The BKZ algorithm

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University of BRISTOL

Outline

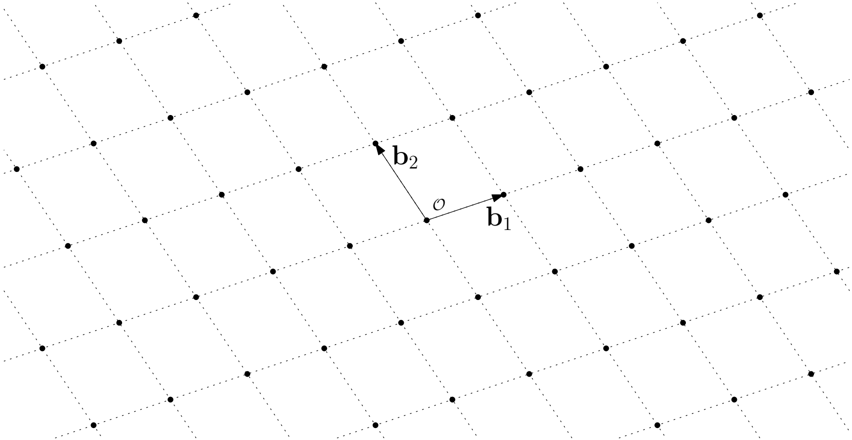
Lattices

Basis Reduction

- LLL
- BKZ
- Enumeration
- BKZ 2.0

Open questions

Lattices



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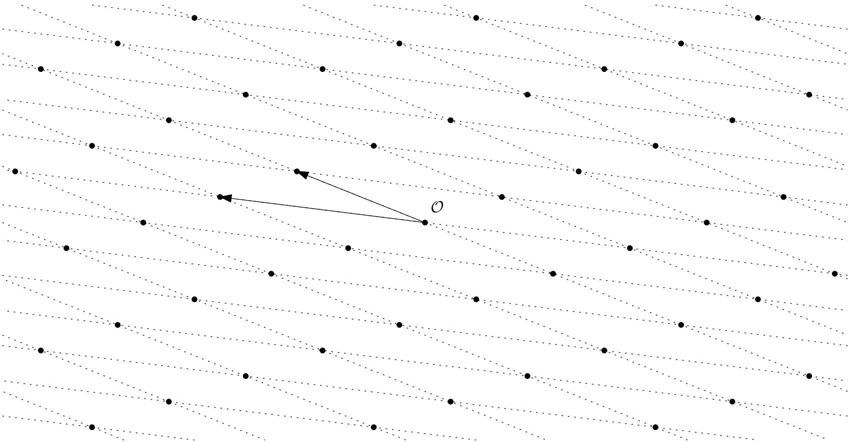
Lattices



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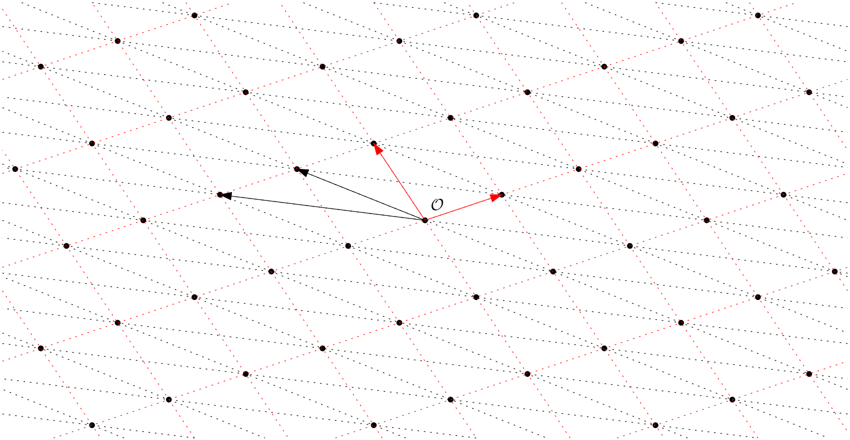
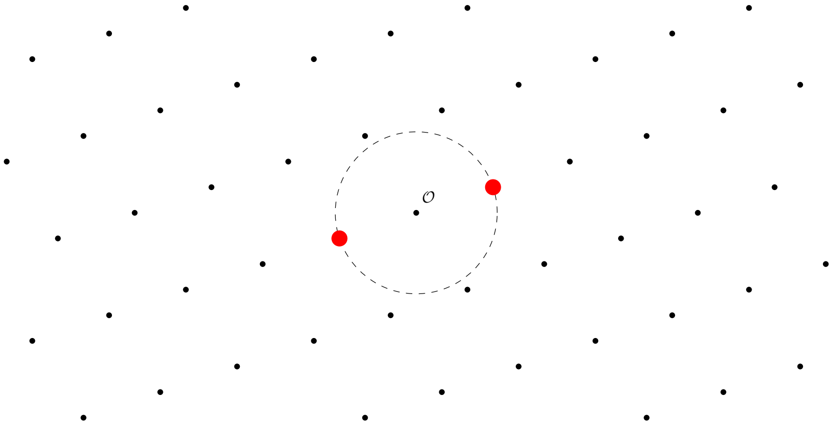


