An introduction to quantum stochastic calculus

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What is Quantum Probability?

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- Quantum probability is the generalisation of the classical theory of probability made necessary by the noncommutative multiplication of quantum observables, which are usually represented by self-adjoint operators in a Hilbert space.
- For example the *momentum* and *position* observables p and q for a one-dimensional particle satisfy the Heisenberg commutation relation

$$[p, q] = -i\frac{h}{2\pi}$$

where h is Planck's constant,

$$h = 6.626 \times 10^{-34} m^2 kg / s,$$

which is quite small in everyday units. It is more convenient to take $h = 4\pi$ so that

$$[p, q] = -2i$$

but to remeber that these are not everyday units.

- Some things don't work in quantum probability. For example if you try to define a *joint probability distribution* $\rho_{p,q}$ for such a canonical pair (p, q) by

$$\int_{\mathbb{R}^2} e^{i(xu+yv)}\rho_{p,q}(u, v)\, du dv = \left\langle e^{i(xp+yq)} \right\rangle_{\psi} =: \left\langle \psi, e^{i(xp+yq)}\psi \right\rangle,$$

you find a nice joint Gaussian distribution

$$\rho_{p,q}(u, v) = (2\pi)^{-1} e^{-\frac{1}{2}\left(u^2+v^2\right)}$$

when ψ is the ground state of the oscillator $\frac{1}{2}\left(p^2 + q^2\right)$, but when it's the first excited state you find

$$\rho_{p,q}(u, v) = (2\pi)^{-1}\left(u^2 + v^2 - 1\right) e^{-\frac{1}{2}\left(u^2+v^2\right)}$$

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which may look OK till you try to calculate the probability that (p, q) lies inside the unit circle.

- Another thing that doesn't always work is the notion of conditional expectation and the associated probabilistic concept of martingale.

- Notice that if $(p_1, q_1), (p_2, q_2), \ldots$ is a sequence of canonical pairs satisfying the canonical commutation relations

$$[p_j, q_k] = -2i\delta_{j,k}, [p_j, p_k] = [q_j, q_k] = 0,$$

then. for $N = 1.2, \ldots$

$$\left[\frac{p_1 + p_2 + \cdots + p_N}{\sqrt{N}}, \frac{q_1 + q_2 + \cdots + q_N}{\sqrt{N}}\right] = -2i.$$

Assuming that the input canonical pairs are "*independent, identically distributed* and of *zero means* and *finite variance*", this new canonical pair converges in distribution as $N \to \infty$ to a Gaussian limit, in the manner of the central limit theorem.

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- But what can these words mean and what is the meaning of convergence in distribution when there is no joint probability distribution?

The answer lies in the Stone-von-Neumann uniqueness theorem, according to which there is, up to unitary equivalence, exactly one irreducible representation of the canonical commutation relations for finitely many degrees of freedom.
In the one degree of freedom case, denoting this representation by (p_0, q_0) and the carrier Hilbert space by $\mathcal{H}_0$, a consequence is that, for an arbitrary canonical pair (p, q) in a Hilbert space $\mathcal{H}$ and an arbitrary state in $\mathcal{H}$, there exists a unique unit vector $\Psi_{(p,q)} \in \mathcal{H}_0 \otimes \bar{\mathcal{H}}_0$ which is invariant under the *flip*, the conjugate-unitary map from $\mathcal{H}_0 \otimes \bar{\mathcal{H}}_0$ to itself for which each $\psi \otimes \bar{\phi} \mapsto \phi \otimes \bar{\psi}$, such that

$$\left\langle \Psi_{(p,q)}, e^{i(xp_0+yq_9)}\Psi_{(p,q)} \right\rangle = \left\langle e^{i(xp+yq)} \right\rangle .$$

$\Psi_{(p,q)}$ is called the *distribution vector* of (p, q) (in this state). Two canonical pairs are *identically distributed* if they have the same distribution vector. A sequence of canonical pairs *converges in distribution* if the sequence of distribution vectors converges in the usual Hilbert space sense.

