Spherical circle coverings and bubbles in foam

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COVERING THE SPHERE WITH EQUAL CIRCLES
The covering problem

How must a sphere be covered by \( n \) equal circles (spherical caps) without interstices so that the angular radius of the circles will be as small as possible?
Known solutions in the 1980’s

• Mathematically proven solutions for $n = 2, 3, 4, 5, 6, 7, 10, 12, 14$

• Conjectural solutions for $n = 8, 9, 32$
Known solutions in the 1980's

- $n = 2, 3, 4, 6, 12$  
  L. Fejes Tóth

- $n = 5, 7, 8$  
  K. Schütte

- $n = 9$  
  E. Jucovič

- $n = 10, 14, 32$  
  G. Fejes Tóth

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1. $n = 2, 3, 4, 6, 12$
2. $n = 5, 7, 8$
3. $n = 9$
4. $n = 10, 14, 32$
SOAP FILMS AND BUBBLES IN FOAM
Minimal nets formed by \( n \) soap-film-like cones

Fig. 1. The ten possible minimal one-dimensional nets formed by the intersection of a film cone by a sphere.

Soap bubbles in Matzke’s observations

Minimum covering:

\[ n = 11 \quad T.T. \& Zs. \ Gáspár \ 1986 \\
\[ n = 13 \quad T.T. \& Zs. \ Gáspár \ 1985 \]
COATED VESICLES
Clathrin-dependent endocytosis

Formation of coated vesicles
Structure of coated vesicles
Barbara Pearse 1975

Fig. 4. Clathrin trimers maintained in 100 mM TEA buffer, pH 8.0, during exposure to mica, illustrating the predominantly counterclockwise orientation characteristic of adsorption under such high-affinity conditions. Some molecules still land in the clockwise “triskelion” orientation (arrowheads); a few even land on edge to give a side view of what may have been their original “pucker” (double arrowheads). × 230 000.

Clathrin triskelions

Fig. 11. Stereo view of the above coated vesicle preparation; a slightly smaller basket of “intermediate” configuration

Clathrin basket coat
Coated vesicle

Clathrin basket

Street lamp in Kyoto
Identified clathrin baskets

- Tetrahedral (16)
- Barrel (20)
- Tennis ball (20)
- Soccer ball (32)
Circle coverings based on identified clathrin baskets

\[ n = 16 \]
\[ n = 20 \]

\( n = 16 \)

\( n = 20 \)
Circle coverings based on identified clathrin baskets

\[ n = 16 \]
\[ n = 16 \]
\[ n = 20 \]
\[ n = 20 \]

\[ n = 16 \] and \[ n = 20 \]  T.T. & Zs. Gáspár 1991
Theoretical clathrin baskets by Katsura

Minimum covering:
\( n = 15, 17, 18, 19 \)
T.T. & Zs. Gáspár 1991

The result for \( n = 19 \) was improved in 1994 by R.H. Hardin, N.J.A. Sloane & W.D. Smith.
Solutions to the covering problem, \( n \leq 20 \)

- \( n = 2, 3, 4, 6, 12 \) L. Fejes Tóth
- \( n = 5, 7, 8 \) K. Schütte
- \( n = 9 \) E. Jucovič
- \( n = 10, 14, 32 \) G. Fejes Tóth
- \( n = 11, 13, 15, 16, 17, 18, 19, 20 \) T.T. & Zs. Gáspár
- \( n = 19, 21 \) to 130 R.H. Hardin, N.J.A. Sloane & W.D. Smith
ISOPARAMETRIC PROBLEM
FOR POLYHEDRA
Isoperimetric problem for polyhedra

Among polyhedra with given surface area and a given number of faces $n$, which has the maximum volume?
Solutions to the isoperimetric problem, $n \leq 20$

Conjectured solutions by M. Goldberg 1935, A. Schoen 1986. The edge graphs of the polyhedra are identical to those of covering except for $n = 11$. For $n = 19$, it is not known.
Identical solutions for both covering and isoperimetric problems

T.T., Zs. Gáspár, K. Hincz 2013
Identical solutions for both covering and isoperimetric problems, $n = 32$

- Minimum circle covering
- Turned ivory, around 1600, Grünes Gewölbe, Dresden
- Roundest soccer ball, the Hyperball
Solutions to covering and isoperimetric problems, $n = 14$

The two solutions are equal under $O_h$ symmetry constraint.

Only the edge graphs of the two solutions are equal if there is no symmetry constraint.
Conclusions

• Soap bubbles in foam, and coated vesicles provided useful ideas to construct locally optimal coverings of the sphere with equal circles.

• The isoperimetric problem for polyhedra, and the problem of minimum covering the sphere with equal circles are related problems, since in both cases the obtained polyhedra have an insphere.

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