{U}{NH}$ are small

small ε - rotation dominated flow

small η - stratification dominated flow

Geostrophic balance

(3) **Geostrophic balance:**
For small Rossby number ε, the rotations terms dominate

$$\implies \quad \frac{1}{\rho}\partial_1 p \sim f u_2, \quad \frac{1}{\rho}\partial_2 p \sim -f u_1$$

Define the geostrophic velocity as the horizontal velocity satisfying the geostrophic balance exactly:

$$\mathbf{v}^g = (v_1^g, v_2^g, 0) \;\; with \;\; v_1^g = -\frac{1}{f\rho}\partial_2 p, \;\; v_2^g = \frac{1}{f\rho}\partial_1 p.$$

There are different possible relative scalings of ε and η within this framework

$\varepsilon = \eta << 1$: the quasi-geostrophic system

The scaling Froude=Rossby, i.e.

$$\frac{H}{L} = \frac{f}{N},$$

gives rise, to first order in the approximation, to the quasi-geostrophic system: conservation of potential vorticity principle, but not valid on large scales (they require constant coefficients)

Results by Bourgeois-Beale on global existence (at least for "well prepared" initial conditions) and validity as a reduction of NS - the scale analysis is rigorously justified.

$\varepsilon \sim \eta^2 << 1$: the semigeostrophic system

This is a second-order accurate approximation.
Only horizontal momentum is approximated $\Longrightarrow$ equations are valid for large scale (f can be variable):

$$
\begin{aligned}
&D_t \mathbf{v}^g - (fu_2, -fu_1) + \frac{1}{\rho}(\partial_1 p, \partial_2 p) = 0 \\
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0, \\
&\partial_1 p = \rho f v_2^g, \quad \partial_2 p = -\rho f v_1^g, \quad \partial_3 p = -\rho g, \\
&D_t \theta = 0, \qquad p^{1-\kappa} = R\rho\theta.
\end{aligned}
$$

where $D = \partial_t + \mathbf{u} \cdot \nabla$ (energy is then conserved)

First derived by Eliassen, then rediscovered by Bretherton and Hoskins as an approximate model for frontogenesis (John Methven's talk on Monday!)

Mathematical vs physical issues

The QG system is a conservation law, but it is not valid on large scales. Also, it is not possible to extend results to variable geometry or variable coefficient cases.

The SG system is valid on large scales, and is also a conservation law in the right variables. It is a more appropriate system to use to understand the predictability of the weather system on large scales, which requires that:

- the equations can be *solved* for given initial and boundary conditions
- the solutions, not necessarily smooth, are well defined at least for some time, ideally on the approximation timescale
- the scale analysis can be rigorously justified - there is a solution of the full NS system close to the semigeostrophic solution in an appropriate asymptotic sense

The semigeostrophic system - 3D incompressible case

The geostrophic velocity is 2D and given by

$$(v_1^g, v_2^g) = (-\partial_2 p, \partial_1 p) \ \ (\textit{geostrophic balance}).$$

Assuming also $\qquad \partial_3 p = -\rho$ (*hydrostatic balance*)

(with Boussinesq approximation and all *constants scaled to* 1):

$$\begin{aligned} &D_t(v_1^g, v_2^g) + (\partial_1 p, \partial_2 p) = (u_2, -u_1) \\ &D_t \rho = 0, \quad \rho = -\partial_3 p, \quad (v_1^g, v_2^g) = (-\partial_2 p, \partial_1 p), \\ &\nabla \cdot \mathbf{u} = 0, \end{aligned}$$

$(D_t = \partial_t + \mathbf{u} \cdot \nabla \qquad$ denotes the *lagrangian derivative*)

Boundary condition: $\mathbf{u} \cdot n = 0$ on $\partial\Omega$.
Unknowns: $\mathbf{u} = (u_1, u_2, u_3)$; $\mathbf{v} = (v_g^1, v_g^2, 0)$; p; ρ.