To define independence we first define the *joint distribution vector* $\Psi_{(p,q),(p',q')} \in \mathcal{H}_0 \otimes \bar{\mathcal{H}}_0 \otimes \mathcal{H}_0 \otimes \bar{\mathcal{H}}_0$ of two mutually commuting canonical pairs using the two-dimensional Stone-von-Neumann theorem analogously. Then (p,q) and (p',q') are *independent* if $\Psi_{(p,q),(p',q')} = \Psi_{(p,q)} \otimes \Psi_{(p',q')}$.
A canonical pair (p,q) is of *zero mean and variance* $\sigma^2 \geq 1$ if $\langle p \rangle = \langle q \rangle = 0$ and the covariance matrix is of form

$$\left\langle \begin{bmatrix} \langle p^2 \rangle & \langle pq \rangle \\ \langle qp \rangle & \langle q^2 \rangle \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} \sigma^2 & -i \\ i & \sigma^2 \end{bmatrix} \right\rangle$$

Finally we say that such a canonical pair (p,q) is *Gaussian* :
If $\sigma^2 = 1$, if $\Psi_{(p,q)} = \psi_0 \otimes \bar{\psi}_0$ where ψ_0 is the harmonic oscillator ground state.
If $\sigma^2 > 1$, if $\Psi_{(p,q)} = \left(\sum_{n=0}^{\infty} e^{-2\beta(2n+1)} \right)^{-1/2} \sum_{n=0}^{\infty} e^{-\beta(2n+1)} \psi_n \otimes \bar{\psi}_n$ where ψ_n is the *n*-th excited oscillator state and the reciprocal temperature is given by $\coth \frac{\beta}{4} = \sigma^2$.

Notes (1) Provided all second moments are finite, an arbitrarily distributed canonical pair (p, q) can be reduced to one of zero mean and variance σ^2 by an inhomogeneous linear canonical transformation $(p, q) \mapsto (\alpha p + \beta q + a, \gamma q + \delta q + b)$, where $\alpha\delta - \beta\gamma = 1$.
(2) For sequence of iid canonical pairs $(p_1, q_1), (p_2, q_2), \ldots$of zero means and finite variance σ^2 the sequence $p_1, p_2, \ldots$ of mutually commuting observables is iid of zero means and finite variance σ^2 in the classical sense. In particular, by the classical central limit theorem, as $N \longrightarrow \infty$, $(N!)^{-\frac{1}{2}} (p_1 + p_2 + \cdots + p_N)$ converges in distribution to the standard Gaussian limit distribution $N(0, \sigma^2)$. Likewise for $q_1, q_2, \ldots$.The limit state for the sequence of pairs is consistent with this.
(3) By Donsker's invariance principle, the process $P_N(t)$ defined by

$$P_N(t) = N^{-\frac{1}{2}} \left(p_1 + p_2 + \cdots + p_{[Nt]} + (Nt - [Nt]) p_{[Nt]+1} \right)$$

must converge to a Brownian motion P of variance σ^2. Similarly

$$Q_N(t) = N^{-\frac{1}{2}} \left(q_1 + q_2 + \cdots + q_{[Nt]} + (Nt - [Nt]) q_{[Nt]+1} \right)$$

converges to a Brownian motion Q, also of variance σ^2.

Quantum planar Brownian motion.

- Since

$$\begin{aligned}
& \left[P_N(s), Q_N(t)\right] \\
= \;& N^{-1}\left[p_1 + p_2 + \cdots + p_{[Ns]}, q_1 + q_2 + \cdots + q_{[Nt]}\right] \\
& + N^{-1}\left[\left(Ns - [Ns]\right) p_{[Ns]+1}, \left(Nt - [Nt]\right) q_{[Nt]+1}\right] \\
= \;& -2i\frac{[N(s \wedge t)]}{N} - 2i\frac{\left(Ns - [Ns]\right)\left(Nt - [Nt]\right)}{N}\delta_{[Ns]+1,[Nt]+1} \\
\underset{N\to\infty}{\longrightarrow} \;& -2is \wedge t
\end{aligned}$$

we expect the limit Brownian motions P and Q to satisfy the commutation relation

$$\left[P(s), Q(t)\right] = -2is \wedge t.$$

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\begin{aligned}
& \left[P_N(s), Q_N(t)\right] \\
= \; & N^{-1}\left[p_1+p_2+\cdots+p_{[Ns]}, q_1+q_2+\cdots+q_{[Nt]}\right] \\
& +N^{-1}\left[(Ns-[Ns])\, p_{[Ns]+1}, (Nt-[Nt])\, q_{[Nt]+1}\right] \\
= \; & -2i\frac{[N(s\wedge t)]}{N}-2i\frac{(Ns-[Ns])(Nt-[Nt])}{N}\delta_{[Ns]+1,[Nt]+1} \\
& \underset{N\to\infty}{\to} -2is\wedge t
\end{aligned}
$$

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\left[P(s), Q(t)\right] = -2is\wedge t.
$$

- How can we construct two such Brownian motions? We again distinguish the cases $\sigma=1$ and $\sigma>1$.

