Robust monitoring of CAPM portfolio betas

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Introduction

Joint works with O. Chochola, Z. Prášková and J. Steinebach (JMVA2013, just submitted)

- Change point problem
- On-line version, monitoring
- Particular regression models
- Multivariate data, functional data
- Dependent observations ($L_p$—approximibility, strong mixing)
- Robust procedures (M-procedures)
CAPM (Capital Assets Pricing Model)

\[ r_{tj} - r_{t,f} = \beta_j (r_{t,M} - r_{t,f}) + \varepsilon_{tj}, \quad j = 1, \ldots, d, \quad t = 1, \ldots, N \]

- \( r_{tj} \) - return of an asset \( j \) at time \( t \)
- \( r_{t,M} \) - return of the market portfolio at time \( t \)
- \( r_{t,f} \) - riskless security
- \( r_{tj} - r_{t,f} \) - access return (risk premium)
- \( \beta_j \) - measure of risk of the security \( j \) with respect to the market portfolio
CAPM (Capital Assets Pricing Model)

\[ r_{tj} - r_{t,f} = \beta_j (r_{t,M} - r_{t,f}) + \varepsilon_{tj}, \; j = 1, \ldots, d, \; t = 1, \ldots, N \]

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When \( r_{t,f} \) is unknown and the access return is not directly observable, we consider a reparametrized model

\[ r_{tj} = \alpha_j + \beta_j r_{t,M} + \varepsilon_{tj} \]
Model with time varying betas:

\[ r_{tj} = \alpha_{tj} + \beta_{tj} r_{t,M} + \varepsilon_{tj}, \ j = 1, \ldots, d, \ t = 1, \ldots, N \]

or more generally,

\[ r_i = \alpha_i + \beta_i r_{i,M} + \varepsilon_i, \ i = 1, \ldots, \]

\[ r_i, \alpha_i, \beta_i, \varepsilon_i \] are \( d \)-dimensional vectors
Model with time varying betas:

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**Sequential monitoring**

We assume stable historical data of size \( m \) such that

\[ \alpha_1 = \ldots = \alpha_m = \alpha_0, \ \beta_1 = \ldots = \beta_m = \beta_0, \]

and with any new observation we want to decide whether the stability in parameters is violated or not.

**Hypothesis testing problem:**

\( H_0 : \beta_1 = \ldots = \beta_0 \)

\( H_1 : \beta_0 = \beta_1 = \ldots = \beta_{m+k^*} \neq \beta_{m+k^*+1} = \ldots \)

where \( k^* = k_m^* \) is an unknown change point.
The null hypothesis is rejected whenever for the first time

\[ \hat{Q}(k, m)/q_\gamma(k/m) \geq c_\alpha \]

where \( \hat{Q} \) is a test statistic, \( q_\gamma(t), t \in (0, \infty) \) is a boundary function and \( c_\alpha \) is an appropriately chosen critical value.
The null hypothesis is rejected whenever for the first time

$$\hat{Q}(k, m)/q_\gamma(k/m) \geq c_\alpha$$

where $\hat{Q}$ is a test statistic, $q_\gamma(t)$, $t \in (0, \infty)$ is a boundary function and $c_\alpha$ is an appropriately chosen critical value.

The stopping rule:

$$\tau_m(\gamma) = \begin{cases} \inf\{1 \leq k < mT + 1 : \hat{Q}(k, m)/q_\gamma(k/m) \geq c_\alpha\}, \\ \infty, \text{ if } \hat{Q}(k, m)/q_\gamma(k/m) < c_\alpha \quad \forall \ 1 \leq k < mT + 1. \end{cases}$$

( closed-end procedure)
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\end{cases}$$

( closed-end procedure)

