Boundary value problems with measure data for nonlinear elliptic equations with absorption.

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Description of the problem.

\[-Lu + g(x, u) = 0, \quad \text{in } \Omega\]  
\[u = \mu, \quad \text{on } \partial \Omega,\]  
\[\Omega \text{ a domain in } \mathbb{R}^N, \quad \mu \text{ a Borel measure on } \partial \Omega,\]  
\[(i) \ L = \Delta, \quad (ii) \ L = \Delta + \frac{a}{\rho^2}, \quad \rho(x) = \text{dist} (x, \partial \Omega), \quad a < \frac{1}{4},\]  
\[g \in C(\mathbb{R}^N \times \mathbb{R}), \quad g \uparrow, \quad g \text{ odd, convex on } [0, \infty) \quad \lim_{t \to \infty} g(x, t)/t = \infty.\]  

*Notation* \((g \circ u)(x) = g(x, u(x)).\)
For $\mu \in \mathcal{M}(\partial D)$ (space of finite Borel measures) $u$ is a solution of the boundary value problem (1-2) with $L = \Delta$ if

(a) $u \in L^1_{loc}(\Omega), \ g \circ u \in L^1_{loc}(\Omega)$
(b) $u$ satisfies the equation in distribution sense,
(c) $u$ has m-boundary trace $\mu$. 
Definition of m-boundary trace:

If $\Omega$ is a bounded $C^2$ domain: let $\{D_n\}$ be an increasing sequence of uniform $C^2$ subdomains of $D$ such that $D_n \uparrow \Omega$.
Assume: $u \in L^1(\partial D_n)$ $\forall n$.
Then
\[ \int_{\partial D_n} hu \, dS \rightarrow \int_{\partial \Omega} h \, d\mu \quad \forall h \in C(\bar{\Omega}). \]

If $\Omega$ is a bounded Lipschitz domain: let $\{D_n\}$ be an increasing sequence of Lipschitz subdomains of $D$ such that $D_n \uparrow \Omega$.
Then,
\[ \int_{\partial D_n} hu \, d\omega_n \rightarrow \int_{\partial \Omega} h \, d\mu \quad \forall h \in C(\bar{\Omega}). \]
The absorption effect.

If \( g(x, t) \to \infty \) sufficiently fast as \( t \to \infty \) then the absorption effect balances the diffusion effect in such a way that

The set of solutions of equation (1) is uniformly bounded in every compact subset of \( D \).

A sharp criterion was supplied by Keller and Osserman (separately, 1957). It holds in arbitrary domains \( \Omega \).

Put \( G(x, s) = \int_0^s g(x, t) dt \). If

\[
\psi(x, a) := \int_a^\infty \frac{ds}{\sqrt{2G(x, s)}} < \infty \quad \forall a > 0,
\]

uniformly in compact subsets of \( \Omega \), then the above statement is valid.
If $\Omega$ is Lipschitz and

$$\sup_{x \in \Omega} \psi(x, a) < \infty, \quad a > 0,$$

there exists $c(g, \Omega)$ s.t., for every positive solution $u$

$$u(x) \leq c\phi(x, \rho(x)), \quad \phi(x, \cdot) = \psi^{-1}(x, \cdot).$$

$$\lim_{x \to \partial \Omega} u = \infty \implies \frac{1}{c} \phi(x, \rho(x)) \leq u(x) \leq c\phi(x, \rho(x)).$$

(M & Bandle 92 for $C^2$ domains; M 2014 for Lipschitz domains.)

Examples:

$$g(t) = |t|^{q-1}t, \quad q > 1 \implies \phi(s) = s^{-\frac{2}{q-1}}$$

$$g(t) = \max(e^t - 1, 0), \implies \phi(s) \sim |\ln s|.$$
Elementary facts about the boundary value problem.

\[-\Delta u + g \circ u = 0 \text{ in } \Omega, \quad u = \nu \text{ on } \partial \Omega, \quad (\nu \in \mathcal{M}(\partial \Omega)).\]

(a) **Uniqueness**: there exists at most one solution.

(b) **Monotonicity**: \(\nu_1 < \nu_2 \rightarrow u_1 \leq u_2\).

(c) **A-priori estimate**: \(\|u\|_{L^1} + \|g \circ u\|_{L^1} \leq c(g, \Omega) \|\nu\|_{\mathcal{M}}\).

(d) **Existence for \(L^1\) data**.

