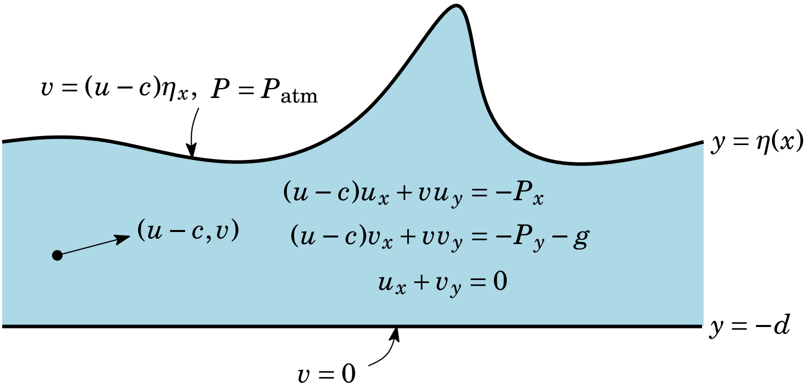
The Froude Number and Solitary Waves with Vorticity

Miles Wheeler

Brown University Department of Mathematics

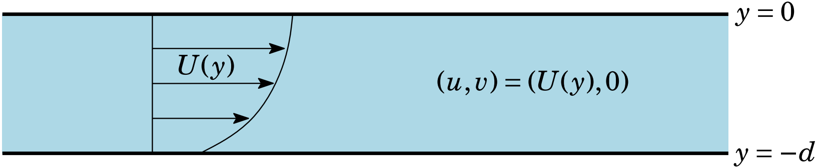
Notation and assumptions

2 space dimensions,
no surface tension,
steady traveling waves

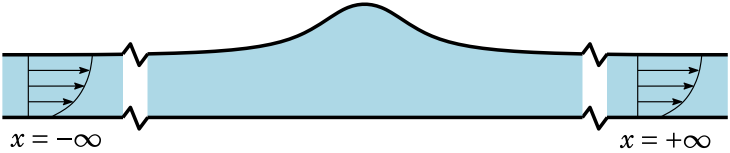


Solitary waves with vorticity

Shear flow solution: $\eta = v = 0$ and $u(x,y) = U(y)$.



Solitary wave: converges to the same shear flow as $x \to \pm\infty$.



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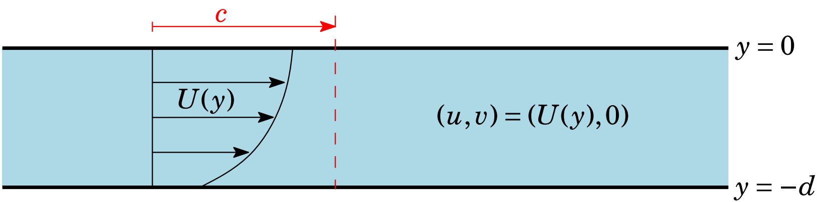

Special case

For irrotational waves, $U \equiv 0$ so the fluid is at rest at infinity.

Restrictions on U

We will always assume $U < c$, but otherwise $U(y)$ is arbitrary.

Small-amplitude solitary waves

Irrotational

Small-amplitude waves have Froude number $F = c/\sqrt{gd}$ slightly greater than 1, and

$$\eta(x) = c_1(F-1)\,\mathrm{sech}^2(c_2\sqrt{F-1}\,x) + \cdots \qquad (*)$$

(Lavrentiev '54, Friedrichs-Hyers '54, ...)

With vorticity

Define a "Froude number" F by

$$\frac{1}{F^2} = g\int_{-d}^{0} \frac{dy}{(c-U(y))^2}. \qquad (**)$$

Assuming $u < c$, again F is slightly bigger than 1 and $(*)$ holds.
(Ter-Krikorov '61, Hur '08a, Groves–Wahlén '08)

Note: If $U \equiv 0$, then $(**)$ simplifies to $F = c/\sqrt{gd}$.

Part I

Bounds on the Froude number (for large-amplitude waves)

The Froude number for irrotational waves

Numerics

Along the branch in Longuet-Higgins–Fenton '74, $1 < F < 1.286$.

Lower bound on the Froude number

Starr '47 gave a formal proof of

$$F^2 = 1 + \frac{3}{2d} \frac{\int \eta^2 \, dx}{\int \eta \, dx}. \qquad (\star)$$

Assuming $\eta \geq 0$, this immediately implies $F > 1$.

Rigorous proofs

Amick–Toland '81: long and complicated proof of $F > 1$ and $(\star)$ without assuming $\int \eta \, dx < \infty$.

McLeod '84: formal proof can easily be made rigorous.

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Upper bound on the Froude number

Bernoulli: $\eta < c^2/2g = F^2 d/2$. Plugging into $(\star)$ gives $F < 2$, $\eta < 2d$.

In fact, Starr showed $F^2 < 1 + \eta_{\max}/d$, which implies $F < \sqrt{2}$, $\eta < d$.

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Consequences of $F > 1$

Even with vorticity (and $u < c$) solitary waves with $F > 1$:

1. are waves of elevation ($\eta > 0$),
2. are symmetric and monotone (η even, $\eta' < 0$ for $x > 0$),
3. have well-described exponential decay as $x \to \pm\infty$.

(Craig–Sternberg '88, Hur '08b, and W '13 for (1) with vorticity)

The Froude number for waves with vorticity

Numerics for constant vorticity

Vanden-Broeck '94: still have $F > 1$, but with large vorticity there are (overhanging?) waves with F arbitrarily large.

A first generalization of Starr's identity

Assume $u < c$ (so no overhanging). We use that the vector fields

$$A = \big(P + (u-c)^2,\ (u-c)v\big), \qquad B = \big((u-c)v,\ P + v^2 + gy\big)$$

are divergence free to compute

$$\iint \operatorname{div}(xA + (y+d)B)\,dx\,dy$$

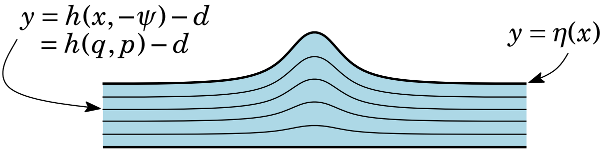
in two ways. Keeping vorticity terms, we eventually get

$$\big((c - U(0))^2 - gd\big)\int \eta\,dx - \frac{3g}{2}\int \eta^2\,dx - 2\iint x\omega v\,dy\,dx = 0.$$

Only involves $U(0)$, so not enough to show the sharp bound $F > 1$.

A standard change of variables

We will extend $U(y)$ etc. to be constant along streamlines. Since $u < c$, these are all graphs.