The constant $c = c_\alpha$ is chosen such that

$$\lim_{m \to \infty} P(\tau_m < \infty|H_0) = \alpha,$$

i.e. the asymptotic level is $\alpha$ and,

$$\lim_{m \to \infty} P(\tau_m < \infty|H_1) = 1,$$

the test is consistent.
Sequential monitoring in linear models
- Chu, Stinchcombe and White, 1996
- Horváth, Hušková, Kokoszka, Steinebach, 2004
- Zeleis, Leisch, Kleiber, Hornik, 2005
- Aue, Hörmann, Horváth, Hušková, and Steinebach, 2011 for CAPM
- many others (Gombay, Kirch,...)
- Koubková 2006 (monitoring, L1-norm)
Motivation for our research

Market portfolio: S&P500
Sequential monitoring in CAPM, with LSE, L1
Historical period m=400, monitoring started on August 6, 2011.

red - $\gamma = 0$, blue - $\gamma = 0.25$, black - $\gamma = 0.45$
solid line - asymptotic critical values, dashed - closed end
Boeing and GE, L1 estimators. No change
Sequential robust testing of stability in CAPM model

Boeing and GE, Huber. Change: 24.9 2001
Robust procedures in linear models:

- Wu (2007) robust estimators, dependent variables
- Koenker, Portnoy, 1990 - multivariate, independent
- Bai et al. 1990, 1992 - multivariate, independent
- Chan, Lakonischok (1992), Genton and Ronchetti (2008) - empirical studies in CAPM
Sequential robust monitoring in CAPM

\[ r_{ij} = \alpha_j^0 + \beta_j^0 \tilde{r}_{iM} + (\alpha_j^1 + \beta_j^1 \tilde{r}_{iM}) I\{i > m + k^*\} + \varepsilon_{ij}, \quad i = 1, 2, \ldots, \]

\( k^* \) - change point, \( \alpha_j^0, \beta_j^0, \alpha_j^1, \beta_j^1, j = 1, \ldots, d \) unknown parameters

\[
\tilde{r}_{iM} = (r_{i,M} - \bar{r}_M), \quad \bar{r}_M = \frac{1}{m} \sum_{i=1}^{m} r_{i,M}.
\]

\( M \)-estimators \( \hat{\alpha}_{jm}, \hat{\beta}_{jm} \) of \( \alpha_j^0, \beta_j^0 \), based on the training sample:

\[
\min_{a_j, b_j} \sum_{i=1}^{m} \rho_j(r_{ij} - a_j - b_j \tilde{r}_{iM})
\]

\( \rho_j \) are convex loss functions with the derivatives \( \psi_j, j = 1, \ldots, d \)

\( M \)-residuals

\[
\psi(\hat{\varepsilon}_i) = (\psi_1(\hat{\varepsilon}_i1), \ldots, \psi_d(\hat{\varepsilon}_id))^T
\]

with

\[
\hat{\varepsilon}_{ij} = r_{ij} - \hat{\alpha}_{jm}(\psi) - \tilde{r}_{iM} \hat{\beta}_{jm}(\psi)
\]
A test statistic based on first $m + k$ observations

$$
\hat{Q}(k, m) = \left( \frac{1}{\sqrt{m}} \sum_{i=m+1}^{m+k} \tilde{r}_{iM} \psi(\hat{\varepsilon}_i) \right)^T \Sigma_m^{-1} \left( \frac{1}{\sqrt{m}} \sum_{i=m+1}^{m+k} \tilde{r}_{iM} \psi(\hat{\varepsilon}_i) \right),
$$

where the matrix $\Sigma_m$ is an estimator of the asymptotic variance

$$
\lim_{m \to \infty} \text{var}\left\{ \frac{1}{\sqrt{m}} \sum_{i=1}^{m} \left( r_{i,M} - Er_{i,M} \right) \psi(\varepsilon_i) \right\}
$$

based on the first $m$ observations, $\tilde{r}_{iM} = r_{i,M} - \bar{r}_M$
Typical score functions $\psi(x) = \rho'(x)$