The *Fock space* $\mathcal{F}(\mathcal{H})$ over a Hilbert space $\mathcal{H}$ is usually defined by physicists as the Hilbert space infinite direct sum

$$\mathcal{F}(\mathcal{H}) = \mathbb{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H})_{\mathrm{sym}} \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H})_{\mathrm{sym}} \oplus \cdots$$

of symmetric parts of the n-fold tensor product of $\mathcal{H}$ with itself.
A more useful definition for our purposes is that $\mathcal{F}(\mathcal{H})$ is a Hilbert space generated by a family $(e(f))_{f \in \mathcal{H}}$ of socalled *exponential vectors,* satisfying

$$\langle e(f), e(g) \rangle_{\mathcal{F}(\mathcal{H})} = \exp \langle f, g \rangle_{\mathcal{H}}.$$

The connection between the two definitions is made by realising the exponential vectors in the first definition as

$$e(f) = 1 \oplus f \oplus \frac{f \otimes f}{\sqrt{2!}} \oplus \frac{f \otimes f \otimes f}{\sqrt{3!}} \oplus \cdots \ .$$

One reason why this second definition is useful is that it makes clear the *exponential property* of Fock spaces, that there is a natural isomorphism allowing us to identify the Fock space over a direct sum with the tensor product of the Fock spaces over the summands;

$$\mathcal{F}(\mathcal{H}_1 \oplus \mathcal{H}_2) = \mathcal{F}(\mathcal{H}_1) \otimes \mathcal{F}(\mathcal{H}_2), \text{ with } e(f_1 \oplus f_2) = e(f_1) \otimes e(f_2).$$

To construct two Brownian motions P and Q satisfying

$$[P(s), Q(t)] = -2i \tag{1}$$

of unit variance $\sigma^2 = 1$ in the Fock space $\mathcal{F}\left(\mathcal{L}^2(\mathbb{R}_+)\right)$ over $\mathcal{L}^2(\mathbb{R}_+)$, first define the mutually adjoint *creation* and *annihilation processes* $A^\dagger = \left(A^\dagger(t)\right)_{t\geq 0}$ and $A = (A(t))_{t\geq 0}$ by their actions on exponential vectors

$$A^\dagger(t)\, e(f) = \frac{d}{dz} e\left(f + z\chi_{[0,t[}\right),\ A(t)\, e(f) = \left\langle \chi_{[0,t[}, f\right\rangle\, e(f).$$

These satisfy the commutation relations

$$\left[A^\dagger(s), A^\dagger(t)\right] = [A(s), A(t)] = 0, \left[A(s), A^\dagger(t)\right] = s \wedge t, \tag{2}$$

in the sense that for example, for arbitrary exponential vectors,

$$\left\langle A^\dagger(s)\, e(f), A^\dagger(t)\, e(g)\right\rangle - \langle A(s)\, e(f), A(t)\, e(g)\rangle = s \wedge t\ \langle e(f), e(g)\rangle$$

Define P and Q by

$$P = i\left(A^\dagger - A\right), Q = A + A^\dagger$$

Then (2) implies (1) and commutativity of the process P (likewise Q).

Moreover in the *vacuum state* $e(0)$ they are both Brownian motions of unit variance in the sense of the following

Theorem: *Denote by* $(\Omega, \mathbb{F}, \mathbb{W})$ *the standard Wiener probability space and by* $X = (X(t))_{t \geq 0}$ *the standard realisation on it of unit variance Brownian motion, so that for* $\omega \in \Omega$, $X(t)(\omega) = \omega(t)$ *and* F *is the* σ*-field generated by the* $X(t)$. *Then there exists a unique Hilbert space isomorphism* D_P *(resp.* D_Q*) from the Fock space* $\mathcal{F}(\mathcal{L}^2(\mathbb{R}_+))$ *onto* $L^2(\Omega, \mathbb{F}, \mathbb{W})$ *with the properties*

$$\begin{aligned} D_P e(0)(\omega) \text{ (resp. } D_Q e(0)(\omega)\text{)} &= 1 \text{ for all } \omega \in \Omega, \\ D_P P(t) D_P^{-1} \text{(resp. } D_Q Q(t) D_Q^{=1}\text{)} &= \mathrm{mult}_{X(t)} \text{ for all } t \in \mathbb{R}_+ \end{aligned}$$

where $\mathrm{mult}_{X(t)}$ *denotes the operator of multiplication by* $X(t)$.