To make the calculations simpler, we then use

$$q = x, \qquad p = -\psi$$

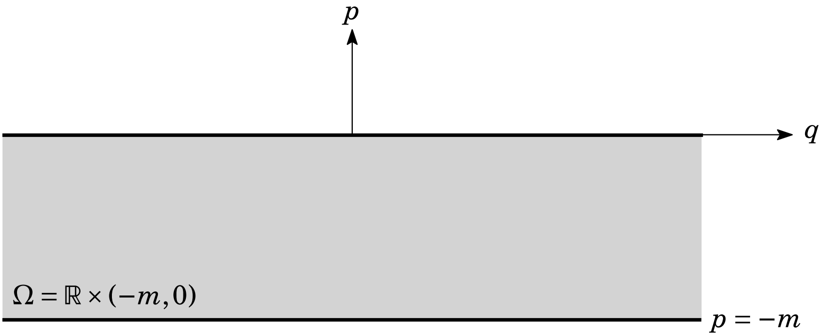
as independent variables, and the “height function”

$$h(q,p) = y + d$$

as the dependent variable (Dubriel-Jacotin ’34).

A standard change of variables

In these variables,



A standard change of variables

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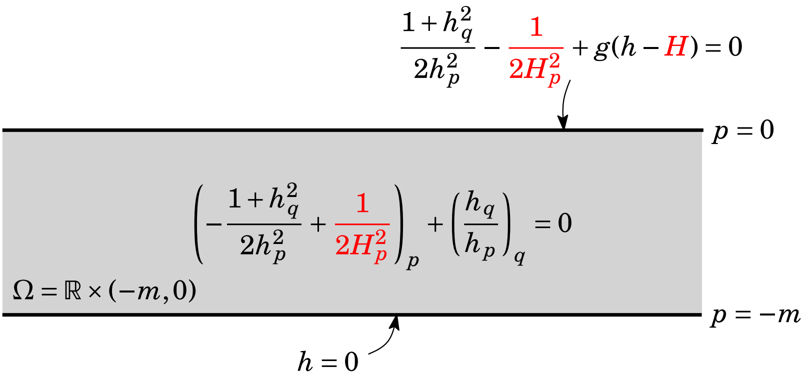


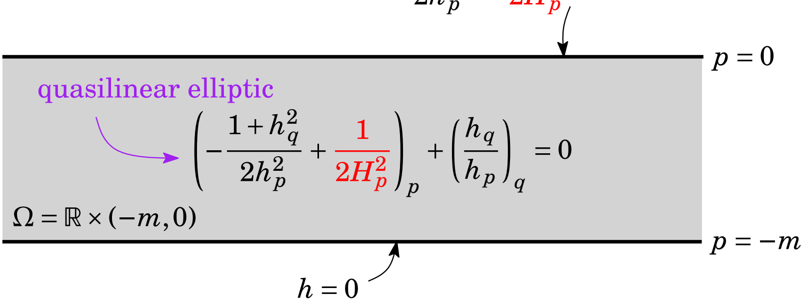
Here $H(p)$ is the height function corresponding to $\eta \equiv 0$ and $u \equiv U$, and $h(q,p) \to H(p)$ as $q \to \pm\infty$.

Since $u < c$, the denominators h_p and H_p are positive.

A standard change of variables

In these variables,

fully nonlinear → $$\frac{1+h_q^2}{2h_p^2} - \frac{1}{2H_p^2} + g(h - H) = 0$$



Here $H(p)$ is the height function corresponding to $\eta \equiv 0$ and $u \equiv U$, and $h(q,p) \to H(p)$ as $q \to \pm\infty$.

Since $u < c$, the denominators h_p and H_p are positive.

Result: Lower bound on the Froude number

Lemma

Any nontrivial solitary wave with $u < c$ satisfies

$$\left(1 - \frac{1}{F^2}\right) \int \eta \, dx = \iint \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2} \, dp \, dq > 0,$$

where the RHS is positive since h_p and H_p are.

From the inequality in the lemma we conclude:

Theorem (W ’14, submitted)

Consider a nontrivial solitary wave with $u < c$. Then $F \neq 1$. Moreover, $F > 1$ if it is a wave of elevation.

Remarks

- The Lemma also shows $F < 1$ for waves of depression.
- The Lemma still holds with surface tension.

Result: Lower bound on the Froude number

Proof of lemma.

Multiplying the equation by

$$\Phi(p) := \int_{-m}^{p} H_p^3(s)\,ds = \int_{-d}^{h(q,p)-d} \frac{dy}{(c-U(y))^2}$$

and integrating by parts, we obtain

$$\frac{1}{F^2}\int \eta\,dx = \iint \Big(\frac{1}{2H_p^2} - \frac{1+h_q^2}{2h_p^2}\Big) H_p^3\,dp\,dq.$$

Some algebra shows that

$$\Big(\frac{1}{2H_p^2} - \frac{1+h_q^2}{2h_p^2}\Big) H_p^3 = (h_p - H_p) - \frac{H_p^3 h_q^2 + (2h_p + H_p)(h_p - H_p)^2}{2h_p^2},$$

the first term of which we can easily integrate

$$\iint (h_p - H_p)\,dp\,dq = \int (h(q,0) - H(0))\,dq = \int \eta\,dx. \qquad \square$$

Result: Upper bound on the Froude number

Using a few more integral identities, we can also prove an upper bound on the Froude number, provided

$$\Lambda := \max_y \frac{c - U(0)}{c - U(y)} < \frac{2}{\sqrt{3}}.$$

Theorem (W '14, submitted)

For any solitary wave of elevation with $u < c$ and $\Lambda < 2/\sqrt{3}$,

$$F < \left(1 - \tfrac{3}{4}\Lambda^2\right)^{-1/2}.$$

Corollary: $\dfrac{\eta}{d} < \dfrac{1}{2}\dfrac{\Lambda^2}{1 - \frac{3}{4}\Lambda^2}.$

Special cases

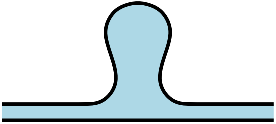
If the vorticity $\omega \le 0$, then $U_y \ge 0$ so that $\Lambda = 1$ and hence the theorem implies $F < 2$, $\eta < 2d$.

Part II

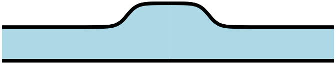
Existence of large-amplitude solitary waves with vorticity

Some previous results

sharp corner at the crest



overhanging waves



broad "tabletops"

Irrotational: "wave of greatest height" (Amick–Toland '81, . . .)

Large constant vorticity: overturning (Vanden-Broeck '94)

Interfacial: "tabletop" solutions (Turner–Vanden-Broeck '88)

Periodic waves with vorticity: $\sup u \to c$ (Constantin–Strauss '04)

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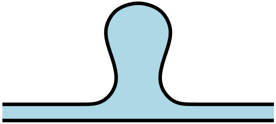

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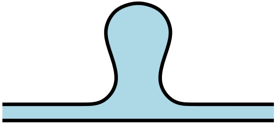

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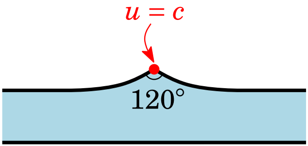

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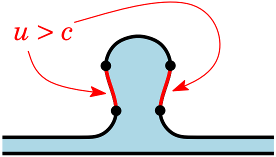

overhanging waves

broad "tabletops"
$\rightsquigarrow$ noncompactness

Irrotational: "wave of greatest height" (Amick–Toland '81, ...)