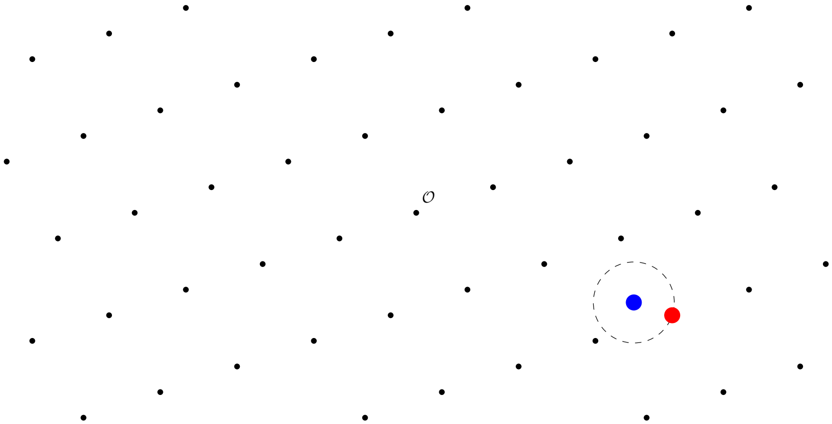
Lattices are represented by a basis. This basis is not unique.
Many bases span the same lattice. Some are ‘better’ than others.

Lattices



Lattice problems are about finding short and close vectors.

Lattices



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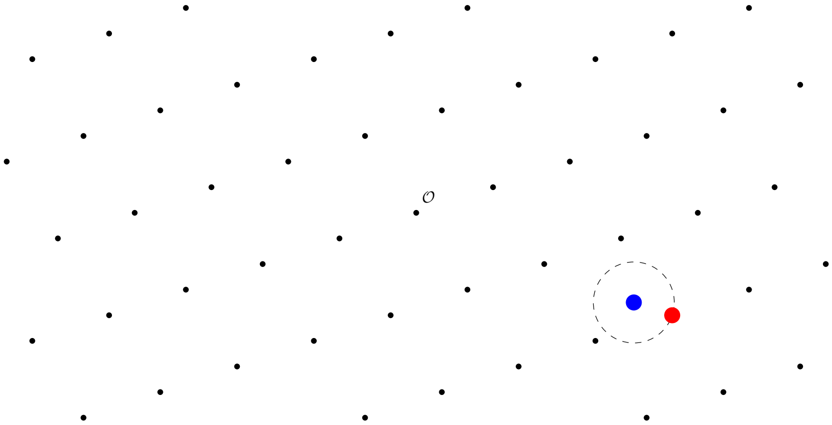

Lattice problems are about finding short and close vectors.
In practice it suffices to find short and orthogonal basis vectors.

Gram-Schmidt

Iterative process to orthonormalize a set of vectors $\mathbf{b}_1, \ldots, \mathbf{b}_d$:

$$\mathbf{b}_1^* := \mathbf{b}_1$$

$$\mathbf{b}_i^* := \mathbf{b}_i - \sum_{j=1}^{i-1} \mu_{ij} \mathbf{b}_j^*, \quad \text{where } \mu_{ij} = \frac{\langle \mathbf{b}_i, \mathbf{b}_j^* \rangle}{\|\mathbf{b}_j^*\|^2} \text{ for all } 1 \leq j < i \leq d.$$

Result: vectors $\mathbf{b}_1^*, \ldots, \mathbf{b}_d^*$ that are pairwise orthogonal.
They span the same *space* as $\mathbf{b}_1, \ldots, \mathbf{b}_d$.