But because P and Q don't commute with eachother they cannot be simultaneously diagonalized; $D_P \neq D_Q$.

Note: Although they don't commute, they have a property amounting to stochastic independence, namely factorization of joint characteristic functions, $\left\langle e^{i(\Sigma_j x_j P(s_j) + \Sigma_k y_k Q(t_k))} \right\rangle = \left\langle e^{i\Sigma_j x_j P(s_j)} \right\rangle \left\langle e^{i\Sigma_k y_k Q(t_k)} \right\rangle$, so we regard them as jointly a *quantum planar Brownian motion*.

Now assume $\sigma^2 > 1$. Then we can write

$$\sigma^2 = \alpha^2 + \beta^2 \text{ where } \alpha^2 - \beta^2 = 1.$$

In the tensor product $\mathcal{F} \otimes \bar{\mathcal{F}}$ of the Fock space $\mathcal{F} = \mathcal{F}\left(\mathcal{L}^2\left(\mathbb{R}_+\right)\right)$ with its dual Hilbert space, equipped with the unit vector $e(0) \otimes \overline{e(0)}$, define

$$P_\sigma(t) = \alpha P(t) \otimes \bar{I} + \beta I \otimes \bar{P}(t),\ Q_\sigma(t) = \alpha Q(t) \otimes \bar{I} + \beta I \otimes \bar{Q}(t),$$

where for example $\bar{P}(t)$ is the dual operator to $P(t)$, $\bar{P}(t)\bar{\psi} = \overline{P(t)\psi}$. Then, in the product state $e(0) \otimes \overline{e(0)}$, $\alpha P \otimes \bar{I}$ is a Brownian motion of variance α^2 and $\beta I \otimes \bar{P}$ is an *independent* Brownian motion of variance β^2 which commutes with it. Hence the sum $\alpha P \otimes \bar{I} + \beta I \otimes \bar{P}$ is itself a Brownian motion of variance $\alpha^2 + \beta^2 = \sigma^2$. Similarly $\alpha Q \otimes \bar{I} + \beta I \otimes \bar{Q}$ is a Brownian motion of variance σ^2. On the other hand

$$\begin{aligned} \left[P_\sigma(s), Q_\sigma(t)\right] &= \alpha^2 \left[P(s), Q(t)\right] \otimes \bar{I} + \beta^2 I \otimes \left[\bar{P}(s), \bar{Q}(t)\right] \\ &= \left(\alpha^2 - \beta^2\right)\left(-2is \wedge t\right) = -2is \wedge t \end{aligned}$$

since $\left[\bar{P}(s), \bar{Q}(t)\right] = \overline{\left[P(s), Q(t)\right]} = \overline{-2is \wedge t} = 2is \wedge t$.
The (P_σ, Q_σ) with $\sigma > 1$ is technically easier to work with than (P, Q) because $e(0) \otimes \overline{e(0)}$ is cyclic *and separating* for $\{P_\sigma, Q_\sigma\}''$.

Quantum stochastic calculus.

Brownian motion X is non-differentiable. Intuitively this is because (for $s > t$) $X(s) - X(t)$ is $N(0, s-t)$, so it is of order of magnitude $\sqrt{s-t}$. So when you form the difference quotient $\frac{X(s)-X(t)}{s-t}$ there can be no sensible limit as $s \to t$.

However the two components P and Q of quantum planar Brownian motion show some hints of differentiability. For example $Q = A^{\dagger} + A$ and so, for exponential vectors $e(f)$ and $e(g)$,

$$\begin{aligned}
\langle e(f), Q(t)e(g)\rangle &= \langle A(t)e(f), e(g)\rangle + \langle e(f), A(t)e(g)\rangle \\
&= \left(\overline{\left\langle \chi_{[0,t[}, f\right\rangle} + \left\langle \chi_{[0,t[}, g\right\rangle\right)\langle e(f), e(g)\rangle \\
&= \int_0^t \left(\bar{f}(s) + g(s)\right)\langle e(f), e(g)\rangle
\end{aligned}$$

which suggests that, at least for well-behaved f and g,

$$\left\langle e(f), \frac{dQ(t)}{dt}e(g)\right\rangle = \left(\bar{f}(t) + g(t)\right)\langle e(f), e(g)\rangle.$$