Large constant vorticity: overturning (Vanden-Broeck '94)

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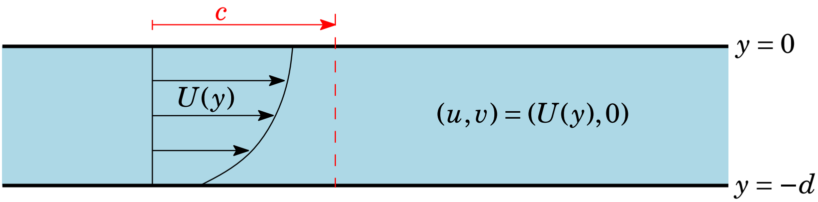
A one-parameter family of shear flows

We will bifurcate from shear flows with

$$U(y) = U(y;F) = c - FU^*(y),$$

for some fixed positive function U^*. These will also be the boundary conditions at $x = \pm\infty$.



We normalize U^* so that $g\int_0^d \frac{dy}{U^*(y)^2} = 1$ and hence F is the Froude number.

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for some fixed positive function U^*. These will also be the boundary conditions at $x = \pm\infty$.

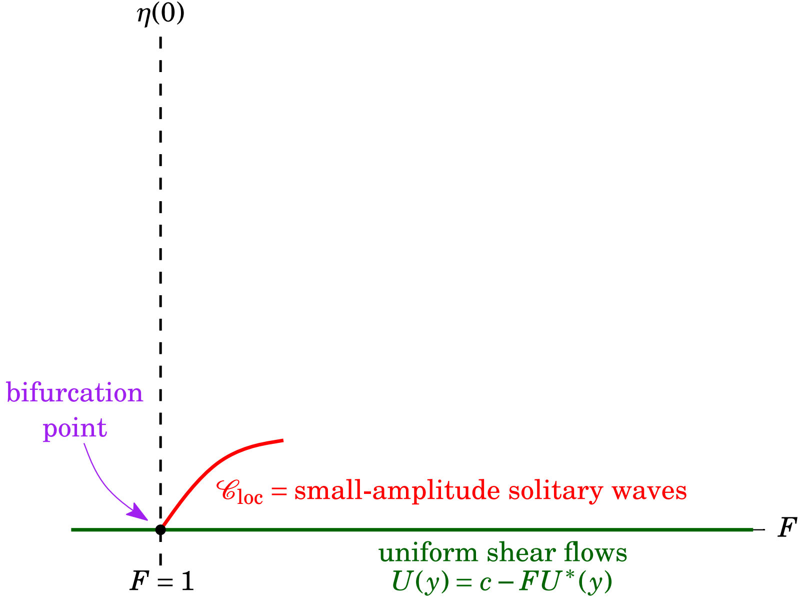
The parameter Λ (from the upper bound on F) is constant,

$$\Lambda := \max_y \frac{c - U(0)}{c - U(y)} = \max_y \frac{U^*(0)}{U^*(y)} =: \Lambda^*.$$

We also define $0 < \eta^* \leq +\infty$ by

$$\eta^* = \int_{-d}^{0} \left(\frac{U^*(y)}{\sqrt{U^*(y)^2 - (U^*_{\min})^2}} - 1 \right) dy$$

Result: Large-amplitude solitary waves with vorticity



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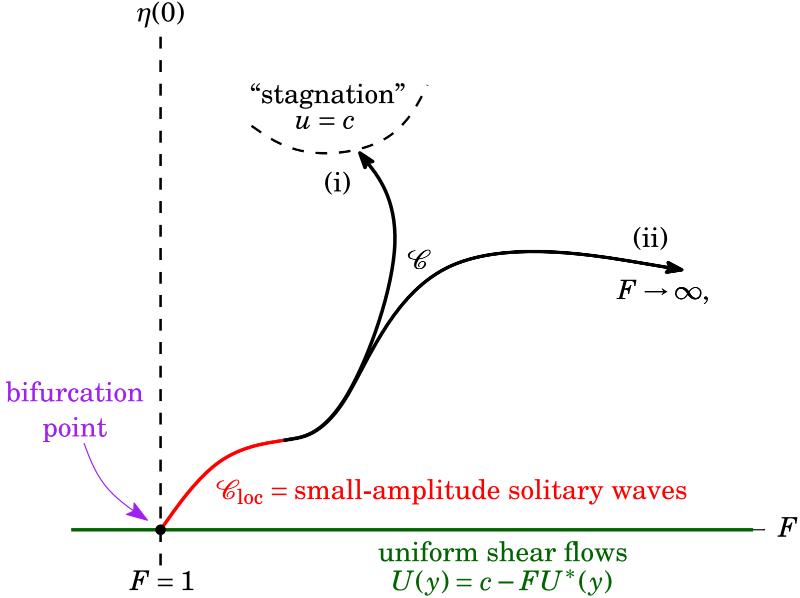

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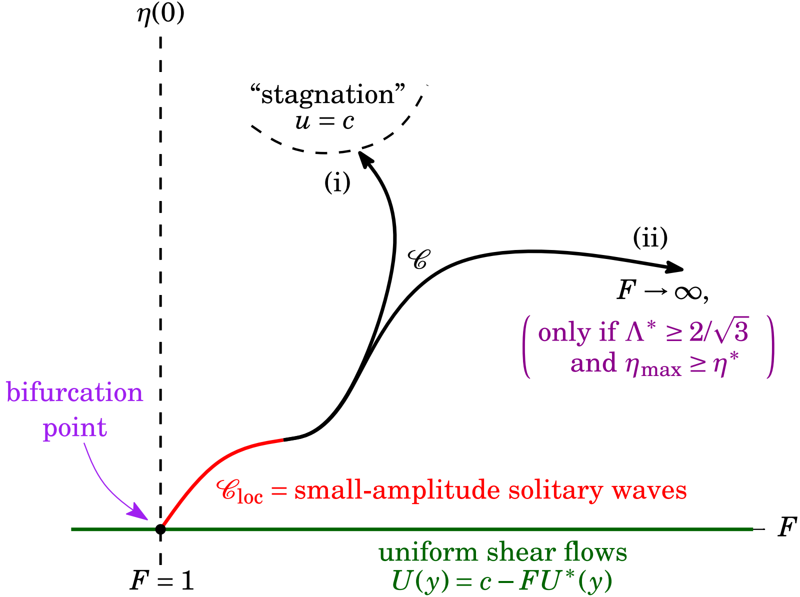

Result: Large-amplitude solitary waves with vorticity

Theorem (W '13, '14)

The local curve $\mathscr{C}_{\mathrm{loc}}$ of small-amplitude solitary waves is contained in a bigger connected set $\mathscr{C}$ of solutions such that

(i) "Stagnation": *There is a sequence in $\mathscr{C}$ with $\sup u_n \to c$; or*

(ii) Large amplitude and Froude number: *There is a sequence in $\mathscr{C}$ with $F_n \to \infty$ and $\lim_{n\to\infty}(\sup \eta_n) \geq \eta^*$.*

Moreover, if $\Lambda^* < 2/\sqrt{3}$, then (ii) cannot occur, so that (i) must hold. Here or is inclusive.

