Volterra Integral Equations of the First Kind with Jump Discontinuous Kernels

Denis Sidorov

Energy Systems Institute, Russian Academy of Sciences
e-mail: contact.dns@gmail.com

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Problem Statement

VIEq with piecewise continuous kernel

\[ \int_{0}^{t} K(t, s)x(s)ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0, \]  
\[ K(t, s) = \begin{cases} 
K_1(t, s), & t, s \in m_1 \\
\ldots \ldots \ldots \ldots \\
K_n(t, s), & t, s \in m_n 
\end{cases} \]

\[ m_i = \{ t, s \mid \alpha_{i-1}(t) < s < \alpha_i(t) \}, \]
\[ \alpha_0(t) = 0, \quad \alpha_n(t) = t, \quad i = 1, n \]

\[ \alpha_i(t), \ f(t) \in C^1_{[0, T]}, \ K_i(t, s) \] have continuous derivatives w.r.t. \( t \) for \( t, s \in m_i \),

\[ K_n(t, t) \neq 0, \quad \alpha_{i}(0) = 0, \quad 0 < \alpha_1(t) < \alpha_2(t) < \cdots < \alpha_{n-1}(t) < t, \]

\( \alpha_{1}(t), \ldots, \alpha_{n-1}(t) \) increase at least in the small neighborhood \( 0 \leq t \leq \tau \),

\[ 0 < \alpha'_{1}(0) \leq \cdots \leq \alpha'_{n-1}(0) < 1 \]
Problem Statement

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VIEq with piecewise continuous kernel

\[ \int_{0}^{t} K(t, s) x(s) ds = f(t), \quad 0 \leq s \leq t \leq T, \quad f(0) = 0, \quad (1) \]

\[ K(t, s) = \begin{cases} 
K_1(t, s), & t, s \in m_1 \\
\ldots \ldots \ldots \\
K_n(t, s), & t, s \in m_n 
\end{cases} \quad m_i = \{ t, s \mid \alpha_{i-1}(t) < s < \alpha_i(t) \}, \]

\[ \alpha_i(t), \quad f(t) \in C_{[0, T]}^1, \quad K_i(t, s) \text{ have continuous derivatives w.r.t. } t \text{ for } t, s \in m_i, \]

\[ K_n(t, t) \neq 0, \quad \alpha_i(0) = 0, \quad 0 < \alpha_1(t) < \alpha_2(t) < \ldots < \alpha_{n-1}(t) < t, \]

\[ \alpha_1(t), \ldots, \alpha_{n-1}(t) \text{ increase at least in the small neighborhood } 0 \leq t \leq \tau, \]

\[ 0 < \alpha'_1(0) \leq \ldots \leq \alpha'_{n-1}(0) < 1 \]
Objectives & Methods

Objective
Our objective is to construct the solution $x(t) \in C_{(0, T]}$

Applications
Mathematical models of evolving dynamical systems: vintage capital models, optimal replacement of equipment under technological change, rational harvesting of biological populations.

Methods
We employ the theory of functional equations\(^a\), power-logarithmic asymptotic expansions, the method of steps from delay ODE theory and conventional successive approximations method. For numerical solution the quadrature methods are used.

\(^a\)Gelfond A.O. *The Calculus of Finite Differences*. 5th Edt, Moscow: URSS Publ., 2012
Previous Results

\[ \int_0^T K(t, s)x(s) \, ds = f(t), \quad 0 \leq s \leq t \leq T \]


New Problem Statement

\[
\begin{align*}
\alpha_1(t) & \quad \int_0^{\alpha_1(t)} K_1(t, s)x(s) \, ds + \int_{\alpha_1(t)}^{\alpha_2(t)} K_2(t, s)x(s) \, ds + \cdots + \\
& \quad \int_{\alpha_{n-1}(t)}^t K_n(t, s)x(s) \, ds = f(t)
\end{align*}
\]