- $\psi(x) = x$, $x \in R^1$ ($\rho(x) = x^2$) - least squares estimators and $L_2$ residuals
- $\psi(x) = \text{sign } x$, $x \in R^1$ ($\rho(x) = |x|$) - $L_1$ estimators and $L_1$ residuals
- Huber (1981)

$$
\psi(x) = \begin{cases} 
|x| & |x| \leq K \\
K \text{sign } x & |x| > K 
\end{cases}
$$

for $x \in R^1$ and some $K > 0$
Assumptions on score functions $\psi_j$’s

1. $\psi_j$ are monotone (nondecreasing) functions,

2. functions $\lambda_j(t) = -\int \psi_j(x - t)dF_j(x)$, $t \in \mathbb{R}$, have the properties $\lambda_j(0) = 0$, $\lambda'_j(0) > 0$, $\lambda'_j(t)$ exists in a neighborhood of 0 and is Lipschitz in neighborhood of 0 for $|t| \leq c_o$ for some $c_o > 0$.

3. $\int |\psi_j(t)|^{2+\Delta}dF_j(t) < \infty$ for some $\Delta > 0$ and

\[
\int |\psi_j(x + t_2) - \psi_j(x + t_1)|^2dF_j(x) \leq C_0 |t_2 - t_1|^{\kappa},
\]

for some $1 \leq \kappa \leq 2$, $c_o > 0$, $C_0 > 0$

$F_j$ distribution function of $\varepsilon_{ij}$
Assumptions on regressors

- For any \( i \in \mathbb{Z} \), \( r_{i,M} = h(\xi_i, \xi_{i-1}, \ldots) \), where \( h \) is a measurable, \( \{\xi_i\}_i \) is a sequence of i.i.d. random vectors and \( E|r_{0M}|^{2+\Delta} < \infty \) for some \( \Delta > 0 \).

- For all \( i \in \mathbb{Z} \),
  \[
  \sum_{L=1}^{\infty} \| r_{iM} - r^{(L)}_{iM} \|_2 < \infty
  \]
  where
  \[
  r^{(L)}_{iM} = h(\xi_i, \xi_{i-1}, \ldots \xi_{i-L+1}, \xi_i^{(L)}, \xi_{i-L}^{(L)}, \ldots),
  \]
  \( \xi_i^{(L)}, \xi_{i-L}^{(L)}, \ldots \) are i.i.d. with the same distribution as \( \xi_i \) independent of \( \{\xi_i\}_i \).

\( \{r_{i,M}\}, \{r^{(L)}_{i,M}\} \) are strictly stationary and ergodic
\( \{r^{(L)}_{i,M}\} \) is \( L \)-dependent, \( r^{(L)}_{i,M} \xrightarrow{D} r_{iM} \forall i \in \mathbb{Z} \).
Assumptions on errors

- For any $i \in \mathbb{Z}$, $\varepsilon_i = g(\zeta_i, \zeta_{i-1}, \ldots)$, where $g$ is a measurable, 
  $\{\zeta_i\}_i$ is sequence of i.i.d. random vectors
- For all $i \in \mathbb{Z}$,
  \[
  \sum_{L=1}^{\infty} \| \psi(\varepsilon_i) - \psi(\varepsilon_i^{(L)}) \|_2 < \infty
  \]
  \[
  \sum_{L=1}^{\infty} \sup_{|a| \leq a_0} \| \psi(\varepsilon_i - a) - \psi(\varepsilon_i^{(L)} - a) \|_2 < \infty
  \]
  for some $a_0 > 0$, where
  \[
  \varepsilon_i^{(L)} = g(\zeta_i, \zeta_{i-1}, \ldots \zeta_{i-L+1}, \zeta_i^{(L)}, \zeta_i^{(L)}, \zeta_i^{(L)}, \ldots)
  \]
  $\zeta_i^{(L)}, \zeta_i^{(L)}, \zeta_i^{(L)}, \ldots$ are i.i.d., independent of $\{\zeta_i\}_i$, with the same distribution as $\zeta_i$. 
Asymptotic results