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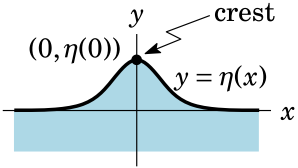

Moreover, $\mathscr{C}$ consists of symmetric and monotone waves of elevation, that is waves with η even in x, $\eta(x) > 0$ for $x \in \mathbb{R}$, and $\eta'(x) < 0$ for $x > 0$.

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Remarks

Constantin–Strauss ’04 show (i) for periodic waves with vorticity.

With $\omega > 0$ large and constant, waves with F arbitrarily large have been observed numerically (Vanden-Broeck ’94), but it is unclear if they are overturning (which rules out $u < c$).

Result: Large-amplitude solitary waves with vorticity

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Moreover, if $\Lambda^* < 2/\sqrt{3}$, then (ii) cannot occur, so that (i) must hold. Here or is inclusive.

Special case

For irrotational waves, $\Lambda^* = 1$ so (ii) is ruled out. Also, the harmonic function u is maximized at the crest, so the sequence in (i) in fact satisfies $u_n|_{\mathrm{crest}} \to c$. This recovers Theorem 3.9 (c) in Amick–Toland '81.

Result: Large-amplitude solitary waves with vorticity

In Amick–Toland '81:

- Use complex analysis techniques to reduce the problem to Nekrasov integral equation on the free surface.
- To get compactness, approximate by either
  (a) truncating a kernel, or
  (b) using conformal mappings to relate to periodic waves
- Make a limiting argument at the level of $\mathscr{C}$.

We, on the other hand:

- Have a semilinear problem in the fluid domain, so cannot reduce to the free surface.
- Have no kernels to truncate, and no canonical way to compare periodic and solitary problems.

Instead, we take a different approach which does not involve any approximate problems.

Finding a point to continue from

To start our continuation, we need a solution whose linearized operator L we understand very well.

For periodic waves

At the bifurcation point L is Fredholm with index 0 and a one-dimensional kernel.

Solitary waves

The essential spectrum of L depends only on the flow at infinity and hence only on the Froude number F:

- If $F > 1$, L is Fredholm with index 0.
- If $F = 1$, L is non-Fredholm.

Lemma (Invertibility)

The linearized operators for the small-amplitude solutions with $F > 1$ *are invertible (have trivial kernel).*

Finding a point to continue from: proving invertibility

Constructions of small-amplitude waves:

Irrotational

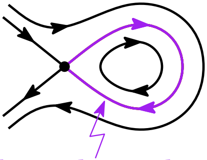
- Lavrentiev '54 and Ter-Krikorov '60: limits of periodic waves
- Friedrichs-Hyers '54: more direct approach
- Beale '77: Nash-Moser iteration
- Mielke '88: spatial dynamics

With vorticity

- Ter-Krikorov '61: method of Friedrichs-Hyers
- Hur '08a: Nash-Moser iteration
- Groves-Wahlén '08: spatial dynamics

  We will use their results and methods.

Finding a point to continue from: proving invertibility



Spatial dynamics

Groves-Wahlén '08 transform the problem into an evolution equation

$$\frac{d\varphi}{dx} = L\varphi + N(F,\varphi), \qquad \varphi\colon \mathbb{R} \to Y.$$

Here Y is a space of functions of the vertical variable, and $L\colon Y \to Y$ is unbounded.

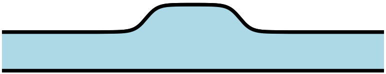
Center-manifold reduction

They construct a 2-dimensional center manifold containing all small-amplitude solitary waves. After rescaling, the reduced system is close to $q'' - q + \frac{3}{2}q^2 = 0$ (KdV).

Proof of the invertibility lemma

Invertibility is easy for the reduced system. We prove that this lifts to our original equation.

Compactness



All global continuation arguments require, at the very least, for bounded sets of solutions to be precompact.

Periodic waves

For periodic waves, this compactness follows from Schauder estimates (Constantin–Strauss).

Solitary waves

In our unbounded domain $\Omega = \mathbb{R} \times (-m, 0)$, Schauder estimates are no longer enough, nor is looking at linearized problems.

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Compactness: monotonicity

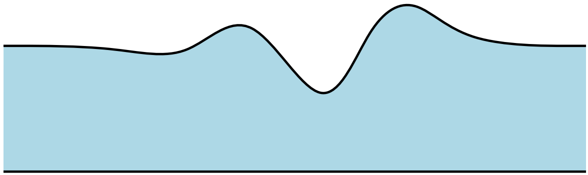
Theorem (Hur '08b)

Solitary waves of elevation with $F > 1$ are symmetric and monotone.

Lemma (W '13)

All solitary waves with $F > 1$ are waves of elevation.

The idea is to use (well-chosen) shear flows as comparison functions in a maximum principle argument.



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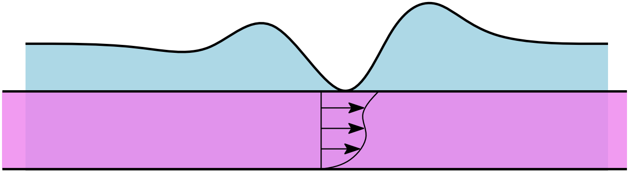


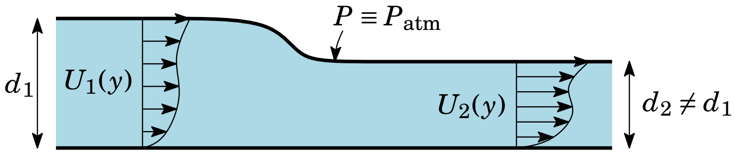
Compactness

Lemma (Compactness)

Subsets of $\mathscr{C}$ which are bounded in $\mathbb{R} \times C^{3+\alpha}(\overline{\Omega})$ are precompact.

Idea of proof.

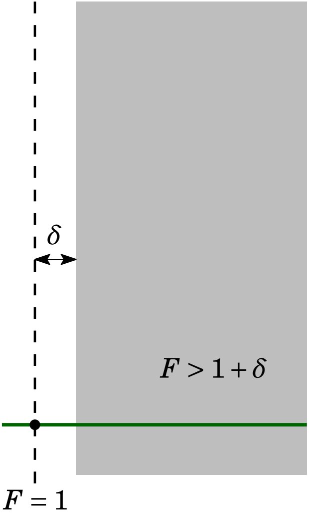
Using monotonicity and a translation argument, it is enough to rule out certain solutions of the form



which we can do using $F > 1$ and momentum conservation,

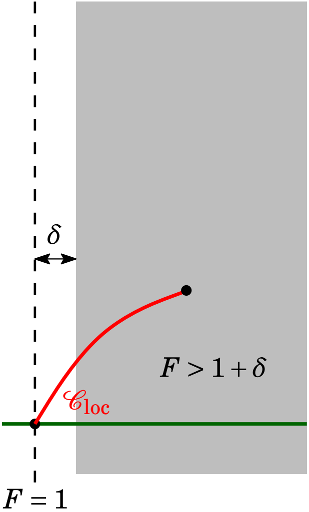
$$\int (P_1 + U_1^2)\,dy = \int (P_2 + U_2^2)\,dy.$$

Outline of the continuation



1. Restricting to $F > 1 + \delta$, compactness allows us to define the Healey–Simpson degree.

Outline of the continuation



1. Restricting to $F > 1 + \delta$, compactness allows us to define the Healey–Simpson degree.

2. For $\delta > 0$ small enough, we can find a solution with $F > 1 + \delta$ and invertible linearization (nonzero local degree).