Example 1

\[
K(t, s) = \begin{cases} 
1, & 0 \leq s < t/2 \\
-1, & t/2 \leq s \leq t
\end{cases}
\]

\[
\int_0^{t/2} x(s) ds - \int_{t/2}^{t} x(s) ds = t
\]

\[
x\left(\frac{t}{2}\right) - x(t) = 1, \quad x(t) = c - \frac{\ln t}{\ln 2}
\]
Existence and Uniqueness of the Local Solution

\[ F(x) \overset{\text{def}}{=} K_n(t, t)x(t) + \sum_{i=1}^{n-1} \alpha'_i(t) \left\{ K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t)) \right\} x(\alpha_i(t)) + \]

\[ + \sum_{i=1}^{n} \frac{\alpha_i(t)}{\alpha_{i-1}(t)} \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} \frac{\partial K_i(t, s)}{\partial t} x(s) ds - f'(t) = 0 \] (2)
Reduction to the VIE of the 2nd kind

\[ x(t) + Ax + Qx = \hat{f}(t) \]  

\( A \) is the functional perturbation operator:

\[ Ax \overset{\text{def}}{=} K_n^{-1}(t, t) \sum_{i=1}^{n-1} \alpha'_i(t) \left\{ K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t)) \right\} x(\alpha_i(t)) \]

\( B \) is the Volterra operator:

\[ Qx := \int_0^t Q(t, s)x(s)ds \overset{\text{def}}{=} \sum_{i=1}^n \alpha_i(t) \int_{\alpha_{i-1}(t)}^{\alpha_i(t)} K_n^{-1}(t, s) \frac{\partial K_i(t, s)}{\partial t} x(s)ds \]

\[ \hat{f}(t) \overset{\text{def}}{=} K_n^{-1}(t, t)f'(t), \quad K_n(t, t) \neq 0 \]

**Objective:** To obtain the sufficient conditions for existence of local solution, i.e. \( \| A + Q \| \leq q, \quad q < 1, \quad t \in (0, \tau] \)
\[ D(t) \overset{\text{def}}{=} \sum_{i=1}^{n-1} |\alpha_i'(t)K_n^{-1}(t, t)| \cdot |K_i(t, \alpha_i(t)) - K_{i+1}(t, \alpha_i(t))| \]

The idea is to use an equivalent norm \( \|x\|_l := \sup_{0<t<\tau} e^{-lt}|x(t)| \), so that the Volterra integral operator \( Q \) becomes contractive.

\[ \|Qx\|_l \leq q(l)\|x\|_l, \quad \lim_{l \to \infty} q(l) = 0, \]

If \( D(0) < 1 \), then \( \forall q_1 < 1 \ \exists \tau > 0 \ D(t) \leq q_1, \ t \in [0, \tau] \).

Then \( \|Ax\|_l \leq q_1\|x\|_l \)

\[ \Rightarrow \|Ax\|_l + \|Qx\|_l \leq (q_1 + q(l))\|x\|_l \leq q\|x\|_l, \ 0 < q < q_1 < 1 \quad \forall l \geq l(q), \ t \in [0, \tau] \]
Existence & Uniqueness of Local Solution

**Theorem 1:**

**Sufficient Conditions of Existence & Uniqueness of Local Solution**

Let for $t \in [0, T]$ the following conditions be fulfilled: continuous $K_i(t, s), i = 1, n$, $\alpha_i(t)$ and $f(t)$ have continuous derivatives wrt $t$, $K_n(t, t) \neq 0$, $0 = \alpha_0(t) < \alpha_1(t) < \cdots < \alpha_{n-1}(t) < \alpha_n(t) = t$ for $t \in (0, T]$, $\alpha_i(0) = 0$, $f(0) = 0$, $D(0) < 1$, then $\exists \tau > 0$ such as eq. (1) has a unique local solution in $C_{[0, \tau]}$.

back to example 1:

$$\int_0^{t/2} x(s) ds - \int_{t/2}^t x(s) ds = t, \quad D(t) \equiv 1 \Rightarrow \text{no unique solution!}$$
Theorem 2:
Sufficient Conditions of Existence & Uniqueness of the Global Solution*

Let the conditions of the Theorem 1 are fulfilled, and moreover let
\[
\min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0.
\]
Then eq. (1) has unique global solution in \( C_{[0,T]} \).