Unfortunately this does not define an operator $\frac{dQ(t)}{dt}$. But it does suggest a theory of operator-valued quantum stochastic integrals, of the form

$$M(t) = \int_0^t \left(F(s)\, dA^\dagger(s) + G(s)\, dA(s) + H(s)\, dT(s) \right)$$

for which the "*first fundamental formula*"

$$\langle e(f), M(t)\, e(g) \rangle = \int_0^t \langle e(f), (F(s)\, \bar{f}(s) + G(s)\, g(s) + H(s))\, e(g) \rangle\, ds$$

holds, **provided that** $dA^\dagger(s)$ may be commuted through $F(s)$ to reach the left of the inner product. Assuming that differentials like $dA^\dagger$ "point into the future" this requires that the integrand processes $F = (F(t))_{t \geq 0}$ are *adapted*, meaning that each operator $F(t)$ is of form

$$F(t) = F_t \otimes I^t,$$

in so far as the Fock space

$$\mathcal{F}\left(L^2(\mathbb{R}_+)\right) = \mathcal{F}\left(L^2([0,t[)\right) \otimes \mathcal{F}\left(L^2([t,\infty[)\right)$$

- Everyone agrees how to define the stochastic integral $\int_0^t F(s)\, dL(s)$, $L \in \{A^\dagger, A, T\}$ of an *elementary* adapted process

$$F(t) = \chi_{[a,b[}(t)\, F(a) = \chi_{[a,b[}(t)\, F_a \otimes I^a;$$

it is $\int_0^t F(s)\, dL(s) = F(a)\,(L(t \wedge b) - L(t \wedge a))$ and hence by additivity also of a *simple* adapted process, ie one taking only finitely many different values.

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- **Note**: The products of unbounded operators occurring here can be defined rigorously on the exponential domain as tensor product operators.
- The first fundamental formula is easily proved for such integrands:
- **Theorem**: For simple adapted processes F, G and H

$$\left\langle e(f), \int_0^t \Big(F(s)\, dA^\dagger(s) + G(s)\, dA(s) + H(s)\, dT(s)\Big)\, e(g) \right\rangle$$

$$= \int_0^t \langle e(f), (F(s)\, \bar{f}(s) + G(s)\, g(s) + H(s))\, e(g)\rangle\, ds.$$

The *second fundamental formula* is the heart of quantum stochastic calculus. It is a rule for expressing the product of two stochastic integrals as a sum of iterated stochastic integrals. The prototype is the rule for Brownian motion X :

$$\begin{aligned}(X(t))^2 &= \left(\int_{0\leq s<t} dX(s)\right)^2 = 2\int_{0\leq s<t} X(s)\,dX(s) + t \\ &= \int_{0\leq s_1<s_2<t} dX(s_1)\,dX(s_2) + \int_{0\leq s_1<s_2<t} dX(s_1)\,dX(s_2) \\ &\quad + \int_{0\leq s<t} dT(s) \qquad (3)\end{aligned}$$

where the unexpected additional "Itô correction" $t = \int_{0\leq s<t} dT(s)$ is rather mysterious in classical Itô calculus. In the quantum generalization it can be seen as a consequence of the basic commutation relations between the annihilation and creation processes. We write (3) in differential form as

$$d\left(X^2\right) = 2XdX + (dX)^2 \text{ where } (dX)^2 = dT.$$

To avoid multiplying unbounded operators, the second fundamntal formula places the two stochastic integrals on the two sides of the Hilbert space inner product between exponential vectors.

Thus let $M_j(t) = \int_0^t \left(F_j(s)\, dA^\dagger(s) + G_j(s)\, dA(s) + H_j(s)\, dT(s)\right)$, $j = 1, 2$ be two stochastic integrals where the F_j, G_j and H_j are (for the moment at least) simple processes, which we assume have adjoint processes defined on the exponential vectors.

$$\textbf{Theorem} : \langle M_1(t)\, e(f), M_2(t)\, e(g)\rangle$$

$$\begin{aligned} = \int_0^t & \{\langle (F_1(s)\,\bar{g}(s) + G_1(s)\, f(s) + H_1(s))\, e(f), M_2(s)\, e(g)\rangle \\ & + \langle M_1(s)\, e(f), (F_2(s)\,\bar{f}(s) + G_2(s)\, g(s) + H(s))\, e(g)\rangle \\ & + \langle F_1(s)\, e(f), F_2(s)\, e(g)\rangle\}\, ds. \end{aligned}$$