(Brezis & Strauss 72, Brezis 75; see also Brezis & Benilan 2003.)
A positive supersolution of the equation is moderate if

\[ \int_{\Omega} (g \circ u) \rho \, dx < \infty. \]  

(Mod)

**Proposition.**

(i) Every moderate positive supersolution has a measure boundary trace.

(ii) If \( u \) is a positive solution of the equation:

\[ u \text{ is moderate} \iff u \text{ is dominated by an harmonic function} \iff u \text{ has a measure boundary trace.} \]

**Definition**

A measure is **good** if the boundary value problem has a solution.

A nonlinearity is **subcritical** if every \( \nu \in \mathcal{M}(\partial \Omega) \) is good.
Three basic problems

I Characterization of good measures.

II Removable singularities

III Classification of positive solutions of the equation in terms of their behavior at the boundary.

The latter requires a proper definition of boundary trace for positive non-moderate solutions of the equation.
These problems have been studied since the early 90’s focusing, in particular, on the equation

$$-\Delta u + u^q = 0, \quad q > 1,$$

(EQ)

in smooth domains. The study was carried on by probabilists and analysts in parallel. Regarding (EQ) these problems have been resolved, in part only recently. For more general nonlinearities and non-smooth domains there are still many challenging open questions.
Subcritical equations.

**Definition.** A measure \( \nu \in \mathcal{M}_+(\partial \Omega) \) is perfect if

\[
\int_{\Omega} g \circ \mathbb{P}[|\nu|] \rho \, dx < \infty.
\]

**Lemma.** If \( \nu \) is perfect then \( \nu \) is good.

But not every good measure is perfect. It is easy to construct \( L^1 \) functions that are not perfect measures.
Theorem. Consider the equation $-\Delta u + g \circ u = 0$ in a bounded Lipschitz domain $\Omega$.

(i) Assume the basic assumptions on $g$ (inc. $g(x, \cdot)$ is convex on $\mathbb{R}_+$ and odd). In addition suppose that, for every positive solution of the equation, $g$ satisfies the following

$$g(u) \leq c(g, \Omega)u\rho^{-2}. \quad (*)$$

Then the condition

$$\int_{\Omega} g(x, \alpha P(x, y))\rho(x)dx < \infty \quad \forall y \in \partial \Omega, \alpha > 0, \quad (\text{SC})$$

is necessary and sufficient for $g$ to be subcritical.

(ii) Suppose that $g(x, t) = \rho(x)^\beta h(t), h \in C(R)$, odd and monotone increasing, $\beta > -2$. If $\Omega$ is of class $C^2$ then (SC) is necessary and sufficient for subcriticality. In addition, the solution is stable w.r. to weak convergence of data.

(M 2014)
**Remark.** If $|g(x, t)| \leq |\bar{g}(x, t)|$, $(x, t) \in \Omega \times \mathbb{R}$ and $\bar{g}$ is subcritical then $g$ is subcritical.

(Gmira & Veron, 1991) provided a *sufficient* condition for subcriticality in $C^2$ domains. If $g$ is independent of $x$ it is equivalent to (SC). In the case $g(t) = t^q$ this condition reduces to $1 < q < q_c = \frac{N+1}{N-1}$ and it was proved that, in this case, it is also necessary.

A necessary and sufficient condition for subcriticality of $g(x, t) = \rho(x)^\beta t^q$, $\beta \geq 0$, in $C^2$ domains was obtained by (M & Veron, 2003). Other results of this type - in $C^2$ domains - were also derived in (M & Veron, 2004) and (Bhakta & M, 2014).
Characterization of good measures for (EQ).

Consider the supercritical case, $q \geq q_c$.

**A. Removable singularities.** Let $u$ be a positive solution of (EQ) vanishing on the boundary outside a compact set $F$. If $C_{2/q,q'}(F) = 0$ then $u \equiv 0$. The condition is sharp.

**B. Good measures** Let $\nu \in \mathcal{M}(\partial \Omega)$. The boundary value problem

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \quad u = \nu \quad \text{on } \partial \Omega,$$

has a solution if and only if $\nu$ vanishes on sets of $C_{2/q,q'}$ capacity zero.
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has a solution if and only if \( \nu \) vanishes on sets of \( C_{2/q,q'/q'} \) capacity zero.

These results have been established during the 90’s, by probabilistic methods (Le Gall, Dynkin and Kuznetsov for \( 1 \leq q \leq 2 \)) and, in parallel, by analytic methods (M & Veron, \( q > 1 \)).
Recently these results have been extended to a general class of nonlinearities and Lipschitz domains (Ancona & M 2013).