Dual variables

Change to dual coordinates:

$$X(t,x) = \nabla P = (v_2^g + x_1, -v_1^g + x_2, -\rho),$$

$$\text{where} \qquad P(t,x) = p(t,x) + \frac{1}{2}(x_1^2 + x_2^2).$$

(*Hoskins' geostrophic variable change*). Then

$$D_t X = J(X - x), \qquad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$P(0,x) = P_0(x);$$

plus the conditions on $\mathbf{u}$: $\nabla \cdot \mathbf{u} = 0; \quad \mathbf{u} \cdot n = 0$ on $\partial\Omega$.
Note: $\mathbf{U} = D_t X = (v_1^g, v_2^g, 0)$ - the dual flow is purely geostrophic

Energy

The geostrophic energy associated with the flow is

$$E = \int_\Omega \left\{ \frac{1}{2} \left[(v_1^g)^2 + (v_2^g)^2 \right] + \rho x_3 \right\} dx.$$

Cullen's stability principle: *stable solutions are energy minimisers with respect to the rearrangements of particles in physical space that conserve absolute momentum and density*

In the dual variables $X = \nabla P(x) = (v_2^g + x_1, -v_1^g + x_2, -\rho)$,

$$E(X) = \int_\Omega \left\{ \frac{1}{2} \left[(x_1 - X_1)^2 + (x_2 - X_2)^2 \right] - x_3 X_3 \right\} dx.$$

In these variables the minimiser condition formally becomes

$$P(t,x) = \frac{1}{2}(x_1^2 + x_2^2) + p(t,x) \quad \text{is a convex function}$$

Minimising the dual energy = optimal transport problem

The change of variable $X = \nabla P(x)$ can be stated in terms of the measure ν on $\mathbb{R}^3$ such that

$$\int_\Omega f(\nabla P(x))dx = \int_{\mathbb{R}^3} f(y)d\nu(y)\ \forall f \in \mathbf{C}_c(\mathbb{R}^3).$$

This is concisely written using the push-forward notation as
$\nu = \nabla P \# \chi_\Omega$
(weak form of the Monge-Ampère equation $\nu = \det(D^2 P^*)$).

The minimisation of $E(X)$ can be phrased as the requirement that the change of variables $X : \Omega \to \mathbb{R}^3$ such that $X\#\chi_\Omega = \nu$ is the optimal transport map with respect to the cost function

$$c(x, X) = \frac{1}{2}\left[(x_1 - X_1)^2 + (x_2 - X_2)^2\right] - x_3 X_3$$

Dual formulation

Assume P convex (*key assumption*) - ∇P is a well defined change of variables.

Let P^* denote the Legendre transform of P:

$$P^*(X) = \sup_x \{x \cdot X - P(x)\} \quad (X = \nabla P(x) \Longleftrightarrow x = \nabla P^*(X))$$

Switch dependent/independent variable $\rightarrow$ dual formulation (conservation law)

$$\nu = \nabla P \# \chi_\Omega; \qquad \partial_t \nu + \nabla \cdot (\mathbf{U}\nu) = 0;$$
$$\mathbf{U} = J(X - \nabla P^*);$$
$$\nu(0, X) = \nu_0(X) \qquad (\textit{with } \ \nu_0 = \nabla P_0 \# \chi_\Omega).$$

Physical vs dual variables

$$D_t(v_1^g, v_2^g) + (\partial_1 p, \partial_2 p) = (u_2, -u_1), \quad \mathbf{v}^g = (-\partial_2 p, \partial_1 p, 0)$$
$$D_t \rho = 0, \quad \rho = -\partial_3 p,$$

$$P(t,x) = p(t,x) + \frac{1}{2}(x_1^2 + x_2^2), \quad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Physical space	Dual space
$(\partial_t + \mathbf{u} \cdot \nabla)X = J(X - x)$	$\partial_t \nu + \nabla \cdot (\mathbf{U}\nu) = 0$
$X = \nabla P$	$\nu = \nabla P \# \chi_\Omega; \quad \mathbf{U} = J(X - \nabla P^*)$
$X = X(x) = \nabla P$	$x = x(X) = \nabla P^*$