Proof. By manipulation of the commutation relations between the creation and annihilation processes.□

In differential form the second fundamental formula becomes

$$d\left(MM'\right) = \left(dM\right)M' + MdM' + dMdM'$$

where $M = M_1^\dagger$, $M' = M_2$ and for $dM = FdA^\dagger + GdA + HdT$ and $dM' = F'dA^\dagger + G'dA + H'dT$, $dMdM'$ is evaluated form the *quantum Itô product table*

	$dA^\dagger$	dA	dT
$dA^\dagger$	0	0	0
dA	dT	0	0
dT	0	0	0

,

or in the $\left(dP, dQ, dT\right)$ basis,

	dP	dQ	dT
dP	dT	$-idT$	0
dQ	idT	dT	0
dT	0	0	0

.

Note: The latter table contains the classical rule $\left(dX\right)^2 = dT$ twice over, with $X = P$ and $X = Q$. But it is manifestly noncommutative.

The real importance of the second fundamental formula is that it provides estimates which allow us to extend stochastic integration beyond simple processes as integrands.

For example, putting $g = f$ and $M_1 = M_2 = M$ and putting $G = H = 0$, we get, for $M(t) = \int_0^t F(s)\, dA^\dagger(s)$,

$$\begin{aligned} & \|M(t)\, e(f)\|^2 \\ = & \int_0^t \left\{ 2\,\mathrm{Re}\,\langle F(s)\, \bar{f}(s)\, e(f), M(s)\, e(f)\rangle + \|F(s)\, e(f)\|^2 \right\} ds. \end{aligned}$$

so $\dfrac{d}{dt}\|M(t)\, e(f)\|^2 = 2\,\mathrm{Re}\,\langle F(t)\, \bar{f}(t)\, e(f), M(t)\, e(f)\rangle + \|F(t)\, e(f)\|^2$

Using the Hilbert space inequality $2\,\mathrm{Re}\,\langle \psi_1, \psi_2\rangle \le \|\psi_1\|^2 + \|\psi_2\|^2$ we get

$$\frac{d}{dt}\|M(t)\, e(f)\|^2 \le |f(t)|^2 \|M(t)\, e(f)\|^2 + 2\|F(t)\, e(f)\|^2.$$

Multiplying by the integrating factor $e^{-\int_0^t |f(s)|^2 ds}$ we obtain the estimate

$$\left\| \int_0^t F(s)\, dA^\dagger(s)\, e(f) \right\|^2 \le 2e^{-\int_0^t |f(s)|^2 ds} \int_0^t \|F(s)\, e(f)\|^2\, ds.$$

This and similar (but simpler) inequalities for the creation and time integrals allow us to extend quantum stochastic integration from simple adapted integrands to adapted integrands F which are *locally square-integrable* in the sense that all seminorms of the form

$$\|F\|_{t,f} = \sqrt{\int_0^t \|F(s)\, e(f)\|^2\, ds}$$

are finite. The two fundamental formulas and the derived estimates hold for the extended integral.

Compare this with classical Itô calculus where the extension is based on the isometry $\mathbb{E}\left[\left|\int_0^t F(s)\, dX(s)\right|^2\right] = \int_0^t \mathbb{E}\left[|F(s)|^2\right] ds$.

In the non-Fock case $\sigma^2 > 1$ the extension uses two isometry relations

$$\begin{aligned}
\left\|\int_0^t F(s)\, dA_\sigma^\dagger(s)\, e(0) \otimes \overline{e(0)}\right\|^2 &= \alpha^2 \int_0^t \left\|F(s)\, e(0) \otimes \overline{e(0)}\right\|^2 ds, \\
\left\|\int_0^t F(s)\, dA_\sigma(s)\, e(0) \otimes \overline{e(0)}\right\|^2 &= \beta^2 \int_0^t \left\|F(s)\, e(0) \otimes \overline{e(0)}\right\|^2 ds
\end{aligned}$$

where $\sigma^2 = \alpha^2 + \beta^2, \alpha^2 - \beta^2 = 1$. Thus the non-Fock theory is simpler.

The gauge process.