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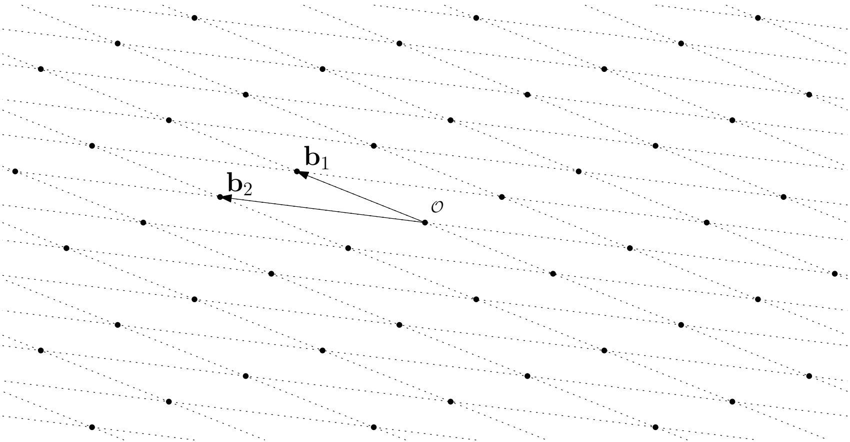
Result: vectors $\mathbf{b}_1^*, \ldots, \mathbf{b}_d^*$ that are pairwise orthogonal.
They span the same *space* as $\mathbf{b}_1, \ldots, \mathbf{b}_d$.

In lattices: only integral combinations are allowed, $\mathbf{b}_1^*, \ldots, \mathbf{b}_d^*$ will not span the same lattice!

Gram-Schmidt



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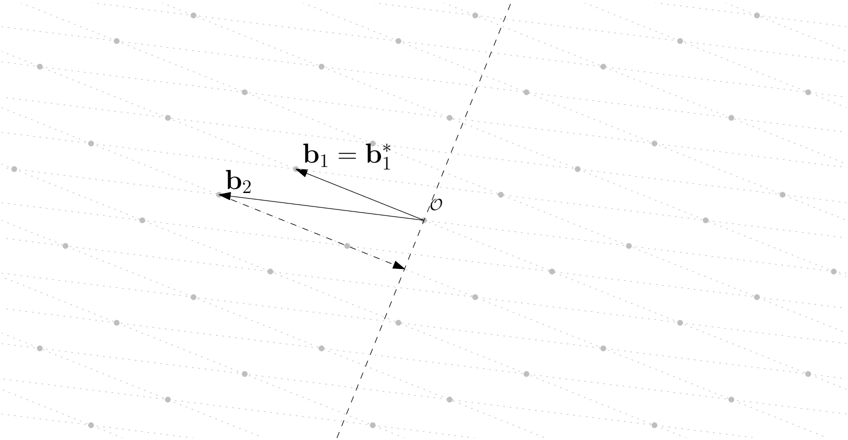
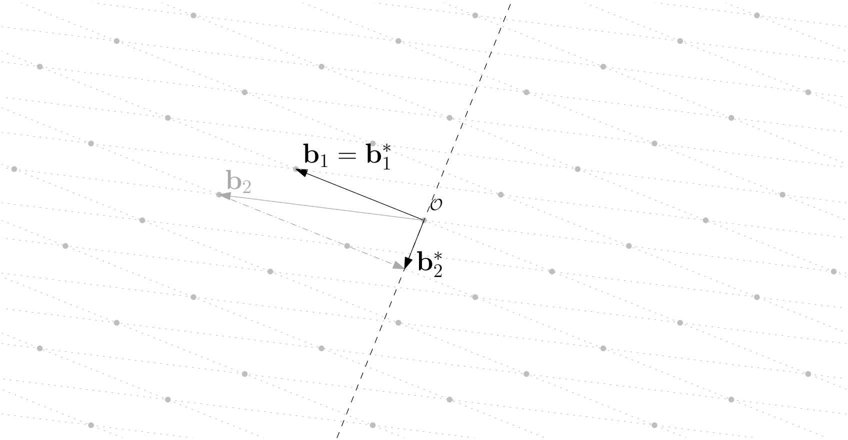


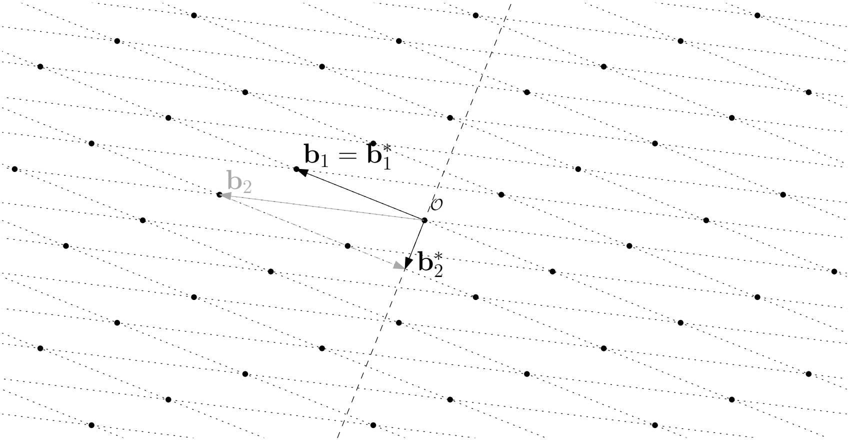
Forget that we are in a lattice.