Aside from the basic conditions on $g$ it is assumed that, for every positive solution $u$ of equation (1),

$$g(x, u) \leq c(g, \Omega) u(x) \rho(x)^{-2}. \quad (*)$$

This is valid for a large class of nonlinearities.

Examples:

$$\rho(x)^{\beta} h(t), \quad \beta > 0$$

where $h$ could be almost every function satisfying the K–O condition.
Under this additional assumption results (A) and (B) remain valid w.r. to
\[-\Delta u + g \circ u = 0,\]
in bounded Lipschitz domains, provided that the Bessel capacity $C_{2/q, q'}$ is replaced by the following:

$$C_{g, \Delta}(F) = \sup\{\tau(F) : \tau \in \mathcal{M}_+(\partial \Omega), \int_{\Omega} (g \circ K[\tau] \psi \, dx \leq 1\}$$

for every Borel set $F \subset \partial \Omega$. Here $K$ is the Martin kernel for $-\Delta$ in $\Omega$ and $\psi$ is the first (positive) eigenfunction normalized at some point $x_0 \in \Omega$. 

An important ingredient is a new result on representation of measures absolutely continuous w.r. to a capacity. This result applies in particular to $C_{g, \Delta}$ and extends results of Feyel & de la Pradelle, Baras & Pierre and Dal Maso that apply only to Bessel capacities.
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BVP with positive unbounded measure data in subcritical case.

I. The boundary trace
Recall that, if $u$ is a positive solution of (EQ) then $u$ has a boundary trace in $\mathcal{M}_+(\partial\Omega)$ if and only if

$$\int_{\Omega} u^q \rho \, dx < \infty.$$  
(Mod)

Motivated by this fact here is a first definition of boundary trace: (M & Veron, 1996)

**Definition**
A point $y \in \partial\Omega$ is regular relative to the positive solution $u$ if there exists a neighborhood $Q$ of $y$ such that:

$$\int_{Q \cap \partial\Omega} u^q \rho \, dx < \infty.$$  

A point is singular relative to $u$ if it is not regular.

The set of regular points $R(u)$ is open. Its complement is denoted by $S(u)$.
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Recall that, if $u$ is a positive solution of (EQ) then $u$ has a boundary trace in $M_+(\partial \Omega)$ if and only if

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**Definition** A point $y \in \partial \Omega$ is *regular* relative to the positive solution $u$ if there exists a neighborhood $Q$ of $y$ such that:

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A point is *singular* relative to $u$ if it is not regular.

The set of regular points $\mathcal{R}(u)$ is open. Its complement is denoted by $S(u)$.
It follows that the boundary trace of $u$ can be described by a pair $(F, \mu)$ where $F \subset \partial \Omega$ is a closed set $\mu$ is a Radon measure on $\partial \Omega \setminus F$. Alternatively we can describe the boundary trace as a measure $\bar{\mu}$ s.t. $\bar{\mu} = \infty$ on $F$ and $\bar{\mu}$ is Radon on $\partial \Omega \setminus F$. The set of positive measures of this type is denoted by $\mathcal{B}_{\text{reg}}$. 
II. Existence and uniqueness.

**Theorem** (M & Veron 1996) If $1 < q \leq q_c$ then the boundary value problem

$$-\Delta u + u^q = 0 \quad \text{in } \Omega, \quad u = \bar{\mu} \quad \text{on } \partial \Omega,$$

has a unique solution for every $\bar{\mu} \in \mathcal{B}_{\text{reg}}$. 

A probabilistic definition of trace and a proof of existence and uniqueness for $q = 2$, $N = 2$, was given by Le Gall in 1995. The proof used the 'Wiener snake' technique. In 1997, Le Gall showed that the uniqueness for $\mu \in \mathcal{B}_{\text{reg}}$ fails if $q_c \leq q$. 

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Boundary values with measure data
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BVP with positive unbounded measure data in supercritical case.