Model under the null hypothesis

**Theorem.** Let the above assumptions be satisfied and

\[ \Sigma_m - \Sigma = o_P(1) \text{ as } m \to \infty. \]

Then under the null hypothesis as \( m \to \infty \)

\[
\max_{1 \leq k \leq mT} \frac{\tilde{Q}(k, m)}{q_\gamma^2(k/m)} \xrightarrow{D} \sup_{0 < t < T/(T+1)} \frac{\sum_{j=1}^{d} W_j^2(t)}{t^{2\gamma}},
\]

where \( \{W_j(t), \ t \in (0, 1)\}, j = 1, \ldots, d \) are independent Brownian motions and

\[ q_\gamma(t) = (1 + t) (t/(t + 1))^\gamma, \ t \in (0, \infty), \ \gamma \in [0, 1/2) \]
Main steps of the proof:

1. Asymptotic representation of M-estimators:

\[
\sqrt{m}(\hat{\alpha}_{mj} - \alpha_j^0) = \frac{1}{\sqrt{m} \lambda_j'(0)} \sum_{i=1}^{m} \psi_j(\varepsilon_{ij}) + o_P(m^{-\eta})
\]

\[
\sqrt{m}(\hat{\beta}_{mj} - \beta_j^0) = \frac{\sqrt{m}}{\lambda_j'(0)} \frac{1}{\sum_{i=1}^{m} \tilde{r}_{iM}^2} \sum_{i=1}^{m} \psi_j(\varepsilon_{ij})\tilde{r}_{iM} + o_P(m^{-\eta})
\]

for some \( \eta > 0 \)
2. The limit behaviour of the weighted partial sums

\[ \hat{H}(m, k) = \frac{1}{\sqrt{m}} \sum_{i=m+1}^{m+k} \tilde{r}_i M \psi(\hat{\varepsilon}_i), \quad k = 1, \ldots, \lfloor mT \rfloor \]

is the same as the limit behaviour of the weighted partial sums

\[ H(m, k) = \frac{1}{\sqrt{m}} \left( \sum_{i=m+1}^{m+k} (r_i M - Er_i M) \psi(\varepsilon_i) \right) - \frac{\sum_{v=1+m}^{m+k} (r_i M - Er_i M)^2}{\sum_{v=1}^{m+k} (r_i M - Er_i M)^2} \sum_{i=1}^{m} (r_i M - Er_i M) \psi(\varepsilon_i), \quad k = 1, \ldots, \lfloor mT \rfloor. \]
3. For

\[ Z_m(t) = \frac{1}{\sqrt{m}} \sum_{i=1}^{\lfloor mt \rfloor} (r_{iM} - E r_{iM}) \psi(\varepsilon_i), \quad 0 \leq t \leq T + 1, \]

\{Z_m(t), 0 \leq t \leq T + 1\} \rightarrow \{W_{\Sigma}(t), 0 \leq t \leq T + 1\}

weakly in the Skorokhod space \( \mathcal{D}^d[0, T + 1] \)

\{W_{\Sigma}(t), 0 \leq t \leq T + 1\} is a centered Gaussian process with covariance function

\[ EW_{\Sigma}(t)W_{\Sigma}^T(s) = \Sigma \cdot \min(t, s) \]
Critical values $c_\alpha$ satisfies

$$P\left( \sup_{0 < t < T/(T+1)} \frac{\sum_{j=1}^d W_j^2(t)}{t^{2\gamma}} \geq c_\alpha \right) = \alpha. \tag{1}$$

The explicit form of the limit distribution is unknown.