Gram-Schmidt



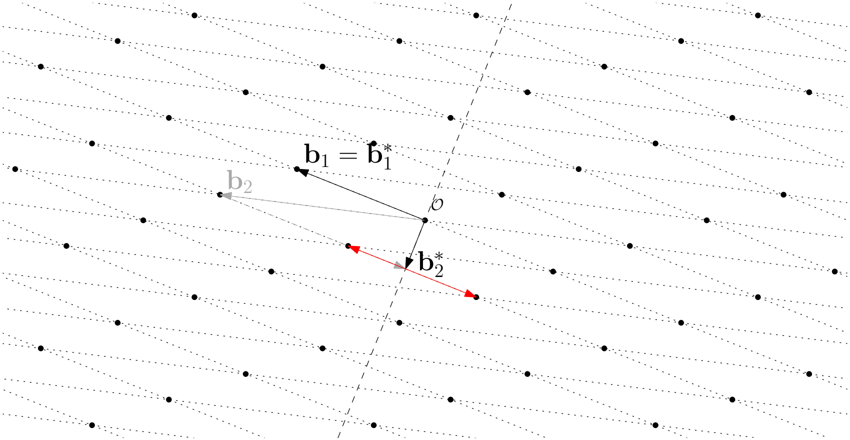
Projecting $\mathbf{b}_2$ gives $\mathbf{b}_2^*$.

Gram-Schmidt



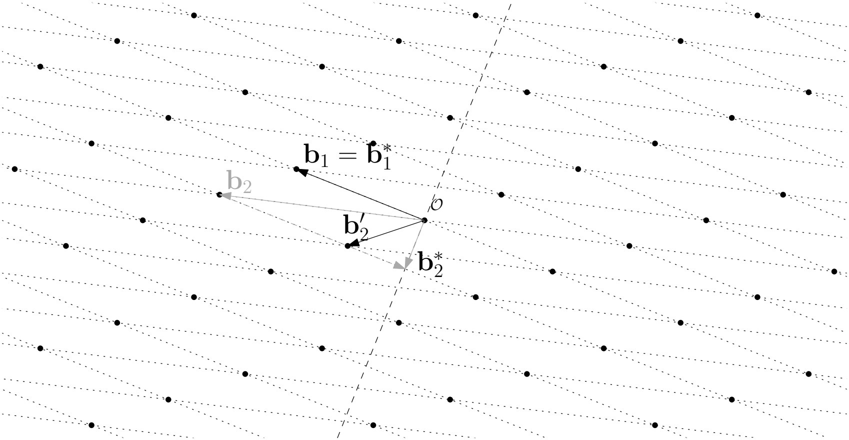
$\mathbf{b}_2^*$ is not a lattice vector.

Gram-Schmidt



But there is a lattice vector within $\frac{1}{2}\|\mathbf{b}_1^*\|$ from $\mathbf{b}_2^*$:
$\mathbf{b}_2' := \mathbf{b}_2 - \lceil \mu_{2,1} \rfloor \cdot \mathbf{b}_1$.

Gram-Schmidt



It is always possible to choose a basis close to the Gram Schmidt vectors. This basis is called size-reduced.

LLL (1982)

First polynomial-time basis reduction algorithm.
Ideas:

- Always take the basis ‘closest’ to Gram-Schmidt.

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- Being greedy when ordering basis vectors is bad for the complexity.

$$\mathbf{b}_1, \ldots, \mathbf{b}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_d$$
$$\mathbf{b}_1^*, \ldots, \mathbf{b}_i^*, \mathbf{b}_{i+1}^*, \ldots, \mathbf{b}_d^*$$

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What happens when $\mathbf{b}_i$ and $\mathbf{b}_{i+1}$ are swapped?
Only $\mathbf{b}_i^*$ and $\mathbf{b}_{i+1}^*$ change. New $\mathbf{b}_i^*$ becomes $\mathbf{b}_{i+1}^* + \mu_{i+1,i}\mathbf{b}_i^*$.

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Swap when $\|\mathbf{b}_{i+1}^* + \mu_{i+1,i}\mathbf{b}_i^*\|^2 < \delta\|\mathbf{b}_i^*\|^2$, for $\delta \in (1/4, 1)$.