The result of Benamou and Brenier

Polar factorization theorem = statement on existence of a unique optimal transport map between χ_Ω and ν, with respect to cost $c(x, X) = \frac{1}{2}|x - X|^2$

This map is given by the gradient of a convex function$\rightarrow$ $X = \nabla P$

$\sim$ minimisation of the geostrophic energy

$$E = \int_\Omega \frac{1}{2}[(x_1 - X_1)^2 + (x_2 - X_2)^2 - x_3 X_3]dx \sim \int_\Omega c(x, X(x))dx$$

Stable solution have to minimise E at each fixed time.

$\Longrightarrow$ at each fixed t. ν determines a convex function P.

Moreover, if $\nu_n \rightarrow \nu$ weakly, then $P_n \rightarrow P$ in $W^{1,p}$(stability).

How does this solve the geostrophic system?

The potential vorticity ν must also satisfy the transport equation

$$\partial_t \nu(t,X) + \nabla \cdot (\mathbf{U}(t,X)\nu(t,X)) = 0, \quad \mathbf{U}(t,X) = J(X - \nabla P^*(t,X))$$

The issue in solving this transport equation is the lack of regularity of the velocity $\mathbf{U}(t,X) = J(X - \nabla P^*(t,X))$.

Polar factorization $\implies$ U is not Lipschitz, only locally bounded, locally BV ($\nabla \psi^* \in \mathbf{W}^{1,\infty}$)

In order to find a weak solution:

time discretisation + regularisation + stability results.

Use standard results to solve the sequence of regularised transport problem, plus the stability given by Brenier's theorem

(Feldman talk)

Note: do not get uniqueness

3D incompressible result in dual variables

Theorem [BB]:
$P_0(x)$ given convex function, such that $\nu_0 = P_0 \# \chi_\Omega$ is in $\mathbf{L}^q(\Omega)$ $q > 1$ ($\Longrightarrow$ compact support $\subset B(0, R_0)$). Then for all $\tau > 0$, there exist

- $\nu(X, t) \in (\mathbf{L}^\infty[0, \tau), \mathbf{L}^q(\mathbb{R}^3))$, *supp* $\nu \subset B(0, R(\tau))$,
- $P(t, X) \in (\mathbf{L}^\infty[0, \tau), \mathbf{W}^{1,\infty}(\Omega))$, *convex*, $P \in \mathbf{W}^{1,r}(\Omega))$ *for all* $r \geq 1$
- P^* *convex, locally bounded in* t *and* X, *with* $\nabla P^* \in (\mathbf{L}^\infty[0, \tau) \cap \mathbf{C}[0, T), \mathbf{L}^\infty(B(0, R(\tau))))$
- $\mathbf{U} \in (\mathbf{L}^\infty[0, \tau), \mathbf{L}^\infty_{loc} \cap BV_{loc}(\mathbb{R}^3))$
- ν *is a weak solution of the transport equation: For all* $\Phi(t, X) \in \mathbf{C}_c([0, \tau) \times \mathbb{R}^3)$

$$\int [\partial_t \Phi + \mathbf{U} \cdot \nabla \Phi] \nu dX dt + \int \nu_0 \Phi(0, X) dX = 0$$

Generalisations

Obvious aims:

- make sense of the solution in physical space (Feldman's talk)
- uniqueness of the solution
- inclusion of more general conditions (compressible equations, variable rotation rate, other boundary conditions,...)