Stochastic integrals against the creation and annihilation processes are always martingales;

$$\mathbb{E}\left[\int_0^t \left(FdA^{\dagger} + GdA\right) |\mathcal{N}_s\right] = \int_0^s \left(FdA^{\dagger} + GdA\right)$$

for $s < t$, where $\mathcal{N}_s$ is the von Neumann algebra $\left\{A^{\dagger}(r), A(r) \,|r \leq s\right\}$
Is the converse true? Can every martingale for the filtration of the quantum planar Brownian motion be represented as a stochastic integral?
Answer: **No** in the Fock case $\sigma^2 = 1$, **Yes** in the non-Fock case $\sigma^2 > 1$.
In the Fock case we define the *gauge process* $\Lambda = (\Lambda(t))_{t \geq 0}$ by

$$\Lambda(t)\, e(f) = \left.\frac{d}{dz} e\left(e^{z\chi_{[o,t[}} f\right)\right|_{z=0}.$$

The gauge process, also called the *number* process, is sometimes attributed by physicists to V P Belavkin but in fact it originates with myself and Parthasarathy [1].

Its matrix elements between exponential vectors are given by

$$\langle e(f), \Lambda(t) e(g) \rangle = \int_0^t \bar{f}(s) g(s) \, ds \, \langle e(f), e(g) \rangle .$$

We compare this with corresponding formulas for $\langle e(f), A^{\dagger}(t) e(g) \rangle$ and $\langle e(f), A(t) e(g) \rangle$ which are

$$\int_0^t \bar{f}(s) \, ds \, \langle e(f), e(g) \rangle , \int_0^t g(s) \, ds \, \langle e(f), e(g) \rangle .$$

And, just as $A^{\dagger} = \int dA^{\dagger}$ and $A = \int dA$ are martingales, **so too is** Λ, essentially because for $s < t$, if f and g vanish on $[s, \infty[$

$$\langle e(f), \Lambda(t) e(g) \rangle = \langle e(f), \Lambda(s) e(g) \rangle \text{ because } \int_0^t \bar{f}(r) g(r) \, dr = \int_0^s \bar{f}$$

But Λ cannot be expressed as a stochastic integral against $dA^{\dagger}$ and dA.

Instead we incorporate Λ as a new fundamental martingale in the Fock case. The first fundamental formula becomes

$$\left\langle e(f), \int_0^t \left(E(s)\, d\Lambda(s) + F(s)\, dA^{\dagger}(s) + G(s)\, dA(s) + H(s)\, dT(s)\right) e(g \right\rangle$$

$$= \int_0^t \langle e(f), (E(s)\bar{f}(s)g(s) + F(s)\bar{f}(s) + G(s)g(s) + H(s))\, e(g)\rangle\, ds.$$

The quantum Itô product table becomes

	$d\Lambda$	$dA^{\dagger}$	dA	dT
$d\Lambda$	$d\Lambda$	$dA^{\dagger}$	0	0
$dA^{\dagger}$	0	0	0	0
dA	dA	dT	0	0
dT	0	0	0	0

.

In this way the classical *Poisson process* of rate l can be incorporated into quantum stochastic calculus as $\Pi_l = \Lambda + \sqrt{l}A^{\dagger} + \sqrt{l}A + lT$, for which $(d\Pi_l)^2 = d\Pi_l$.

But in the non-Fock case when $\sigma > 1$ there is no gauge process.

Quantum stochastic differential equations for unitary processes.

In physical applications of quantum stochastic calculus the physics usually takes place in a separate "initial" Hilbert space $\mathcal{H}$ which is coupled to the "noise" space carrying the calculus as a Hilbert space tensor product. Consider the quantum stochastic differential equation

$$(\mathrm{id} \otimes d)\, U = U \left(E \otimes d\Lambda + F \otimes dA^{\dagger} + G \otimes dA + H \otimes dT \right), \ U(0) = I$$

where E, F, G and H are bounded operators on $\mathcal{H}$. Writing

$$E \otimes d\Lambda + F \otimes dA^{\dagger} + G \otimes dA + H \otimes dT = \sum_{j=1}^{4} E_j \otimes d\Lambda_j$$

