The BKZ algorithm

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Outline

Lattices

Basis Reduction
  LLL
  BKZ
  Enumeration
  BKZ 2.0

Open questions
Lattices are represented by a basis.
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Lattice problems are about finding **short** and close vectors.
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Lattice problems are about finding short and close vectors. In practice it suffices to find short and orthogonal basis vectors.
Gram-Schmidt

Iterative process to orthonormalize a set of vectors $\mathbf{b}_1, \ldots, \mathbf{b}_d$:

\[
\begin{align*}
\mathbf{b}_1^* & := \mathbf{b}_1 \\
\mathbf{b}_i^* & := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{ij} \mathbf{b}_j^*, \quad \text{where } \mu_{ij} = \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\| \mathbf{b}_j^* \|^2} \text{ for all } 1 \leq j < i \leq d.
\end{align*}
\]

Result: vectors $\mathbf{b}_1^*, \ldots, \mathbf{b}_d^*$ that are pairwise orthogonal. They span the same space as $\mathbf{b}_1, \ldots, \mathbf{b}_d$. 

In lattices: only integral combinations are allowed, $\mathbf{b}_1^*, \ldots, \mathbf{b}_d^*$ will not span the same lattice!
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Gram-Schmidt
Forget that we are in a lattice.
Projecting $\mathbf{b}_2$ gives $\mathbf{b}_2^*$. 

$\mathbf{b}_1 = \mathbf{b}_1^*$

$\mathbf{b}_2$

$\mathbf{b}_2^*$
Gram-Schmidt

\[ \mathbf{b}_2^* \text{ is not a lattice vector.} \]
Gram-Schmidt

But there is a lattice vector within \( \frac{1}{2} \| b_1^* \| \) from \( b_2^* \):

\[
b'_2 := b_2 - \left\lfloor \mu_{2,1} \right\rfloor \cdot b_1.
\]
It is always possible to choose a basis close to the Gram Schmidt vectors. This basis is called size-reduced.
LLL (1982)
First polynomial-time basis reduction algorithm. Ideas:
▶ Always take the basis ‘closest’ to Gram-Schmidt.
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- Being greedy when ordering basis vectors is bad for the complexity.

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&b_1, \ldots, b_i, b_{i+1}, \ldots, b_d \\
&b^*_1, \ldots, b^*_i, b^*_{i+1}, \ldots, b^*_d
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$$b_1, \ldots, b_i, b_{i+1}, \ldots, b_d$$

$$b_1^*, \ldots, b_i^*, b_{i+1}^*, \ldots, b_d^*$$

What happens when $b_i$ and $b_{i+1}$ are swapped?

Only $b_i^*$ and $b_{i+1}^*$ change. New $b_i^*$ becomes $b_{i+1}^* + \mu_{i+1,i}b_i^*$. 
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Swap when \( \| \mathbf{b}_{i+1}^* + \mu_{i+1,i} \mathbf{b}_i^* \|^2 < \delta \| \mathbf{b}_i^* \|^2 \), for \( \delta \in (1/4, 1) \).
BKZ (1987, 1994)

Trade-off between basis quality and time.

\[ b_1, \ldots, b_i, b_{i+1}, \ldots, b_{i+\beta-1}, \ldots, b_d \]

\[ b_1^*, \ldots, b_i^*, b_{i+1}^*, \ldots, b_{i+\beta-1}^*, \ldots, b_d^* \]

Compute \( b_{\text{new}} \), a combination of vectors \( b_i, b_{i+1}, \ldots, b_{i+\beta-1} \) such that it becomes the shortest possible \( i \)'th Gram-Schmidt vector.

If \( \|b_{\text{new}}\|_2 < \delta \|b_i^*\|_2 \), insert \( b_{\text{new}} \) into the basis:

\[ b_1, \ldots, b_{i-1}, b_{\text{new}}, b_i, b_{i+1}, \ldots, b_d \]

Now LLL is used to remove the linear dependency created by the extra vector. BKZ moves cyclically through the basis indices \( i \).

Note: we do not have a good bound on the time complexity of BKZ.
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In practice enumeration is used: enumerate all lattice points within a certain radius around the origin.
Enumeration

\[ \mathbf{b}_1 \]

\[ \mathbf{b}_2 \]
1) Choose a bound.
2) Do the Gram-Schmidt.
3) ‘Project’ whole lattice.
Enumeration

Lattice vector within bound $\Rightarrow$ its projection within bound.
4) Enumerate all vectors in projected lattice within bound.
5) For each vector in projected lattice, enumerate all lattice vectors.
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6) Pick the shortest vector.
Enumeration as a tree

Enumeration is like a tree search.
Enumeration as a tree

Enumeration is like a tree search. Each level corresponds to a projected lattice.
Enumeration as a tree

Enumeration is like a tree search. Each level corresponds to a projected lattice. The leaves correspond to lattice vectors.

$b_1 = b_1^*$

$b_2^*$
Extreme pruning

Branches near the edge yield fewer leaves.
Extreme pruning

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\[ b_2 = b_2^* \]

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Branches near the edge yield fewer leaves. Pruning decreases the size of the tree. It might also remove the solutions.
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BKZ 2.0

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Chen and Nguyen proposed BKZ 2.0 with the following improvements over the original:

- Better enumeration bound
- Extreme pruning
- Aborting BKZ after a fixed number of rounds
- Better preprocessing of the blocks
Open questions

Regarding BKZ (2.0):

- Many heuristics. What can we prove?
- Destroys local structure for global improvement. Can this be done better?
- What about structured (ideal) lattices?
- Can we speed it up using a quantum computer?

In general:

- Are there better classical algorithms?
- What about quantum algorithms?
Questions?