What has been done:

- Incompressible shallow water equations (2D) with a free boundary
  (Cullen and Gangbo 2001)
- Solution of the compressible case in dual coordinates
  (Cullen and Maroofi 2003)
- Solution in physical rather than dual coordinates
  (Cullen and Feldman 2006)

A robust approach: the compressible dual result

Energy:

$$E(t) = \int_\Omega \left[\frac{1}{2}|v^g|^2 + \Phi(x) + \theta(t,x)p(t,x)^{\frac{\kappa-1}{\kappa}} \right] \rho(t,x)dx.$$

The set-up is similar: in dual variables, it is the coupling of two problems

1. formulate the energy minimisation as an optimal transport problem at each fixed time t
2. solve (weakly) the transport equation for the dual density ν

The energy to be minimised can be written in dual variables as

$$E_\nu(\sigma) = \mathcal{E}(\nu,\sigma) + K\int_\Omega \sigma^\kappa dx, \qquad \mathcal{E}(\nu,\sigma) = \inf_{T\#\sigma=\nu}\int_\Omega c(x, T(x))dx$$

where $\sigma = \theta\rho$ is a physical density, and the cost is

$$c(x,y) = \frac{\left[\frac{1}{2}(x_1-y_1)^2 + \frac{1}{2}(x_2-y_2)^2 + \Phi(x)\right]}{y_3}.$$

- Show that the optimal transport problem for $\mathcal{E}(\nu,\sigma)$ has a unique solution $T:\Omega\to\Lambda$, $T = \nabla P$ with P convex
- Show that, for fixed ν, there exists a unique $\sigma\in P_{ac}(\Omega)$ that minimizes the energy $E_\nu(\sigma)$ - in addition $\sigma\in \mathbf{W}^{1,\infty}(\Omega)$
- Uniformity estimates
- solutions to the transport problem for ν constructed by approximation+regularisation (discrete time stepping procedure, as in Benamou-Brenier)

Extensions, some in progress, I want to discuss

- Alternative proof using Ambrosio-Gangbo Hamiltonian flows in spaces of probability measures
- 3D free boundary value problem for incompressible case
- Justification of the scaling reduction for sG

- Variable rate of rotation - *in progress upstairs*

1 - Alternative proof of the solvabiliy of the transport problem

Very general approach (Ambrosio and Gangbo): study of hamiltonian ODEs in space $\mathcal{P}_2^{ac}$ of probability measures - metric space with Wasserstein metric W^2

The differentiable structure of this space was developed following the work many people (Mc Cann, Otto, Ambrosio, Gangbo, Pacini). In particular one has a notion of $(\lambda-)$displacement convexity .

The notion of subdifferential mimicks the usual notion in convex analysis
This is used to define rigourously *Hamiltonian flows* in the space of probability measures (Gangbo-Pacini).

Given a lower sc Hamiltonian $H : \mathcal{P}_2^{ac} \to \mathbb{R}$, and given the probability measure $\mu(t = 0)$, find a solution to the transport

$$\partial_t \mu_{(t)} + \nabla \cdot (J \nabla H(\mu_{(t)}) \mu_{(t)}), \quad \mu_0 \textit{ given}$$

(a path in probability space from μ_0 to $\mu_{(t)}$)

∇H *denotes the element of minimal norm in the subdifferential*

Solutions exist under a growth condition and continuity conditions for the gradient of H - under additional convexity assumption, H is also constant along solutions.

Hamiltonian ODEs

Let $H : \mathcal{P}_2^{ac} \rightarrow \mathbb{R}$ be lsc, and satisfy
(H1): $|\nabla H(\mu)(z)| \leq C(1+|z|)$ for a.e. z
(H2): for $\mu_n \rightarrow \mu$ narrowly, there exists a subsequence and L^2 functions $w^k = \nabla H(\mu_{n_k})$, $w = \nabla H(\mu)$ a.e. such that $w_k \rightarrow w$ in L^2.