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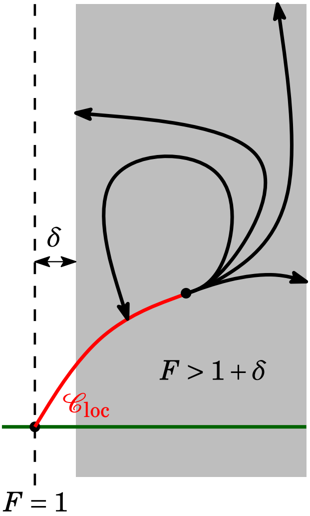
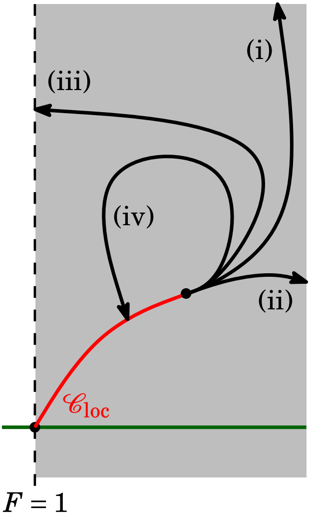


1. Restricting to $F > 1 + \delta$, compactness allows us to define the Healey–Simpson degree.
2. For $\delta > 0$ small enough, we can find a solution with $F > 1 + \delta$ and invertible linearization (nonzero local degree).
3. Then we make a global continuation argument in the style of Rabinowitz (Kielhöfer, Healey–Simpson, Constantin–Strauss).

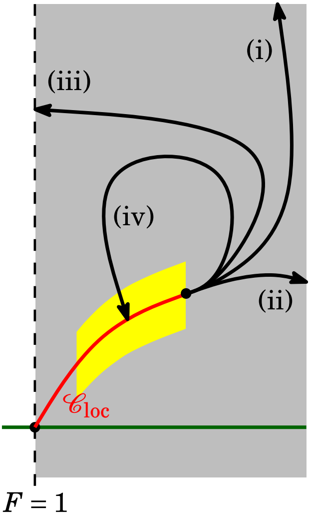
Outline of the continuation



4. Sending $\delta \to 0$, we are left with four alternatives: either one of
   - (i) $\|h\|_{C^{3+\alpha}} \to \infty$
   - (ii) $F \to \infty$
   - (iii) $F \to 1$

   occurs along a sequence in $\mathscr{C} \setminus \mathscr{C}_{\mathrm{loc}}$, or
   - (iv) there is a "loop".

Outline of the continuation



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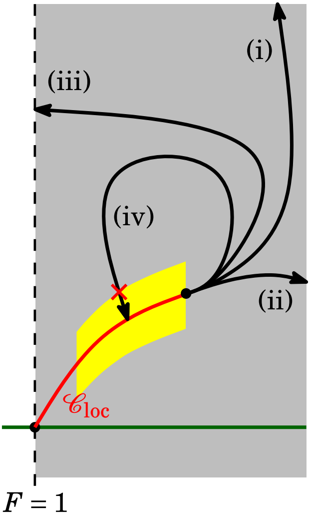
5. Alternative (iv) is ruled out by the invertibility along $\mathscr{C}_{\mathrm{loc}}$.

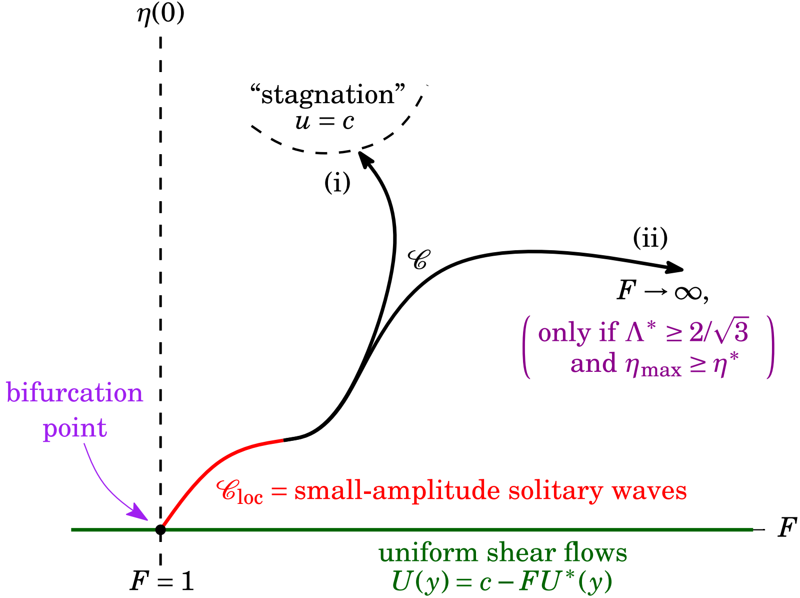
Outline of the continuation



(i) $\|h\|_{C^{3+\alpha}} \to \infty$
(ii) $F \to \infty$
(iii) $F \to 1$

6. To simplify (i), we use a lower bound on the pressure and nonlinear elliptic regularity results (Varvaruca, Constantin–Strauss, Lieberman).
7. For (ii) we use our upper bound on F or else a lower bound on the height of the crest.
8. To deal with (iii) we use our bound $F > 1$ and uniqueness for the center manifold from Groves-Wahlén.

Result: Large-amplitude solitary waves with vorticity

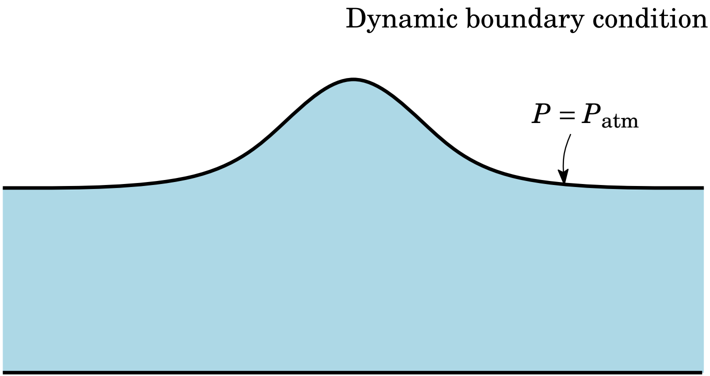


Thank you!