*D.N.Sidorov, E.V.Markova. One the Volterra integral model of evolving
\[
\min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0
\]
Lemma 1

Lemma

Let $\alpha : [0, T] \to \mathbb{R}^+$, $\alpha(t) \in C^1_{[0, T]}$, $\alpha(0) = 0$, $0 \leq \alpha^{(1)}(0) < 1$. Let $h \in (0, T)$, $l_0 := [0, h]$, $\cdots$, $l_k := [(1 + (k-1)\varepsilon)h, (1 + k\varepsilon)h]$, $\varepsilon > 0$, $[0, T] = \bigcup_{k=0}^{m} l_k$. If $0 < \alpha(t) < t$, $\alpha^{(1)}(t) \leq \frac{1}{1+\varepsilon}$ for $0 \leq t \leq T$ then

$$\alpha : l_j \to \bigcup_{k=0}^{j-1} l_k, j = 1, m.$$  \hspace{1cm} (4)
Proof of the Lemma 1

**Idea:** proof by induction and mean value theorem.

Since \( \alpha : l_j \to \bigcup_{k=0}^{j-1} l_k \iff \alpha(t) \leq (1 + (j - 1)\varepsilon)h \ \forall t \in l_j \), then it's enough to proof by induction that

\[
\alpha(t) \mid_{t \in l_j} \leq (1 + (j - 1)\varepsilon)h, \ j = 1, m. \tag{5}
\]

\( j = 1 \): Let \( a \in l_1, \ b = l_1 \cap l_0 = h \), then \( a - b \leq \varepsilon h \). \( \exists \xi \in [0, h], \exists \xi_1 \in [h, a] : \alpha(a) = \alpha(b) + \alpha'(\xi)(a - b) = \alpha(0) + \alpha'(\xi_1)h + \alpha'(\xi)(a - b) \leq \frac{1}{1+\varepsilon}h + \frac{1}{1+\varepsilon}\varepsilon h = h. \)
Proof of the Lemma 1

Let (5) be fulfilled for $j = 2, \ldots, m - 1$.

Then $\forall a \in I_m$ and $b = I_m \cap I_{m-1} = (1 + (m - 1)\varepsilon)h$:

$$
\alpha(a) = \alpha(b) + \alpha'(\xi)(a - b) \leq (1 + (m - 2)\varepsilon)h + \frac{1}{1 + \varepsilon} \varepsilon h
$$

$$
1 + (m - 2)\varepsilon + \frac{\varepsilon}{1 + \varepsilon} < 1 + (m - 1)\varepsilon
$$

Therefore $\forall a \in I_m : \alpha(a) \leq (1 + (m - 1)\varepsilon)h$.  ■
Proof of the Th. 2

The local solution exists on $[0, \tau]$ due to conditions of the Theorem 1. Let $\Delta > 0$, $l_0 = [0, \tau], \ldots, l_k = [\tau + (k - 1)\Delta, \tau + k\Delta], k = 1, N,$ $[0, T] \subset \bigcup_{k=0}^{N} I_k$. $\alpha_i(t) : [0, \tau] \rightarrow [0, \tau_i'], \tau_i' < \tau, i = 1, n - 1$. For $t \in [\tau, T]$ due to the Lemma’s conditions we have $\alpha_i(t) \leq t - h$. It allows us to select $\Delta \leq h$, such as

$$\alpha_i(t) : l_k \rightarrow \bigcup_{j=0}^{k-1} l_j.$$  \hfill (6)
Proof of the Th. 2

Inclusion (6) allows us to extend the local solution $x_0(t)$ onto the whole $[\tau, T]$ with step $\Delta$ using the method of steps. Indeed, let us construct the desired continuation of the solution on section $I_1$. For that objective we solve the VIE of the 2nd kind

$$x(t) + \int_{\tau}^{t} Q(t, s)x(s)ds = -Ax_0 - \int_{0}^{\tau} Q(t, s)x_0(s)ds + \hat{f}(t), \quad (7)$$

$$t \in [\tau, \tau + \Delta] = I_1,$$

where in the right hand side we already have the known $x_0(t)$. 