Theorem [AG] (absolutely continuous case)
Given a Hamiltonian H as above, there exists a Hamiltonian flow μ_t starting from a given $\mu_0 \in \mathcal{P}_2^{ac}$, whose velocity field coincides with $\nabla H(\mu_t)$ for a.e. t, i.e. a solution of

$$\partial_t \mu_{(t)} + \nabla \cdot (J \nabla H(\mu_{(t)})\mu_{(t)}) = 0, \quad \mu_{(0)} = \mu_0$$

$t \rightarrow \mu - t$ is a Lipschitz map, and the support of the measures is controlled.
If H is λ-convex, the Hamiltonian is constant along trajectories.

Alternative proof of dual space result

Incompressible: coupling of two problems

1. find P convex: $\nabla P \# \chi_\Omega = \nu$ optimal transport problem at each fixed time t
2. evolution equation for ν

Dual equations:

$$U = J(X - \nabla P^*(X))$$
$$\partial_t \nu + \nabla \cdot (U\nu) = 0, \quad \nu = T\#\sigma$$

Use the result of Ambrosio and Gangbo by

- proving directly that the semigeostrophic energy is subdifferentiable, (-2)-convex and lower s.c.
- computing explicitly ∇H and showing that for fixed t

$$\nabla H = X - \nabla P^*(X)$$

Consider the 3D incompressible semigeostrophic system in a domain

$$\Omega_h(t) = \Omega_2 \times [0, h(t, x_1, x_2)], \qquad \Omega_2 \subset \mathbb{R}^2.$$

where h describe the (free) top boundary (rigid bottom at $x_3 = 0$).

Incompressibility: $|\Omega_h(t)| = 1$ for all $t < \tau \Rightarrow h \in L^1 \cap L^2(\Omega_2)$

Boundary conditions:

$$\mathbf{u} \cdot \mathbf{n} = 0 \qquad \mathbf{x} \in \partial\Omega_h(t) \setminus \{x_3 = h\},$$

$$\begin{cases} \partial_t h + u_1 \frac{\partial h}{\partial x_1} + u_2 \frac{\partial h}{\partial x_2} = u_3, \\ p(t, x_1, x_2, h(x_1, x_2)) = p_h, \end{cases} \qquad \mathbf{x} \in \partial\Omega_h(t): \ x_3 = h(t, x_1, x_2),$$

Dual variables problem

$$\mathbf{x} \in \Omega_h(t) \to \mathbf{T}(t,\mathbf{x}) = (x_1 + u_2^g, x_2 - u_1^g, -\rho) \in \Lambda \subset \mathbb{R}^3,$$

Potential density $\nu = \mathbf{T}\#\sigma_h \in \mathcal{P}_{ac}(\Lambda)$, with $\sigma_h = \chi_{\Omega_h}$

Cullen's stability principle

Given $\nu \in \mathcal{P}^2_{ac}(\Lambda)$, a stable solution corresponds to the following minimal value for the energy:

$$\mathcal{E}(t,\nu) = \inf_{\sigma_h \in \mathcal{H}} E_\nu(h)$$

$\mathcal{H} \subset \mathcal{P}_{ac}(\mathbb{R}^3)$ is an appropriate subset of $\mathcal{P}_{ac}(\mathbb{R}^3)$, and

$$E_\nu(h) = \inf_{\mathbf{T}:\, \mathbf{T}\#\sigma_h = \nu} \int_{\mathbb{R}^3} \left[\frac{1}{2} \left(|x_1 - T_1|^2 + |x_2 - T_2|^2 \right) - x_3 T_3 \right] \sigma_h(\mathbf{x})\, d\mathbf{x}$$