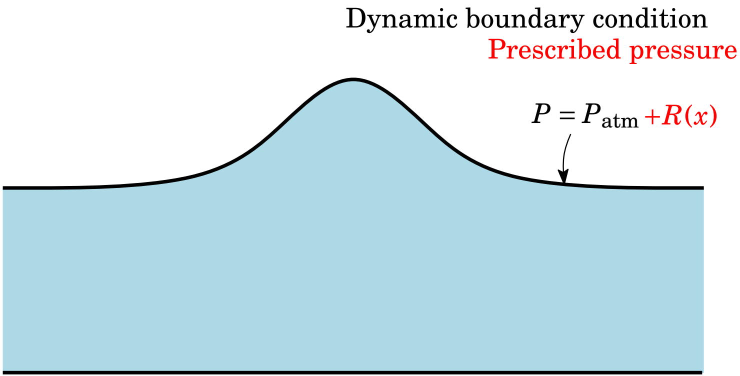
Part III

Large-amplitude waves with surface pressure

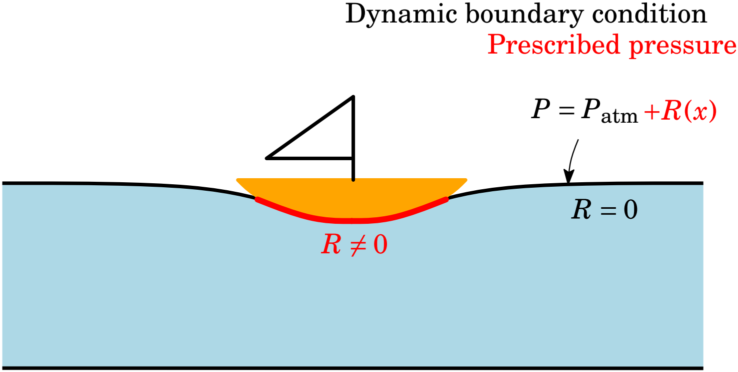
Solitary waves generated by surface pressure



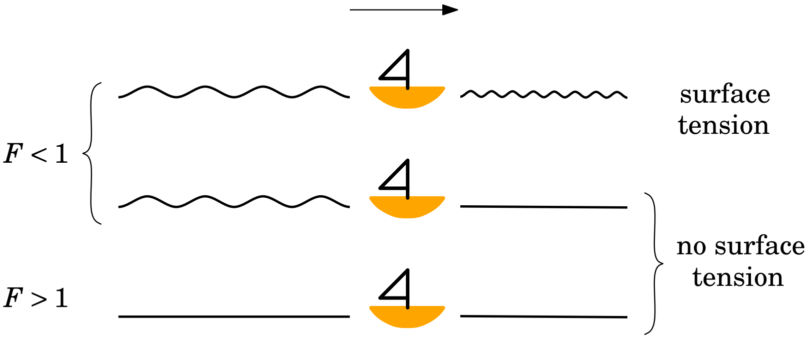
Solitary waves generated by surface pressure



Solitary waves generated by surface pressure



History: Small-amplitude ship waves (irrotational)



Pressure disturbances

Beale ’80 ($F < 1$, surface tension), also Mielke ’86 and Sun–Shen ’93 ($F \approx 1$).

Wetted hull part of a streamline

Pagani–Pierotti ’99 ($F > 1$) and ’04 ($F < 1$)

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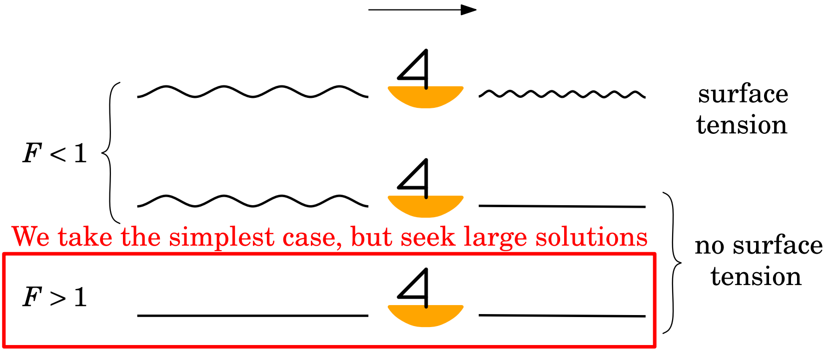

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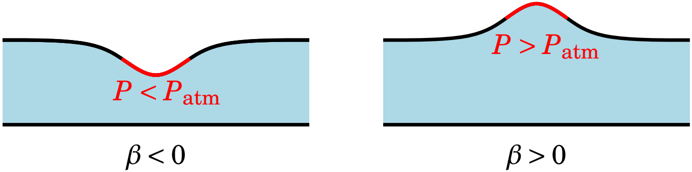
The prescribed pressure R

Introducing a pressure parameter $\beta \in \mathbb{R}$, we assume that the prescribed pressure $R(x)$ on the free surface is given by

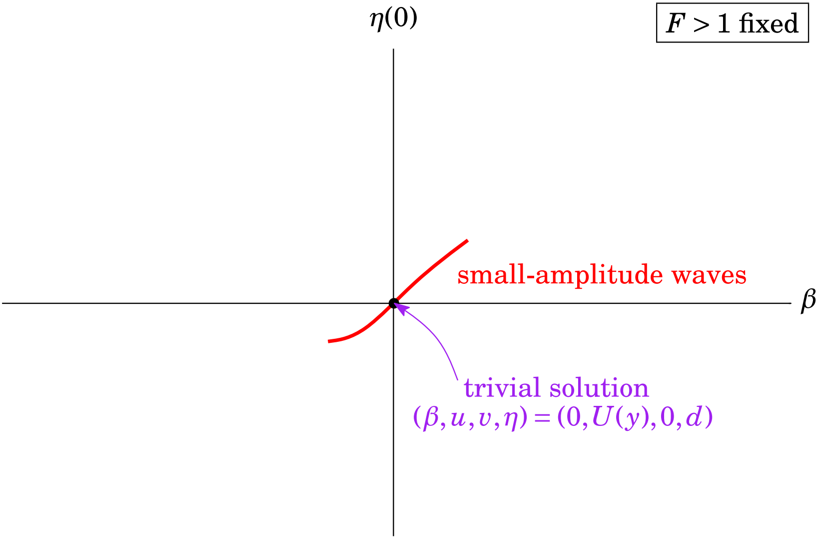
$$R(x) = \beta R^*(x),$$

where $R^* \in L^1$ is smooth, positive, even, and monotone decreasing for $x \geq 0$.



For small-amplitude symmetric waves and $F > 1$, we expect waves of elevation for $\beta > 0$ and waves of depression for $\beta < 0$ (kinetic term in Bernoulli's law dominates potential term).