Denis N. Sidorov contact.dns@gmail.com

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Proof of the Th. 2

Continuous function $x_1(t)$, which satisfy eq. (7) for $t \in I_1$ is continuous extension of the solution $x_0(t)$ onto $[\tau, \tau + \Delta]$. Let us fix $q \in (0, 1)$, let

$$\sup_{s,t} |Q(t, s)| = c < \infty \text{ and } \Delta \leq \min\left(h, \frac{q}{c}\right).$$

Then operator $\int_{\tau}^{t} Q(t, s)x(s)ds$ in $C[\tau, \tau + \Delta]$ will be contracting, and $q$ is contraction coefficient. Hence eq. (7) has the unique solution $x_1(t) \in C[\tau, \tau + \Delta]$.

Therefore the function

$$\hat{x}_1 = \begin{cases} x_0(t), & 0 < t \leq \tau, \\ x_1(t), & \tau \leq t < \tau + \Delta, \\ 
\end{cases}$$

is constructed and it continuously extends the solution on $[\tau, \tau + \Delta]$. 
Proof of the Th. 2

The next continuation \( x_2(t) \in C_{[\tau+\Delta, \tau+2\Delta]} \). Now we have to solve the VIEq with known \( \hat{x}_1 \):

\[
x(t) + \int_{\tau+\Delta}^{t} Q(t,s)x(s)ds = -A\hat{x}_1 - \int_{0}^{\tau+\Delta} Q(t,s)\hat{x}_1(s)ds, \quad t \in [\tau+\Delta, \tau+2\Delta].
\]

Finally, we will construct the function

\[
\hat{x}_2(t) = \begin{cases} 
    x_0(t), & 0 < t \leq \tau, \\
    x_1(t), & \tau \leq t \leq \tau + \Delta, \\
    x_2(t), & \tau + \Delta \leq t \leq \tau + 2\Delta.
\end{cases}
\]

We can continue this process and within finite number of steps on \((0, T]\) the desired \( x(t) \in C_{(0, T]} \), will be constructed.
Supplementary Condition of Local Smoothness

So, in case of $D(0) < 1$ the global solution exists and unique: local solution is constructed by successive approximations and continued by method of steps with successive approximations on each step. Therefore theoretical instest is to study the case $D(0) \geq 1$

A. Exists polynomial $P_i(t, s) = \sum_{\nu+\mu=0}^{M} K_{i\nu\mu} t^\nu s^\mu, i = 1, n$, ,

$f^M(t) = \sum_{\nu=1}^{M} f_\nu t^\nu, \alpha_i^M(t) = \sum_{\nu=1}^{M} \alpha_i^\nu t^\nu, i = 1, n - 1$, where

$0 < \alpha_{11} < \alpha_{12} < \cdots < \alpha_{n-1,n} < 1$ such as for $t \to +0, s \to +0$ the following estimates hold:

$|K_i(t, s) - P_i(t, s)| = O((t+s)^{M+1}), i = 1, n,$

$f(t) - f^M(t) = O(t^{M+1})$,

$|\alpha_i(t) - \alpha_i^M(t)| = O(t^{M+1}), i = 1, n - 1.$
Supplementary Condition of Local Smoothness