BKZ (1987, 1994)

Trade-off between basis quality and time.

$$\mathbf{b}_1, \ldots, \mathbf{b}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{i+\beta-1}, \ldots, \mathbf{b}_d$$
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Compute $\mathbf{b}_{\mathrm{new}}$, a combination of vectors $\mathbf{b}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{i+\beta-1}$ such that it becomes the shortest possible i'th Gram-Schmidt vector.

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Now LLL is used to remove the linear dependency created by the extra vector. BKZ moves cyclically through the basis indices i.

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Note: we do not have a good bound on the time complexity of BKZ.

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"Compute $\mathbf{b}_{\text{new}}$, a combination of vectors $\mathbf{b}_i, \mathbf{b}_{i+1}, \ldots, \mathbf{b}_{i+\beta-1}$ such that it becomes the shortest possible *i*'th Gram-Schmidt vector."

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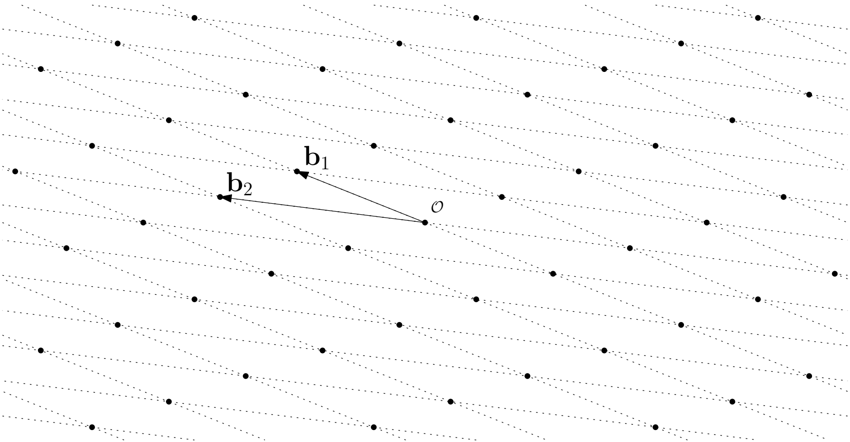
In practice enumeration is used: enumerate all lattice points within a certain radius around the origin.

Enumeration



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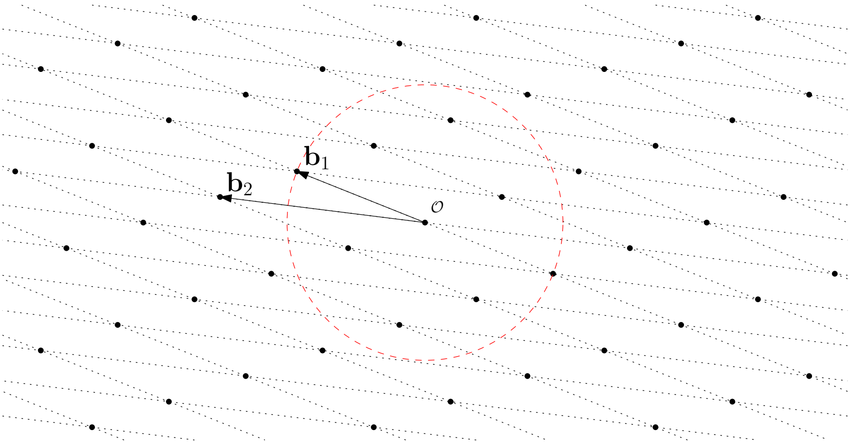
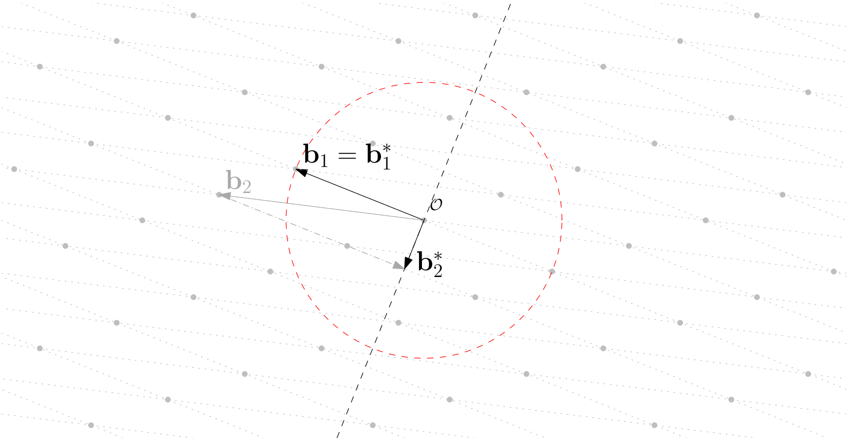


1) Choose a bound.

Enumeration



2) Do the Gram-Schmidt.