Result: Continuation in pressure



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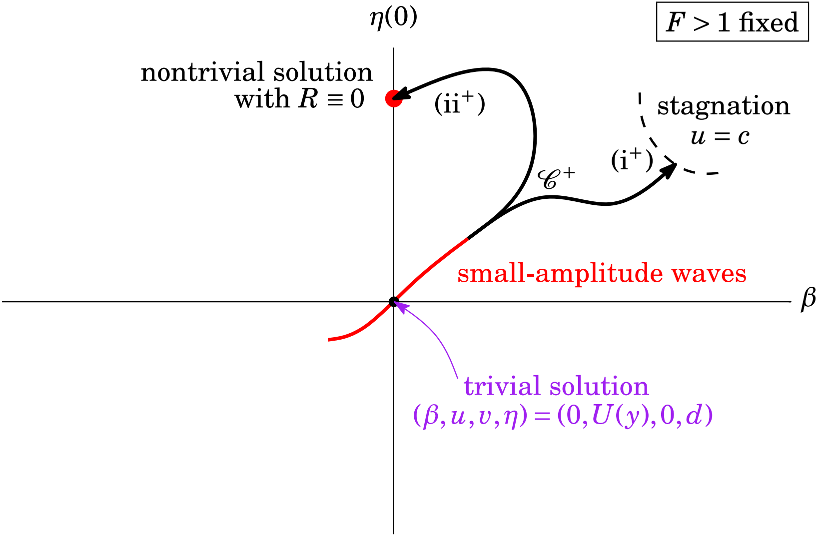

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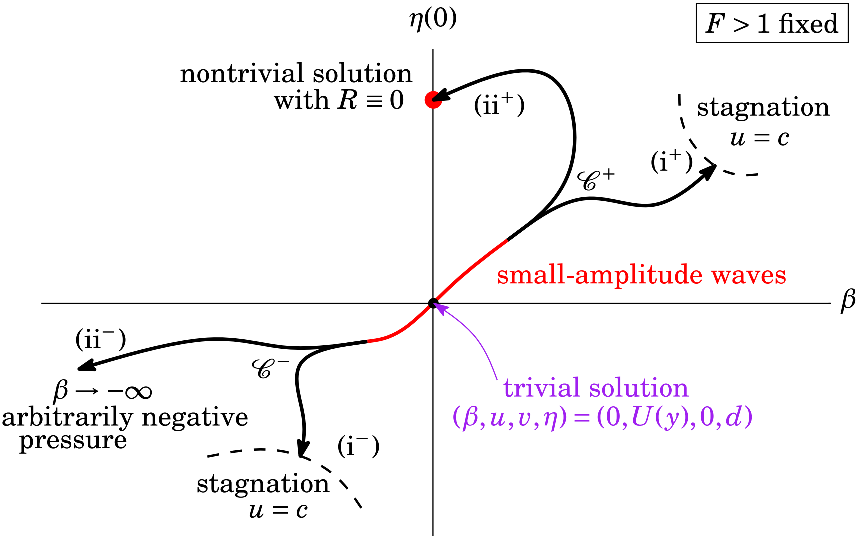


Result: Continuation in pressure

Theorem (W '13, submitted)

Fix $F > 1$. Then the trivial solution $(\beta, u, v, \eta) = (0, U, 0, d)$ lies on a connected set $\mathscr{C}$ of symmetric solitary waves. The set $\mathscr{C}^+ = \mathscr{C} \cap \{\beta \geq 0\}$ consists of monotone waves of elevation and satisfies one of the two alternatives:

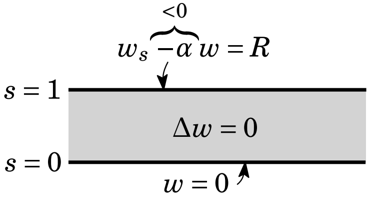
- (i^+) **Stagnation:** *There exists a sequence in $\mathscr{C}^+$ with $\sup u_n \to c$; or*
- (ii^+) **Solution with $R \equiv 0$:** *There exists a solution other than $(0, U, 0, d)$ in $\mathscr{C}^+$ with $\beta = 0$.*

The set $\mathscr{C}^- = \mathscr{C} \cap \{\beta < 0\}$ consists of monotone waves of depression and satisfies one of the two alternatives:

- (i^-) **Stagnation:** *There exists a sequence in $\mathscr{C}^-$ with $\sup u_n \to c$; or*
- (ii^-) **Arbitrarily negative pressure:** *There exists a sequence in $\mathscr{C}^-$ with $\beta_n \to -\infty$.*

Nodal properties

Want to use monotonicity to get compactness. Key idea:



Lemma (Maximum principle)

Fix $\alpha \in (0,1)$. If $R \geq 0$, then $w > 0$ in $\Omega \cup \{s = 1\}$ or $w \equiv 0$.

Proof 1: Craig–Sternberg '88

Cannot directly apply Hopf lemma, so instead write $w(\cdot, 1) = K_\alpha * R$ where $K_\alpha \geq 0$ by inspection.

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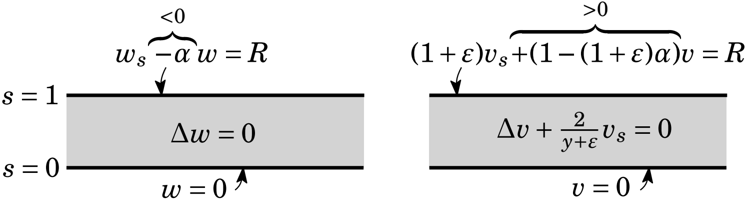

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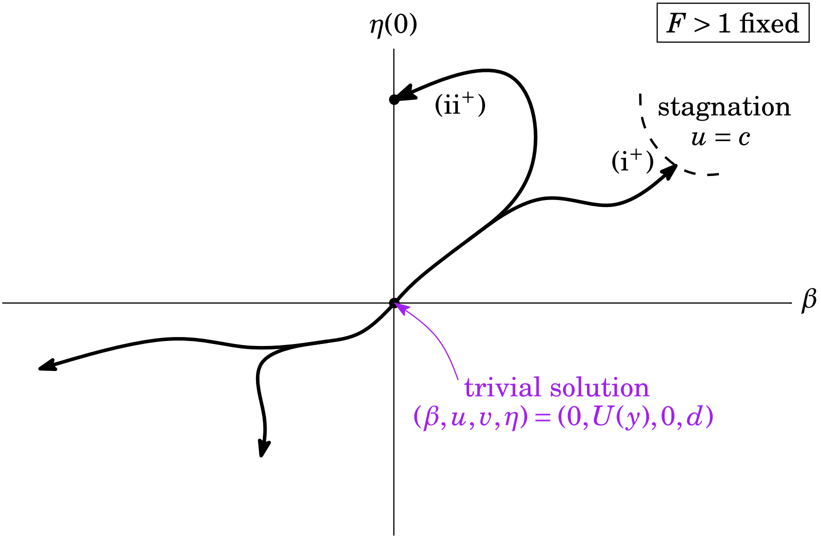
Proof 2: extends to the nonlinear problem, vorticity

Apply Hopf lemma to $v := w/(y+\varepsilon)$ for $0 < \varepsilon \ll 1$.