Let \( 0 \leq \alpha'_i(0) < 1, \alpha_i(0) = 0, i = 1, n - 1 \). Then \( \forall \varepsilon \in (0, 1) \)
\[
\exists \tau \in (0, T] : \max_{i=1, n-1, t \in [0, \tau]} |\alpha'_i(t)| \leq \varepsilon, \quad \sup_{t \in (0, \tau]} \frac{\alpha_{n-1}(t)}{t} \leq \varepsilon.
\]

B. For fixed \( q \in (0, 1) \), \( \tau \in (0, T] \), \( 0 < \varepsilon < 1 \)
\[
\max_{t \in [0, \tau]} \varepsilon^M D(t) \leq q < 1
\]

Estimate is true for big enough \( M \)
Let condition $D(0) < 1$ is not fulfilled.

$$C. \text{ Select } N^* \text{ such as } \lim_{t \to 0} \frac{\left( \int_0^t K(t,s)\hat{x}(s) \, ds - f(t) \right)'}{t^{N^*}} = 0$$

**Theorem** 3

Let $\hat{x}(t)$ be the known function such as the discrepancy $C$ is true for $N^* \geq M$. Then eq. (1) has the solution $x(t) = \hat{x}(t) + t^{N^*} u(t)$, where $u(t) \in C[0,T]$ is unique and can be constructed by means of the successive approximations method.

Regularization

For $u(t)$ we have the equation

$$\int_0^t K(t, s)s^N u(s) \, ds = g(t) \quad (\ast)$$

$$g(t) := -\int_0^t K(t, s)\hat{x}(s) \, ds + f(t),$$

$$g(t) \in C_{[0, T]}^{(1)}, g(0) = 0, |g'(t)| = o(t^N), \text{ as } t \to +0$$

**Definition**

The equation $(\ast)$ has unique solution and we call it as *regularization* of the equation (1). Function $\hat{x}(t)$ is approximation of solution to eq. (1).
Question Remains:
How to Construct an Approximation $\hat{x}(t)$ to meet the Condition $\mathbf{C}$?

Idea:
To construct $\hat{x}(t)$ as power-logarithmic asymptotic expansion. We would need an additional smoothness of all the given functions according to the condition $\mathbf{A}$ since we need the Taylor coefficients.

No need in condition $D(0) < 1$
Relaxation of the condition $D(0) < 1$

**Theorem 4 (Relaxed Sufficient Condition for Existence & Uniqueness)**

Let conditions (B) and (C) be fulfilled, and

$$B(j) := K_n(0, 0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j} (K_i(0, 0) - K_{i+1}(0, 0)) \neq 0$$

for $j \in \mathbb{N} \cup \{0\}$. Then eq. (1) has unique solution $x(t) = x^M(t) + t^{N^*} u(t)$ in $C_{[0, T]}$, $M \geq N$. Moreover, for $t \to +0$ polynomial

$$\hat{x}(t) \equiv x^M(t) = \sum_{i=0}^{M} x_i t^i$$

is an $M$th order asymptotic approximation of such solution.
Since $B(j) \neq 0$, $j \in \mathbb{N} \cup \{0\}$ then all the coefficients in the solution $x_M(t) = \sum_{i=0}^{M} x_i t^i$ can be determined. We can construct the solution in the form $x(t) = x_M(t) + t^{N^*} u(t)$, $M \geq N^*$. We get the integral-functional equation satisfying the contraction mapping principle on $(0, \tau]$ and the function $u(t) \in C_{(0, T]}$ can be uniquely determined with successive approximations. ■
We search for the approximation of solution as \( \hat{x}(t) = \sum_{j=0}^{N} x_j t^j \).