Model under local alternatives:

$$r_{ij} = \alpha_j^0 + \beta_j^0 \tilde{r}_{iM} + (\alpha_j^1 + \beta_j^1 \tilde{r}_{iM}) \delta_m I\{i > m + k^*\} + \varepsilon_{ij}$$

$\delta_m \to 0$ and $k^* < Tm + 1$

Theorem (consistency): When $\delta_m \to 0$, $|\delta_m|m^{1/2} \to \infty$, $\beta_j^1 \neq 0$ for at least one $j$ and $k^* = \lfloor ms \rfloor$, $0 < s < T$, as $m \to \infty$

$$\max_{1 \leq k \leq mT} \frac{\hat{Q}(k, m)}{q_\gamma(k/m)} \to \infty, \quad \text{in probability.}$$
Estimator of asymptotic variance matrix $\Sigma$

$$\Sigma = \sum_{i=\infty}^{\infty} E[(r_{0M} - Er_{0M})(r_{iM} - Er_{iM})\psi(\varepsilon_0)\psi(\varepsilon_i)^T]$$

Bartlett-type estimator

$$\Sigma_m = \sum_{|k|\leq q} \omega_q(k)\hat{\Gamma}_k$$

$$\hat{\Gamma}_k = \begin{cases} 
\frac{1}{m} \sum_{j=1}^{m-k} \tilde{r}_jM\tilde{r}_{j+k},M\psi(\hat{\varepsilon}_j)\psi(\hat{\varepsilon}_{j+k})^T & k \geq 0 \\
\hat{\Gamma}_{|k|}^T & k < 0 
\end{cases}$$

$$\omega_q(x) = (1 - \frac{|x|}{q}) I\{|x| \leq q\}$$

$q(m) \to \infty$ as $m \to \infty$, $q(m)/m \to 0$
Numerical results

Simulations

- Critical values computed from simulated limit distribution
- Empirical quantiles of sample values of test statistic by Monte Carlo method
- Empirical quantiles of sample values of test statistic generated by pair bootstrap from historical period
- Empirical level of test
<table>
<thead>
<tr>
<th>$T = 10$</th>
<th>Huber</th>
<th>L1</th>
</tr>
</thead>
</table>

$rM \sim AR(1/4)$


$rM \sim AR(1/2)$


e $\sim AR(1/4)$


e $\sim AR(1/2)$

| Boot | 6.724 | 7.867 | 11.600 | 6.631 | 7.735 | 10.620 |

e $\sim AR(1/4), rM \sim AR(1/4)$

| MC | 7.104 | 8.088 | 10.508 | 6.777 | 7.813 | 10.216 |

Empirical critical values, $\alpha = 0.05$, $T=10$. First row: 2000 Monte Carlo simulations, Second row: 500 pair bootstrap samples and 250 replicas
<table>
<thead>
<tr>
<th>L2</th>
<th>Huber</th>
<th>L1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$e \sim N_2$</td>
<td></td>
</tr>
<tr>
<td>0.0735</td>
<td>0.0810</td>
<td>0.0690</td>
</tr>
<tr>
<td>0.0470</td>
<td>0.0585</td>
<td>0.0495</td>
</tr>
<tr>
<td>0.0430</td>
<td>0.0500</td>
<td>0.046</td>
</tr>
<tr>
<td>$r_m \sim AR(1/4)$</td>
<td></td>
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<tr>
<td>0.0645</td>
<td>0.0775</td>
<td>0.0700</td>
</tr>
<tr>
<td>0.0545</td>
<td>0.0640</td>
<td>0.0565</td>
</tr>
<tr>
<td>0.0510</td>
<td>0.0625</td>
<td>0.0580</td>
</tr>
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<tr>
<td>0.0820</td>
<td>0.0960</td>
<td>0.086</td>
</tr>
<tr>
<td>0.0605</td>
<td>0.0710</td>
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</tr>
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<td>0.0515</td>
</tr>
</tbody>
</table>