I. The boundary trace.
Dynkin (1998) introduced the useful concept of \( \sigma \)-moderate solution:

*a positive solution of the equation is \( \sigma \)-moderate if it is the limit of an increasing sequence of moderate solutions.*

Theorem. The boundary value problem for (EQ) has a unique solution in the family of \( \sigma \)-moderate solutions. (Dynkin & Kuznertsov 1998 for \( q_{c} \leq q \leq 2 \); M & Veron 2004/7 for every \( q \geq q_{c} \).)
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Following Le Gall’s observation two definitions of boundary trace were proposed for (EQ) in the supercritical case by Dynkin & Kuznertsov for \(q_c \leq q \leq 2\) and later by M & Veron for every \(q \geq q_c\). For \(\sigma\)-moderate solutions it is evident that the two definitions agree up to a set of \(C_{2/q,q'}\)-capacity zero. Later Verbitsky showed that, in fact, they agree in this sense for every positive solution.
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**Theorem.** The boundary value problem for (EQ) has a unique solution in the family of σ-moderate solutions.

(Dynkin & Kuznertsov 1998 for $q_c \leq q \leq 2$; M & Veron 2004/7 for every $q \geq q_c$).
To complete the classification of positive solutions we have to address the question:

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**Proposition.** Let $F \subset \partial \Omega$ be a compact set and let $\mathcal{U}(F)$ denote the family of solutions of (EQ) that are continuous in $\bar{\Omega} \setminus F$ and vanish on $\partial \Omega \setminus F$. Let $U_F = \sup \mathcal{U}(F)$. Then $U_F$ is a solution that vanishes on $\partial \Omega \setminus F$.

$U_F$ is called the **maximal solution** relative to $F$. 
M’selati (Ph.D. thesis 2002 published in Memoirs AMS, 2004) proved: If $q = 2$, $N \geq 3$, then every positive solution is $\sigma$-moderate.

M’selati used the Wiener snake technique of Le Gall combined with analytic methods.

M & Veron (JEMS 2003) proved: For every compact set $F \subset \partial \Omega$ and all $q \geq q_c$ the maximal solution $U_F$ is $\sigma$-moderate. The proof was based on sharp capacitary estimates for maximal solutions.

Dynkin (Colloquium Publications, AMS 2004) proved: For $q_c \leq q \leq 2$, every positive solution of (EQ) is $\sigma$-moderate. The proof used the two results mentioned above.

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M (J.D’Anal. 2012) proved: 
For all \( q \geq q_c \), every positive solution of (EQ) is \( \sigma \)-moderate.
The proof falls into two main parts:

(i) For every $q \geq q_c$, every positive solution of (EQ) dominates a positive moderate solution.

(ii) Let $u^* := \sup\{w : w \text{ is a } g\text{-moderate solution, } 0 < w \leq u\}$. Then $u^*$ is (obviously) $\sigma$-moderate and (non-obviously) $u^* = u$. 
The main idea of the proof is as follows. Let $u$ be a positive solution of (EQ) and denote

$$V = u^{q-1}.$$ 

Observe that $u$ satisfies:

$$L^V u := -\Delta u + Vu = 0.$$ 

The Keller–Osserman estimate implies that

$$V(x) \leq c\text{dist}^{-2}(x, \partial \Omega).$$ 

Therefore one can apply to $L^V$ classical results on Schrödinger equations.
A basic result of Ancona (Annals 87) states that any $L^V$ harmonic function - in particular $u$ - can be represented in the form

$$u(x) = \int_{\partial \Omega} K^V(x, \zeta) d\nu(\zeta) \quad \forall x \in \Omega.$$ 

where $\nu \in \mathcal{M}_+(\partial \Omega)$. Here $K^V$ denotes the *Martin kernel* for $L^V$ in $\Omega$. 

In the case of a solution $u$ of (EQ) it can be shown that $\nu$ vanishes on sets of $\mathcal{C}_2/\mathcal{Q}$, $\mathcal{Q}'$ capacity zero. Therefore, by a result of Baras and Pierre $\nu$ is the limit of an increasing sequence of measures $\{\nu_n\} \subset W^{-2/q, q} + (\partial \Omega)$. Every measure in this space is a perfect measure.
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The crucial step is the following:

**Lemma** If $\tau \in W^{-2/q,q}_+(\partial \Omega)$ then, either $\min(u, P[\tau]) > 0$ or $\tau \perp \nu$. 

This implies that $\min(u, P[\tau]) > 0$ and proves the first step. The proof of the second step relies, among other things, on the fact that the maximal solution is $\sigma$-moderate.
The crucial step is the following:

**Lemma** If $\tau \in W_+^{-2/q,q}(\partial \Omega)$ then, either $\min(u, P[\tau]) > 0$ or $\tau \perp \nu$.

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