To find \( x_j \) using method of undetermined coefficients we must solve the recurrent sequence of linear algebraic equations

\[
B(j)x_j = M_j(x_0, \cdots, x_{j-1}), \quad M_0 = f'(0)
\]

**Characteristic Equation**

\[
B(j) \equiv K_n(0,0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j} (K_i(0,0) - K_{i+1}(0,0)) = 0
\]

\( B(j) \neq 0 \) for \( j \in \mathbb{N} \cup \{0\} \)

If \( B(j^*) = 0 \) then coefficient \( x_{j^*} \) of power-logarithmic expansion is solution to the difference eq.: 

\[
K_n(0,0)x_{j^*}(z) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j^*} \{ K_i(0,0) - K_{i+1}(0,0) \} \cdot x_{j^*}(z + \ln \alpha'_i(0)) = M_j(x_0(z), \cdots, x_{j^*-1}(z)), \quad z := \ln t
\]
Construction of the Power-Logarithmic Asymptotic of Solution

We search for the approximation of solution as \( \hat{x}(t) = \sum_{j=0}^{N} x_j t^j \).

To find \( x_j \) using method of undetermined coefficients we must solve the recurrant sequence of linear algebraic equations

\[
B(j)x_j = M_j(x_0, \cdots, x_{j-1}), \quad M_0 = f'(0)
\]

**Characteristic Equation**

\[
B(j) \equiv K_n(0, 0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j}(K_i(0, 0) - K_{i+1}(0, 0)) = 0
\]

\( B(j) \neq 0 \) for \( j \in \mathbb{N} \cup \{0\} \)

If \( B(j^*) = 0 \) then coefficient \( x_{j^*} \) of power-logarithmic expansion is solution to the difference eq.: \( K_n(0, 0)x_{j^*}(z) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j^*}(K_i(0, 0) - K_{i+1}(0, 0)) \cdot x_{j^*}(z + \ln \alpha'_i(0)) = M_j(x_0(z), \cdots, x_{j^*-1}(z)), \quad z := \ln t \)
We search for the approximation of solution as \( \hat{x}(t) = \sum_{j=0}^{N} x_j t^j \).

To find \( x_j \) using method of undetermined coefficients we must solve the reccurent sequence of linear algebraic equations

\[
B(j)x_j = M_j(x_0, \cdots, x_{j-1}), \quad M_0 = f'(0)
\]

**Characteristic Equation**

\[
B(j) \equiv K_n(0, 0) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j}(K_i(0, 0) - K_{i+1}(0, 0)) = 0
\]

\( B(j) \neq 0 \) for \( j \in \mathbb{N} \cup \{0\} \)

If \( B(j^*) = 0 \) then coefficient \( x_{j^*} \) of power-logarithmic expansion is solution to the difference eq.: \( K_n(0, 0)x_{j^*}(z) + \sum_{i=1}^{n-1} (\alpha'_i(0))^{1+j^*}(K_i(0, 0) - K_{i+1}(0, 0)) \cdot x_{j^*}(z + \ln \alpha'_i(0)) = M_j(x_0(z), \cdots, x_{j^*-1}(z)), \quad z := \ln t \)
Example 1

Back to example 1: \( \int_0^{t/2} x(s)ds - \int_{t/2}^{t} x(s)ds = t \)

Characteristic eq. \( B(j) \equiv -1 + (1/2)^j = 0 \), root: \( j^* = 0 \) \( \Rightarrow \)

\[ x\left(\frac{t}{2}\right) - x(t) = 1, \]

\[ x(t) = C_1 + C_2 \ln t \]

\[ C_1 + C_2 \left( \ln \frac{t}{2} \right) - C_1 - C_2 \ln t = 1 \Rightarrow C_2 = -\frac{1}{\ln 2} \]

Solution:

\[ x(t) = -\frac{\ln t}{\ln 2} + \text{const}. \]
Example 1

Back to example 1: \( \int_{0}^{t/2} x(s) \, ds - \int_{t/2}^{t} x(s) \, ds = t \)

Characteristic eq. \( B(j) \equiv -1 + (1/2)^j = 0 \), root: \( j^* = 0 \) \( \Rightarrow \)

\[
x \left( \frac{t}{2} \right) - x(t) = 1,
\]

\[
x(t) = C_1 + C_2 \ln t
\]

\[
C_1 + C_2 \left( \ln \frac{t}{2} \right) - C_1 - C_2 \ln t = 1 \Rightarrow C_2 = -\frac{1}{\ln 2}
\]