Lagrangian dual form

$$
\begin{aligned}
&\frac{\partial \nu}{\partial t} + \nabla \cdot (\nu \mathbf{w}) = 0, \quad \text{in } [0,\tau) \times \Lambda, \\
&\mathbf{w}(t,\mathbf{y}) = J(\mathbf{y} - \mathbf{T}^{-1}(t,\mathbf{y})), \quad \text{in } [0,\tau) \times \Lambda, \\
&\mathbf{T}(t,\cdot) = \nabla P(t,\cdot), \qquad t \in [0,\tau), \\
&\nabla P \# \sigma_{\bar{h}} = \nu, \\
&\sigma_{\bar{h}} \text{ minimises } E_{\nu(t,\cdot)}(\cdot) \text{ over } \mathcal{H}, \qquad t \in [0,\tau).
\end{aligned}
$$

plus initial conditions

$$h(0,\cdot) = h_0(\cdot) \in W^{1,\infty}(\Omega_2), \qquad \nu(0,\cdot) = \nu_0(\cdot) \ \in L^r(\Lambda_0),\ r \in (1,\infty),$$

$$P_0(\mathbf{x}) \in W^{1,\infty}(\Omega_{h_0}) : \qquad \nabla P_0 \# \sigma_{h_0} = \nu_0.$$

Unknowns: $h(t,x_1,x_2)$ and $P(t,\mathbf{x})$.

Then $\nu = \nabla P \# \sigma_h$; $\quad p(t,\mathbf{x}) = P(t,\mathbf{x}) - \frac{1}{2}(x_1^2 + x_2^2)$

Tools for the proof

Same idea as the proof for the problem in a fixed domain:

1. Prove that the minimising problem has a unique solution for P and h - optimal transport problem but with $T = \nabla P$ depending on $h \Longrightarrow$ minimisation in both T and h
   Uses strict convexity with respect to the usual linear structure of L^1, plus $h > 0$, $\rho > 0$.
2. Show that the Hamiltonian satisfies the continuity and growth conditions required by Ambrosio-Gangbo result so that the transport problem with velocity $J\nabla H(\nu)$ can be solved.
   Then show that $J\nabla H(\nu) = \mathbf{w}$

3 - Justification of the scaling reduction

Quantifying the relation with the solutions of Euler (or NS) is crucial for the rigorous justification of the scaling reduction.

QG:

- Bourgeois-Beale: existence of global solutions in physical space (periodic or Neumann conditions)
- Based on refined energy estimates (control of velocity gradients) - basis is fundamental work of Temam on Euler
- $\Longrightarrow$ Existence of Euler solutions close to QG solution to the same order as that of approximation

SG:

- Brenier-Cullen results - periodic 2d - but order of approximation is not optimal ($\sqrt{\varepsilon}$ instead of ε).
- Applies only to smooth solutions (that exist only locally) and periodic BC

2D result is based on the Bregman functional

$$\eta_{P^*}(t,y,y^\epsilon) = P^*(t,y^\epsilon) - P^*(t,y) - (\nabla P^*)(y^\epsilon - y)$$

and the (brute force) estimation of the related energy functional

$$e(t) = \int_B \left[\varepsilon^2 \frac{+|v^\epsilon(t,x) - v(t,x)|}{2} + \eta_{P^*}(t,y,v^\epsilon) \right] dx.$$

Recently extended to 3D periodic problem:

Consider a periodic box in $\mathbb{R}^3$. Let $(y^\epsilon, v^\epsilon, P^\epsilon)$ and $(y = \nabla P, v)$ be smooth local solutions to Euler and 3DSG respectively on a finite time interval $[0,T]$. Assume further that P has a smooth convex extension. Then the L^2 distance between y^ϵ and y stays uniformly of order ϵ as $\epsilon \to 0$, provided it does at $t = 0$ and the initial velocity $v^\epsilon(0,x)$ is uniformly bounded in L^2.

Exploiting Lagrangian trajectories in a clever way, the optimal order of approximation can be obtained (*work in progress*).

Conclusions - what mathematics has to offer

Relatively technical mathematical analysis can confirm rigorously the validity of models reduced on the basis of physical considerations.

- Existence of solutions - the model is a useful representation of reality
- Smooth or weak solutions? Reality is not very smooth.... (difference between QG and SG)
- Global(=long time) existence → predictability, on large scales, at least on certain (long) time intervals

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