Local models for Size-Topology Correlations

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Outline – Tissues, Tilings, Packings

I. Motivation from Biological Tissues
II. A simple local geometric model
III. Terminal Disorder, Bidispersity
IV. Lewis’ Law
**Drosophila** wing development

Remodeling after proliferation
Both neighbor and area distributions narrow simultaneously over time: Size-topology correlations!

quantify width by coefficients of variation $c_n$, $c_A$

($= \Delta n/\langle n \rangle$, $\Delta A/\langle A \rangle$)
Distribution width correlation

from Quilliet et al., Phil. Mag. Lett. 2008
Another empirical correlation: Lewis’ Law

[F. T. Lewis, *The Anatomical Record* (1928)]
Statistical Physics approach

“shuffling”: T1s / local conservation laws (→ isotropy of domains!)

[M.Durand, J.Käfer, C.Quilliet, S.Cox, S.Talebi, F.Graner *PRL* (2011)]
Statistical Physics approach

“shuffling”: T1s / local conservation laws (→ isotropy of domains!)

Local Geometry (GM) approach

Disk templates / local geometry (→ isotropy of domains!)

[M.Durand, J.Käfer, C.Quilliet, S.Cox, S.Talebi, F.Graner *PRL* (2011)]

[SH & M.Miklius, *PRL* (2012)]
The Granocentric Model in 2D


\[ \phi = 2 \arcsin \left( \frac{1}{1 + \sqrt{A_c/A}} \right) \]

given \( P(A) \Rightarrow f_c(\phi) \)

For fixed \( A_c \), draw neighbors until \( \phi > \phi_{\text{max}} \)

- disk areas \( \Leftrightarrow \) regular polygonal areas
- packings \( \Leftrightarrow \) tilings

Euler’s theorem: mean number of neighbors \( \bar{n} = 6 \), fixes \( \phi_{\text{max}} \approx 13\pi/6 \); no free parameter!
Computing probabilities

Conditional probability:

\[ P(n|A_c) = \int_0^{\phi_{\text{max}}} R_{c,n}(\phi) F(\phi_{\text{max}} - \phi) d\phi \]

\( R_{c,n}(\phi) \): PDF of sum of \( n \) angles; \( F(\psi) \equiv \int_0^\infty f_c(\phi) d\phi \)

Unconditional probability:

\[ P_n = \int P(n|A_c) P(A_c) dA_c \quad \Rightarrow c_n, \bar{A}_n \]

explicit results (“GM simulations”) using \( P(A) \) as input

But correlations use (at most) \( c_A \)

\( \Rightarrow \) use Gaussian PDFs: integrals become \textit{analytically solvable}!

Result: Neighbor probabilities

\[ P^n \]

\[ \text{C}_A \]

Packings: Hard Disk Crystallization

Analytical result: \( c_{A,\text{crit}} = \pi \sqrt{2/1755} \approx 0.106 \)


Terminal Bidispersity

These results are empirical!

Critical size ratio for 50/50 mixtures ($p=1/2$)

Terminal Bidispersity

Only three angles!

Two parameters: size ratio $\sigma = R_2/R_1$, frequency $p = N_1/(N_1 + N_2)$;

[S. Hilgenfeldt, Phil. Mag., 2013]
Result: Neighbor probabilities

Analytical result: \( \sigma_{\text{crit}} = \csc \left( \frac{13\pi}{84} \right) - 1 \approx 1.1401... \)

[S. Hilgenfeldt, Phil. Mag., 2013]
Back to Polydisperse Tilings
Result: Distribution width correlation

Result: Distribution width correlation

[M. Miklius and SH, PRL, 2012]
Tilings not obeying the correlation

Poisson-Voronoi

relaxed foam

anisotropic domains

near-isotropic domains

Introducing (sufficient) interfacial energy establishes near-isotropic domains!
From GM theory: nonlinear Lewis’ Law

Nonlinear, leading order: independent of $c_A$!

[M. Miklius and SH, PRL, 2012]
The original linear Lewis’ Law

Cell Anisotropy

\[ \alpha = \frac{b}{a} \]

Orientational Disorder

Experiment

Model: rectangular cells of uniform $\alpha$

More neighbor configurations for a given size distribution because of varying orientation!

Oorientational Disorder

...explains deviations from isotropic model prediction

Orientational Disorder

...explains linear appearance of Lewis’ Law

Conclusions

• Strictly local neighbor statistics capture size-topology correlations in 2D: foams, biological tissues,…

• Terminal poly- and bidispersity

• Correlations diagnose disorder due to size, position, and orientation: surface energy matters qualitatively

• Lewis’ law nonlinear for near-isotropic domains, near-linear for anisotropic domains