Solution:

\[
x(t) = -\frac{\ln t}{\ln 2} + \text{const.}
\]
Example 1

Back to example 1: $\int_{0}^{t/2} x(s) ds - \int_{t/2}^{t} x(s) ds = t$

Characteristic eq. $B(j) \equiv -1 + (1/2)^j = 0$, root: $j^* = 0 \Rightarrow$

$$x\left(\frac{t}{2}\right) - x(t) = 1,$$

$$x(t) = C_1 + C_2 \ln t$$

$$C_1 + C_2 \left(\ln \frac{t}{2}\right) - C_1 - C_2 \ln t = 1 \Rightarrow C_2 = -\frac{1}{\ln 2}$$

Solution:

$$x(t) = -\frac{\ln t}{\ln 2} + \text{const}.$$
Example 1

Back to example 1: \( \int_0^{t/2} x(s)ds - \int_{t/2}^{t} x(s)ds = t \)

Characteristic eq. \( B(j) \equiv -1 + (1/2)^j = 0 \), \( \text{root: } j^* = 0 \Rightarrow \)

\[
x \left(\frac{t}{2}\right) - x(t) = 1, \\
x(t) = C_1 + C_2 \ln t \\
C_1 + C_2 \left(\ln \frac{t}{2}\right) - C_1 - C_2 \ln t = 1 \Rightarrow C_2 = -\frac{1}{\ln 2} \\
\]

Solution:

\[
x(t) = -\frac{\ln t}{\ln 2} + \text{const}. \\
\]
1. Generalization to the Nonlinear VIE

\[
\int_{0}^{t} K(t, s, x(s)) \, ds, \quad 0 \leq s \leq t \leq T, \quad f(0) = 0
\]  

\(K(t, s, x(s)) = \begin{cases} 
K_1(t, s)G_1(s, x(s)), & t, s \in m_1, \\
\quad \ldots \quad \ldots \ldots \\
K_1(t, s)G_1(s, x(s)), & t, s \in m_n,
\end{cases}\)

\(m_i = \{t, s| \alpha_{i-1}(t) < s < \alpha_i(t)\}, \ \alpha_0(t) = 0, \ \alpha_n(t) = t, \ i = 1, n, \ K_i(t, s), \ f(t), \ \alpha_i(t) \) have continuous derivatives w.r.t. \(t\) for \(t, s \in m_i, \ K_n(t, t) \neq 0, \ \alpha_i(0) = 0, \ 0 < \alpha_1(t) < \alpha_2(t) < \cdots < \alpha_{n-1}(t) < t, \ \alpha_1(t), \ldots, \alpha_{n-1}(t) \) increase at least in the small neighborhood \(0 \leq t \leq \tau.\)
Problem Statement

Asymptotic Approximation of the Solution

Singular Case

Generalizations

Numerical Solution

(D) Lipschitz cond.: \(|G_i(s, x_1) - G_i(s, x_2) - (x_1 - x_2)| \leq q_i|x_1 - x_2|, \forall x_1, x_2 \in \mathbb{R}^1\)

(E) \(q_n + \sum_{i=1}^{n-1} \alpha_i'(0)|K_n^{-1}(0, 0)(K_i(0, 0) - K_{i+1}(0, 0))|(1 + q_i) < 1\)

Theorem 2' (Sufficient Condition for Existence & Uniqueness for NLVIEq)

Let for \(t \in [0, T]\) the following conditions be fulfilled:

\(K_i(t, s), G_i(s, x(s)), i = 1, n\) are continuous, \(\alpha_i(t)\) and \(f(t)\) have continuous derivatives wrt \(t\), \(K_n(t, t) \neq 0, 0 = \alpha_0(t) < \alpha_1(t) < \cdots < \alpha_{n-1}(t) < \alpha_n(t) = t\), for \(t \in (0, T]\), \(\alpha_i(0) = 0, f(0) = 0, (D)\) and (E).