Empirical levels of the test, $T = 10$, nominal level $\alpha = 0.05$.
Row blocks: $m = 100, 200, 500$, columns: $\gamma = 0, 0.25, 0.49$, various types of dependence, 2000 simulations.
Conclusions from simulations:

- Test based on L2 rejects $H_0$ too often, many false alarms
- Test based on L1 is too conservative
- Huber function can be recommended
- Bootstrap critical values can be used, very time consuming
- Centering with the mean over the monitoring period gives much better results than that over the historical period
Application to real data

Data: not traded indices computed from sector data of global world economy, serve as benchmarks for investors:

- World Consumer Staples (NDWUCSTA)- food, beverages, tobacco, prescription drugs, households necessities
- World Financials (NDWUFNCL)
- World Health Care (NDWUHC)
- Market portfolio: MSCI World Daily Index (NDDUWI)
- Monitoring period: 1.12.2006-1.10. 2010
- Huber function
Sequential robust testing of stability in CAPM model

Financials (6), Health Care (7)
Sequential robust testing of stability in CAPM model

Index: 4, 6  
Type: 3

Consumer Staples (4), Financials (6)
Index: 4, 7    Type: 3

Sequential robust testing of stability in CAPM model

Consumer Staples (4), Health Care (7)
Remarks

- Similar assertions were proved for $\alpha$-mixing instead of $L_p - m$ mixing (Chochola)

- Analogous results proved for off-line robust procedure for general regression models (Chochola, Prášková)

- Further possibility to extend, e.g. more general loss function (technically quite complex). Also to functional setup- shortly it will be discussed next.

- $\psi_\beta(x) = \beta I\{x > 0\} - (1 - \beta) I\{x \leq 0\} - \beta$-quantile, $\beta \in (0, 1)$

- score function related likelihood ratio
Robust monitoring for CAPM for high-frequency portfolio betas

\[ r_i(s) = \alpha_i + \beta_i r_{i,M}(s) + e_i(s), \quad i \in \mathbb{Z}, \quad s \in [0,1], \]

\[ r_i(s) = (r_{i,1}(s), \ldots, r_{i,d}(s))^T - d\text{-dimensional vector of (functional) log-returns at (say) “day” } i \text{ and “intra-day time” } s, \]

\[ r_{i,M}(s) - \text{the log-return of the market portfolio at day } i \text{ and time } s, \]

\[ e_i(s) = (e_{i,1}(s), \ldots, e_{i,d}(s))^T - d\text{-dimensional (functional) error terms} \]

\( \alpha_i \)'s and \( \beta_i \)'s are \( d \)-dimensional unknown parameters

\( \beta_i \)'s are the parameters of interest, usually called the “portfolio betas”
We assume a training sample of size $m$ with no instabilities is available, i.e.,

$$\alpha_1 = \ldots = \alpha_m =: \alpha_0 = (\alpha^0_1, \ldots, \alpha^0_d)^T,$$
$$\beta_1 = \ldots = \beta_m =: \beta_0 = (\beta^0_1, \ldots, \beta^0_d)^T,$$

$\alpha_0$ and $\beta_0$ – unknown parameters

Null hypothesis

$$H_0 : \beta_1 = \ldots = \beta_m = \beta_{m+1} = \ldots$$

of no “change versus” the alternative

$$H_A : \beta_1 = \ldots = \beta_{m+k^*} \neq \beta_{m+k^*+1} = \ldots$$

a “structural break” at an unknown change-point $k^* = k^*_m$.

$$r_{i,j}(s) = \alpha^0_j + \beta^0_j r_{i,M}(s) + (\alpha^1_j + \beta^1_j r_{i,M}(s))\delta_m I\{i > m+k^*\} + e_{i,j}(s), \ j = 1, \ldots, d$$  \hspace{1cm} (1)