we always interpret such sde's as the equivalent integral equation

$$U(t) = I + \int_0^t U(s) \sum_{j=1}^{4} E_j \otimes d\Lambda_j(s) .$$

We may try to solve this iteratively, starting with $U_0(t) \equiv I$ and

$$U_N(t) = I + \int_0^t U_{N-1}(s) \sum_{j=1}^{4} E_j \otimes d\Lambda_j(s).$$

Thus

$$\begin{aligned} U_1(t) &= I + \sum_{j=1}^{4} E_j \otimes \int_0^t d\Lambda_j(s), U_2(t) = I + \sum_{j=1}^{4} E_j \otimes \int_0^t d\Lambda_j(s) \\ &+ \sum_{j_1, j_2=1}^{4} E_{j_1} E_{j_2} \otimes \int_{0<s_1<s_2<t} d\Lambda_{j_1}(s_1) d\Lambda_{j_2}(s_2), \ldots, \\ U_N(t) &= I + \sum_{M=1}^{N} \sum_{j_1, j_2, \ldots, j_M=1}^{4} E_{j_1} E_{j_2} \ldots E_{j_M} \\ &\otimes \int_{0<s_1<s_2<\cdots<s_M<t} d\Lambda_{j_1}(s_1) d\Lambda_{j_2}(s_2) \ldots d\Lambda_{j_M}(s_M) \end{aligned}$$

Provided that the coefficient operators E_j are bounded you can then use estimates for iterated stochastic integrals derived from the fundamental estimates to show that the limit

$$U(t) = I + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N = 1}^{4} E_{j_1} E_{j_2} \ldots E_{j_N} \otimes \int_{0<s_1<s_2<\cdots<s_N<t} d\Lambda_{j_1}(s_1)\, d\Lambda_{j_2}(s_2) \ldots d\Lambda_{j_N}(s_N)$$

is a well defined operator on the span of vectors of form $\psi \otimes e(f)$, and that it is a solution of the original sde and that the solution is unique.
If we start with the corresponding "left-driven" sde

$$(\mathrm{id} \otimes d)\, V = EV \otimes d\Lambda + FV \otimes dA^{\dagger} + GV \otimes dA + HV \otimes dT, \ \ V(0) = I$$

we obtain similarly the solution

$$V(t) = I + \sum_{N=1}^{\infty} \sum_{j_1, j_2, \ldots, j_N = 1}^{4} E_{j_N} E_{j_{N-1}} \ldots E_{j_1} \otimes \int_{0<s_1<s_2<\cdots<s_N<t} d\Lambda_{j_1}(s_1)\, d\Lambda_{j_2}(s_2) \ldots d\Lambda_{j_N}(s_N).$$

When is the solution U of

$$(\mathrm{id} \otimes d)\, U = U \left(E \otimes d\Lambda + F \otimes dA^\dagger + G \otimes dA + H \otimes dT \right), U(0) = I$$

unitary-valued? The adjoint process $U^\dagger$ satisfies

$$(\mathrm{id} \otimes d)\, U^\dagger = E^\dagger U^\dagger \otimes d\Lambda + G^\dagger U^\dagger \otimes dA^\dagger + F^\dagger U^\dagger \otimes dA + H^\dagger U^\dagger \otimes dT, \ \ U^\dagger (0)$$

By applying the Leibniz-Itô formula $d(MM') = (dM)\, M' + MdM' + dMdM'$ to the products $UU^\dagger$ and $U^\dagger U$ using the quantum Itô product rule, we find that the conditions

$$\begin{aligned} E + E^\dagger + EE^\dagger &= 0, F + G^\dagger + EG^\dagger = 0, \\ G + F^\dagger + GE^\dagger &= 0, H + H^\dagger + GG^\dagger = 0 \end{aligned}$$

are *sufficient* for coisometry ($UU^\dagger = I$) while the conditions

$$\begin{aligned} E^\dagger + E + E^\dagger E &= 0, G^\dagger + F + E^\dagger F = 0, \\ F^\dagger + G + F^\dagger E &= 0, H + H^\dagger + F^\dagger F = 0 \end{aligned}$$

are *necessary* for isometry ($U^\dagger U = I$).

But by using *backward-adapted quantum stochastic calculus*, which is completely equivalent to forward adapted for iterated integrals such as occur in the solutions above, you can show that both conditions are both *necessary and sufficient [2]*. Both are equivalent to the condition

$$(E, F, G, H) = \left(S - 1, L, -SL^{\dagger}, ih - \frac{1}{2} L^{\dagger} L \right), \; S \text{ unitary, } h \text{ self-adjoint}$$

which I understand is known nowadays in the physics communiity as the (S, L, h) formalism.

Quantum stochastic differential equations for flows.

References.

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