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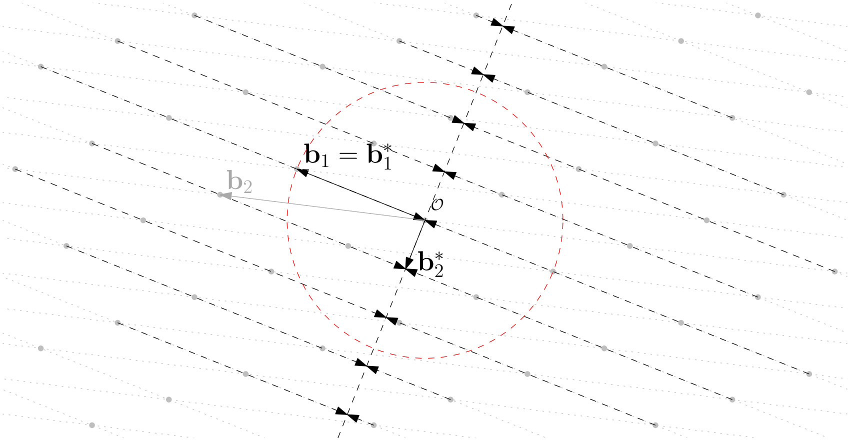
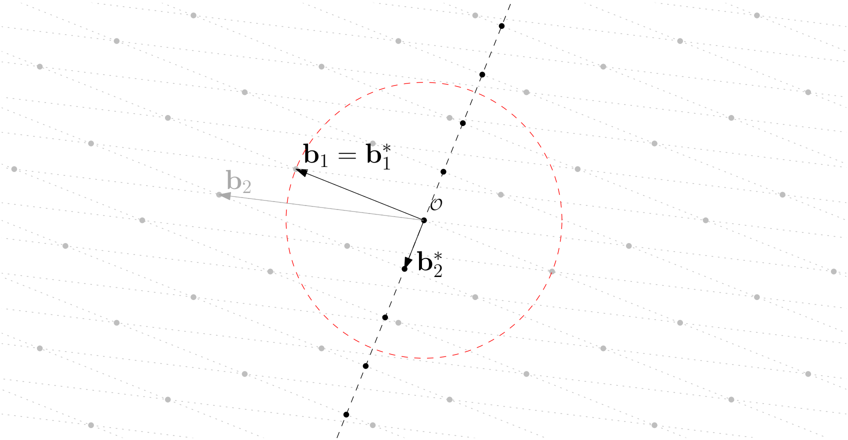


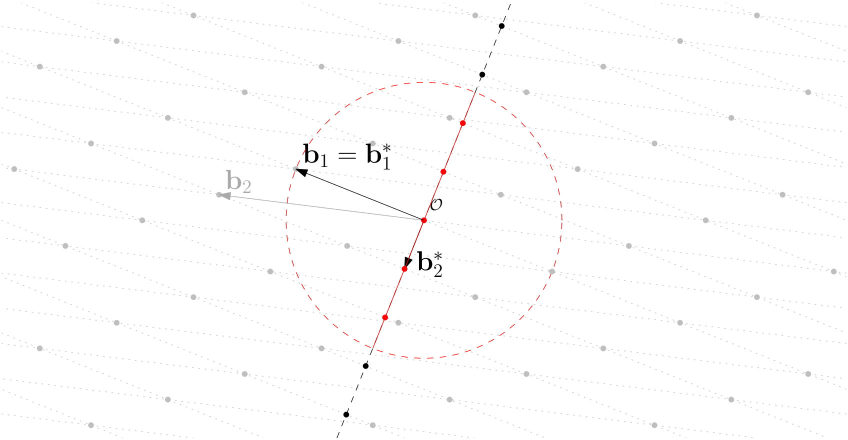
3) ‘Project’ whole lattice.

Enumeration



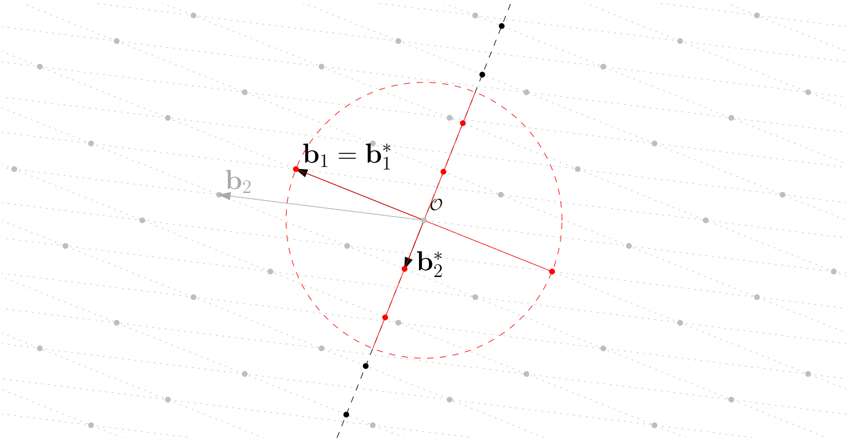
Lattice vector within bound ⇒ its projection within bound.