Define

\[ \psi(\hat{e}_i(s_\nu)) = (\psi_1(\hat{e}_{i,1}(s_\nu)), \ldots, \psi_d(\hat{e}_{i,d}(s_\nu)))^T \]

\[ \hat{e}_i(s_\nu) = (\hat{e}_{i,1}(s_\nu), \ldots, \hat{e}_{i,d}(s_\nu))^T, \]

\[ \hat{e}_{i,j}(s_\nu) = r_{i,j}(s_\nu) - \hat{\alpha}_j m - \hat{\beta}_j m r_{i,M}(s_\nu). \]

\[ s_\nu = \nu/n, \nu = 1, \ldots, n, \ n = n(m) \]

test statistic based on the first \( m + k \) (functional) observations:

\[ \hat{Q}(k, m) = \left( \frac{1}{\sqrt{m}} \sum_{i=m+1}^{m+k} \frac{1}{n} \sum_{\nu=1}^{n} r_{i,M}(s_\nu) \psi(\hat{e}_i(s_\nu)) \right)^T \left( \hat{\Sigma}_m \right)^{-1} \]

\[ \left( \frac{1}{\sqrt{m}} \sum_{i=m+1}^{m+k} \frac{1}{n} \sum_{\nu=1}^{n} r_{i,M}(s_\nu) \psi(\hat{e}_i(s_\nu)) \right) \]
\( \hat{\Sigma}_m \) is an estimator of the asymptotic variance (matrix)

\[
\Sigma = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \int_{0}^{1} r_{i,M}(s) \psi(e_i(s)) ds
\]

based on the first \( m \) observations.

The null hypothesis is rejected if

\[
\hat{Q}(k, m)/q_{\gamma}(k/m) \geq c
\]

for properly chosen \( c \).

For a vector-valued random variable \( \mathbf{X} \) define

\[
\|\mathbf{X}\|_p = \left( \mathbb{E}|\mathbf{X}|^p \right)^{1/p}, \quad p \geq 1,
\]

the \( L_p \)-norm of \( \mathbf{X} \), where \( |\mathbf{X}| \) denotes the Euclidean norm of \( \mathbf{X} \).
Assumptions

(B.1) For any $i \in \mathbb{Z}$, $r_{i,M}(\cdot) = h(\xi_i(\cdot), \xi_{i-1}(\cdot), \ldots)$, where $h(\cdot)$ is a measurable function, $\{\xi_i(\cdot)\}$ is a sequence of i.i.d. random functions, and $\sup_{s \in [0,1]} E|r_{i,M}(s)|^3 < \infty$.

(B.2) For any $i \in \mathbb{Z}$, $e_i(\cdot) = g(\zeta_i(\cdot), \zeta_{i-1}(\cdot), \ldots)$, where $g(\cdot)$ is a measurable function, $\{\zeta_i(\cdot)\}$ is a sequence of i.i.d. random functions having some further properties to be specified later.

(B.3) The sequences $\{\xi_i(\cdot)\}$ and $\{\zeta_i(\cdot)\}$ are independent.
(B.4) For all $i \in \mathbb{Z}$,

$$\sup_{s \in [0,1]} \sum_{L=1}^{\infty} \| r_{i,M}(s) - r_{iM}^{(L)}(s) \|_2 < \infty,$$

$$r_{iM}^{(L)}(\cdot) = h(\xi_i(\cdot), \xi_{i-1}(\cdot), \ldots, \xi_{i-L+1}(\cdot), \xi_{i-L}(\cdot), \xi_{i-L-1}(\cdot), \ldots),$$

with $\xi_{i-L}(\cdot), \xi_{i-L-1}(\cdot), \ldots$ being i.i.d. with the same distribution as $\xi_0(\cdot)$ and independent of $\{\xi_i(\cdot)\}$.