Going further in the pressure continuation



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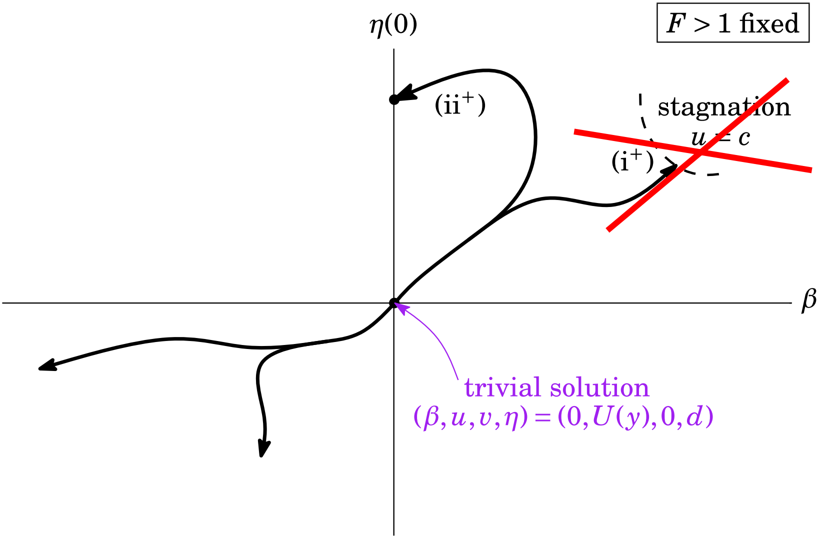

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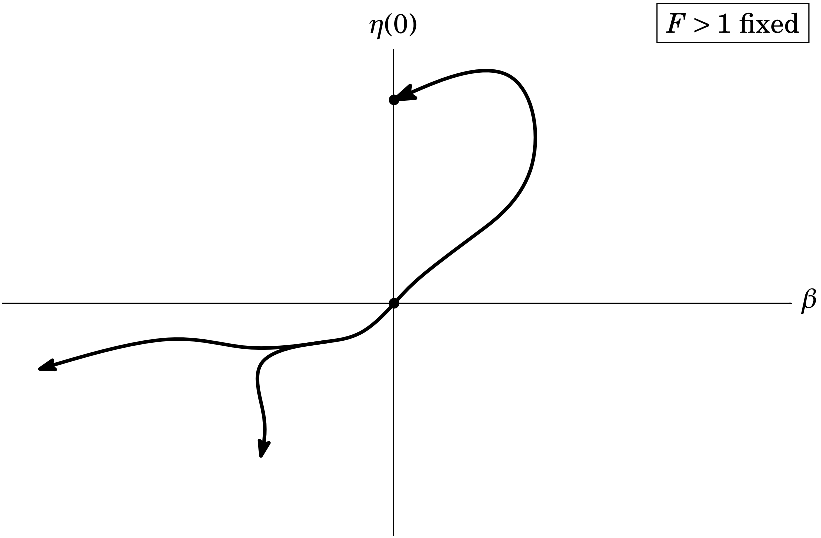

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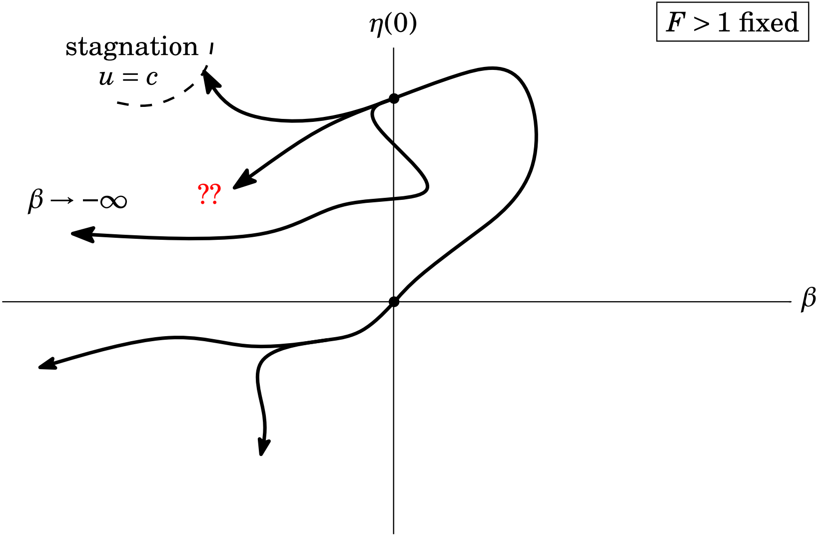

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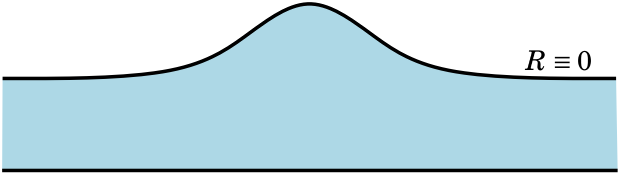

The difficulty in continuing past (ii$^+$) is that we no longer expect solutions to be monotone. Indeed, the above behavior was observed numerically by Asavanant and Vanden-Broeck ’94 in a related problem.

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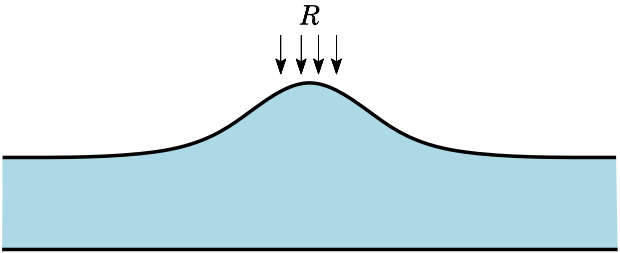

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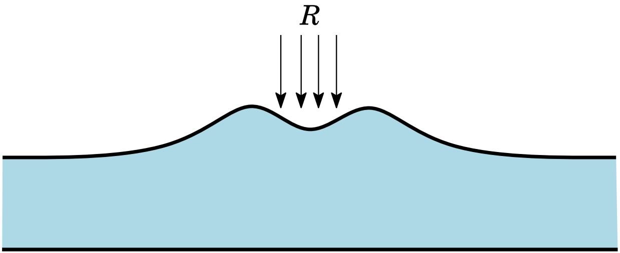

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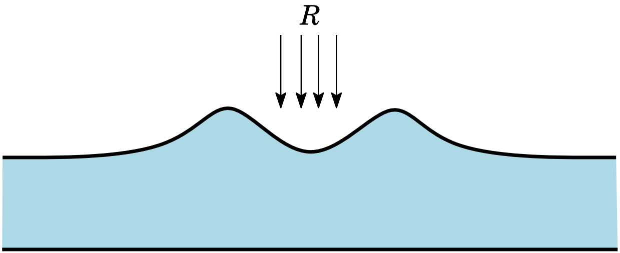

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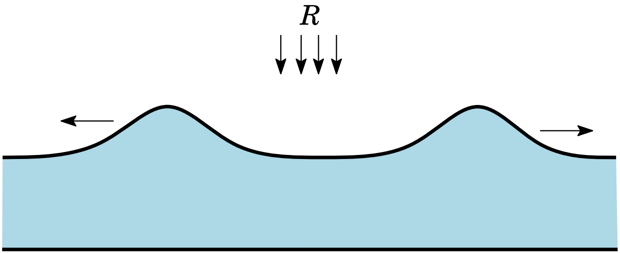

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