Enumeration



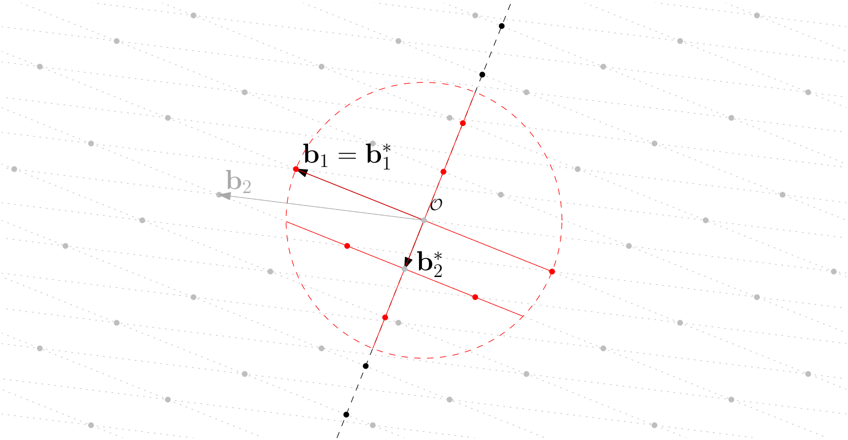
4) Enumerate all vectors in projected lattice within bound.

Enumeration



5) For each vector in projected lattice, enumerate all lattice vectors.

Enumeration



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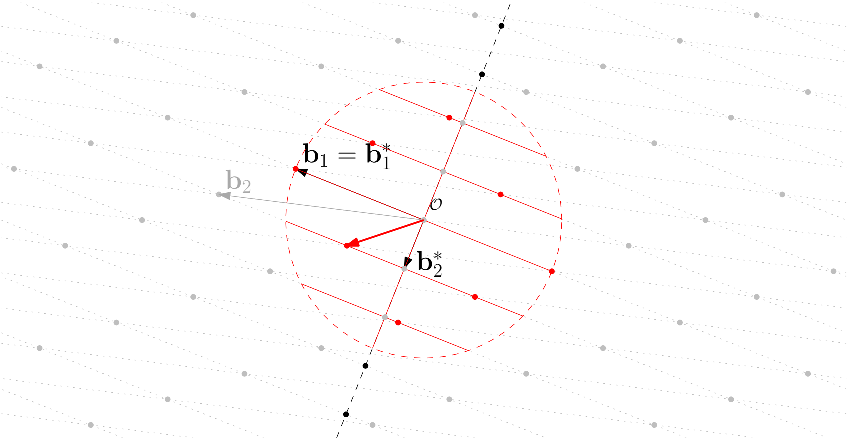
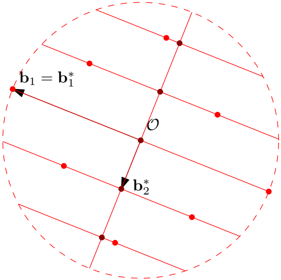
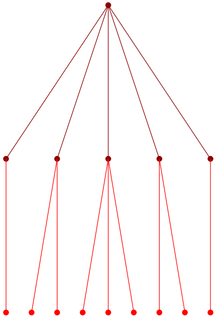


6) Pick the shortest vector.

Enumeration as a tree





Enumeration is like a tree search.

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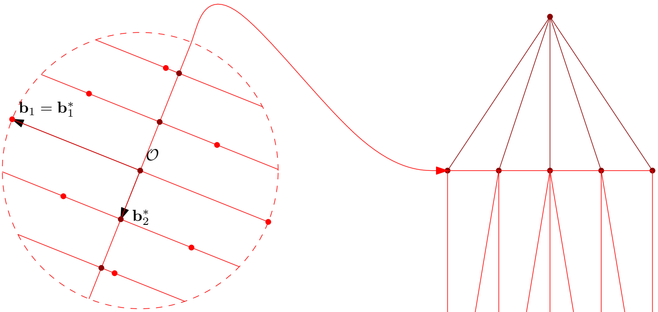