(B.5) With $\psi(e_i(\cdot)) = (\psi_1(e_{i,1}(\cdot)), \ldots, \psi_d(e_{i,d}(\cdot)))^T$, for all $i \in \mathbb{Z}$, it holds that

$$\sup_{s \in [0,1]} \sup_{|a| \leq a_0} \sum_{L=1}^{\infty} \| \psi(e_i(s) - a) - \psi(e_i^{(L)}(s) - a) \|_2 < \infty$$

for some $a_0 > 0$, where

$$e_i^{(L)}(\cdot) = g(\zeta_i(\cdot), \zeta_{i-1}(\cdot), \ldots, \zeta_{i-L+1}(\cdot), \zeta_{i-L}(\cdot), \zeta_{i-L-1}(\cdot), \ldots),$$

with $\zeta_{i-L}(\cdot), \zeta_{i-L-1}(\cdot), \ldots$ being i.i.d. with the same distribution as $\zeta_0(\cdot)$ and independent of $\{\zeta_i(\cdot)\}$. 
(B.6) We let \( n = n(m) \rightarrow \infty \) as \( m \rightarrow \infty \).

(B.7) For all \( i \in \mathbb{Z}, j = 1, \ldots, d, \) with \( s_\nu = \nu/n \) as above and \( n = n(m) \rightarrow \infty \),

\[
\lim_m (\log m) \frac{1}{n} \sum_{\nu=1}^{n} \sup_{h \in [0, 1/n]} \| r_{i,M}(s_\nu) - r_{i,M}(s_\nu - h) \|_2 = 0
\]

and

\[
\lim_m (\log m) \frac{1}{n} \sum_{\nu=1}^{n} \sup_{h \in [0, 1/n]} \| \psi_j(e_{i,j}(s_\nu)) - \psi_j(e_{i,j}(s_\nu - h)) \|_2 = 0.
\]
### Application

**Sectors:** Boeing (BA), Bank of America (BAC), Microsoft (MSFT), AT&T (T), and Exxon Mobile (XOM)

Market portfolio, the S&P 100 index itself

The intra-day behavior of the process \( \{r_i(s) : s \in [0; 1]; i \in \mathbb{Z}\} \) is defined at time \( s \) as the difference between the log-prices of the stocks at time \( s \) and \( s + 15 \) min, is thus sampled every 15 minutes during any trading day \( i \).

The process \( r_{iM}(\cdot) \) is defined analogously.

Historical training period January 29, 2001 and consists of 120 trading days (the portfolio betas appear reasonably stable).

The monitoring horizon for the closed-end procedure was selected as 360 days, corresponding to \( T = 3 \). This covers the 9/11/2001 event.
Figure 1 shows $L_2$ estimates of portfolio betas based on moving windows of 10 trading days for each company throughout the historical and monitoring periods. The solid black vertical line marks the end of the historical period (120 days), whereas the dashed black line marks the last day, when the estimate is not influenced by the observations from the monitoring period. The grey lines refer in the same way to the 9/11 event.

The BAC and T estimates seem to be stable throughout the whole period, whereas there is a small temporary influence of the 9/11 event on MSFT and a very big one on BA. Finally there seems to be a shift in the portfolio beta of XOM right after the end of the training period. We come back to these observations later on.

Figure 2 shows $\hat{Q}(k, m)/(c_{0.25}(0.05) q_{\gamma}(k/m))$, for the $L_2$ (dashed line), Huber (solid line) and $L_1$ (dotted line).
Figure: $L_2$ estimates of portfolio beta based on moving windows of 10
Robust monitoring of CAPM

Introduction

Figure: Normalized test statistics for the $\ell_2$ (dashed line), Huber (solid line) and $\ell_1$ (dotted line) monitoring procedures, various combinations of stocks - given in the heading of each chart. x-axis shows number of trading days from the beginning of the monitoring on.
Figure: Boeing stock, normalized test statistics for $L_2$ (dashed line), Huber (solid line) and $L_1$ (dotted line) monitoring procedures. 5 or 10 days excluded from the monitoring after the 9/11.
THANK YOU !!!