Then \(\exists \tau > 0\) such as eq. (8) posseses unique local solution. Moreover if \(\min_{\tau \leq t \leq T} (t - \alpha_{n-1}(t)) = h > 0\) then such local solution is continuously extendable on the entire \([\tau, T]\) using the method of steps and successive approximations. Therefore exists unique global solution in \(C_{[0, T]}\).
2. Generalizations:

**Systems of linear VIEs with piecewise continuous kernels:**


**Abstract integral-operator Volterra equations in Banach spaces:**


\[ K_i(t, s) \in \mathcal{L}(E_1 \to E_2), \quad G_i : E_1 \to E_1 \]

Numerical Solution

\[ \int_0^{\alpha(t)} K_1(t, s)x(s)ds + \int_{\alpha(t)}^{t} K_2(t, s)x(s)ds = f(t), \quad t \in [0, T], \quad (9) \]

\[ 0 < \alpha(t) < t \quad \forall t \in (0, T], \quad \alpha(0) = 0, \quad K_1(t, s), K_2(t, s), f(t) \text{ are continuous and smooth sufficiently, } f(0) = 0, \quad K_2(t, t) \neq 0 \quad \forall t \in [0, T]. \]

\[ t_i = t_0 + ih, \quad i = 1, n, \quad t_0 = 0, \quad nh = T, \quad l = \left[ \frac{\alpha(t_i)}{h} \right] + 1 \]

\[ h \sum_{j=1}^{l-1} K_1(t_i, t_j)x^h(t_j) + (\alpha(t_i) - t_{l-1})K_1(t_i, \alpha(t_i))x^h(\alpha(t_i)) + \]

\[ + (t_l - \alpha(t_i))K_2(t_i, t_l)x^h(t_l) + h \sum_{j=l+1}^{i} K_2(t_i, t_j)x^h(t_j) = f(t_i) \quad i = 1, n \]

Denis N. Sidorov contact.dns@gmail.com
\[ x^h(\alpha(t_1)) = \frac{f(t_1)}{(\alpha(t_1) - t_0)K_1(t_1, \alpha(t_1)) + (t_1 - \alpha(t_1))K_2(t_1, \alpha(t_1))} \] (11)

\[ x(0) = \frac{f'(0)}{\alpha'(0)[K_1(0, 0) - K_2(0, 0)] + K_2(0, 0)} \] (12)

If \( D(0) < 1 \) (Theorem 2) then
\[
\alpha'(0)[K_1(0, 0) - K_2(0, 0)] + K_2(0, 0) \neq 0
\]
1st Example

\[
\int_{0}^{\frac{t}{3}} (1 + t - s)x(s)\,ds - \int_{\frac{t}{3}}^{t} x(s)\,ds = \frac{t^4}{108} - \frac{25t^3}{81}, \quad t \in [0, 2],
\]

\[
\bar{x}(t) = t^2
\]

\[
\varepsilon_1 = \max_{1 \leq i \leq n} |\bar{x}(t_i) - x_1^h(t_i)|, \quad \varepsilon_2 = \max_{0 \leq i \leq n} |\bar{x}(t_i) - x_2^h(t_i)|,
\]

\(x_1^h\) is solution computed with (11) in \(a(t_1)\)

\(x_2^h\) is solution computed with (12), (11) and extrapolation in the 1st node.

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<th>(\varepsilon_1)</th>
<th>(\varepsilon_2)</th>
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2nd Example

\[
2 \int_0^{\sin(\frac{t}{2})} x(s) \, ds - \int_{\sin(\frac{t}{2})}^t x(s) \, ds = \frac{1}{3} \sin^3 \frac{t}{2} + \frac{t^3}{3}, \quad t \in [0, 2\pi], \\
\bar{x}(t) = t^2
\]

<table>
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THANK YOU

FOR YOUR ATTENTION
Special Session: “Optimization in Inverse Problems”