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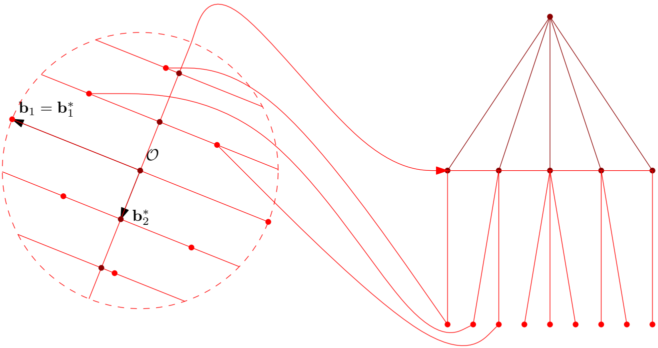
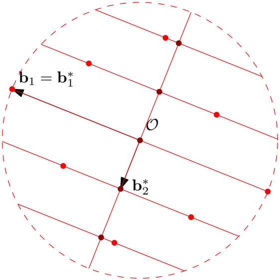
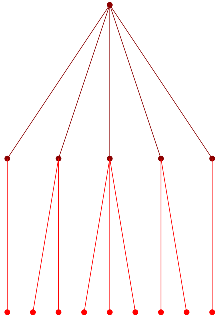


Enumeration is like a tree search. Each level corresponds to a projected lattice. The leaves correspond to lattice vectors.

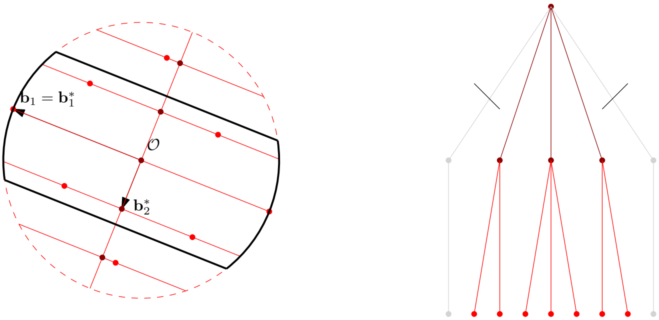
Extreme pruning





Branches near the edge yield fewer leaves.

Extreme pruning



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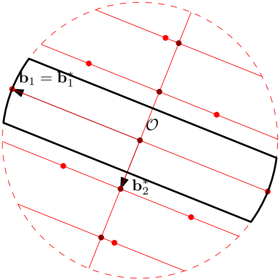
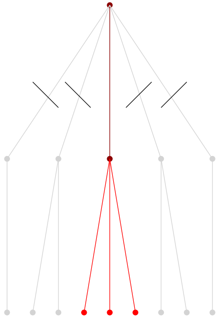
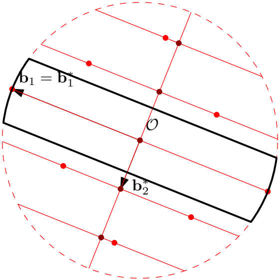
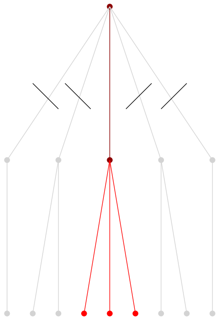




Branches near the edge yield fewer leaves. Pruning decreases the size of the tree. It might also remove the solutions.

Extreme pruning





Extreme pruning: probability p of finding the solution, but more than p^{-1} times faster.

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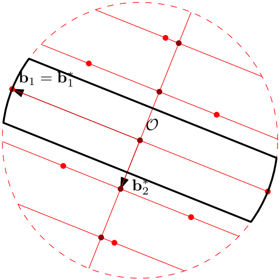

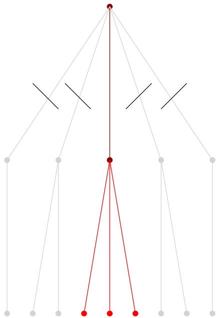

Extreme pruning: probability p of finding the solution, but more than p^{-1} times faster. This gives a speed-up of $\approx 2^{d/2}$.

BKZ 2.0

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For $\beta > 40$, the projected lattices behave like random lattices. The Gaussian Heuristic gives us a good bound.

Chen and Nguyen proposed BKZ 2.0 with the following improvements over the original:

- Better enumeration bound
- Extreme pruning
- Aborting BKZ after a fixed number of rounds
- Better preprocessing of the blocks

Open questions

Regarding BKZ (2.0):

- Many heuristics. What can we prove?
- Destroys local structure for global improvement. Can this be done better?
- What about structured (ideal) lattices?
- Can we speed it up using a quantum computer?

In general:

- Are there better classical algorithms?
- What about quantum algorithms?

Questions?