Violent Wave Motion due to Impact

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Theory of Water Waves, Cambridge 14 July 2014

at Isaac Newton Institute for Mathematical Sciences, Cambridge, UK.

Acknowledgments: *Howell Peregrine [1938–2007], A.A. Korobkin, Alan Tassin, Moritz Reinhard, ...*

Outline of Talk:

1. Sea Wave Impact Images:

2. Pressure-Impulse Theory

3. Example: Wave impact inside rectangular box

4. Results

5. Conclusions from Talk

References:

Euler's equations of motion in an impact:

Scale **v** with U; scale t with Δt; scale p with p_0; scale **x** with L where $\Delta t << L/U$: non-dimensionalise to get

$$\frac{\partial \mathbf{v}}{\partial t} + \left\{ \frac{U\Delta t}{L} \right\} (\mathbf{v} \cdot \nabla)\mathbf{v} = - \left\{ \frac{p_0 \Delta t}{\rho L U} \right\} \nabla p - \left\{ \frac{g \Delta t}{U} \right\} \mathbf{j} \qquad (1)$$

$$\nabla \cdot \mathbf{v} = 0. \qquad (2)$$

In eq. (1) neglect 2nd and 4th terms, and the pressure p is scaled with respect to $p_0 = \frac{\rho L U}{\Delta t}$.

Approximate equations in an impact analysed:

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p \qquad ((1))$$

$$\mathbf{v}_a - \mathbf{v}_b = -\frac{1}{\rho}\nabla \left\{ \int_{t_b}^{t_a} p(x, y, z, t) \, \mathrm{d}t \right\}$$

$$\mathbf{v}_a - \mathbf{v}_b = -\frac{1}{\rho}\nabla P(x, y, z) \ , \qquad (3)$$

where P is the pressure-impulse. Also from incompressibility (2):

$$\nabla \cdot \mathbf{v}_a = 0 = \nabla \cdot \mathbf{v}_b$$

Mixed b.v.p. for pressure-impulse P:

Definition: $P(x,y,z) = \int_{t_b}^{t_a} p(x,y,z,t)\,\mathrm{d}t$

is the *pressure-impulse*, at point (x,y,z) in fluid.

In the fluid domain at time t_b of impact: $P_{xx} + P_{yy} + P_{zz} = 0$,
subject boundary conditions:

(i) At designated free surface: $P = 0$;
(ii) Impermeable rigid boundary: $\frac{\partial P}{\partial n} = \rho \mathbf{u}_b \cdot \mathbf{n} = u_{bn}$,
where u_{bn} is given velocity component normal to structure **b**efore impact;
(iii) Far field: $P(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$.

After solving this b.v.p. for P, the fluid velocity after impact $\mathbf{v}_a(x, y, z)$ can be found from it and the velocity $\mathbf{v}_b(x, y, z)$ before impact. From eq. (3):

$$\mathbf{v}_a = \mathbf{v}_b - \frac{1}{\rho}\nabla P. \qquad (3)$$

From P we can estimate the distribution of the temporal peak in pressure, p_{pk}:

$$p_{pk}(x, y, z) \approx \frac{2P(x, y, z)}{t_a - t_b} \quad . \qquad (4)$$

Typically $P = \rho U L \times$ a dimensionless function that depends on the geometry. We find that P is large if the fluid domain's boundary has a small proportion of free surface ($P = 0$), and is confined by rigid impermeable walls, ($\partial P / \partial n = u_{bn}$ or 0).

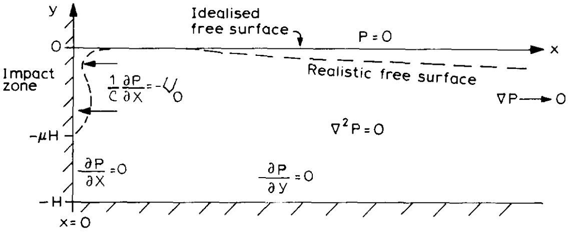
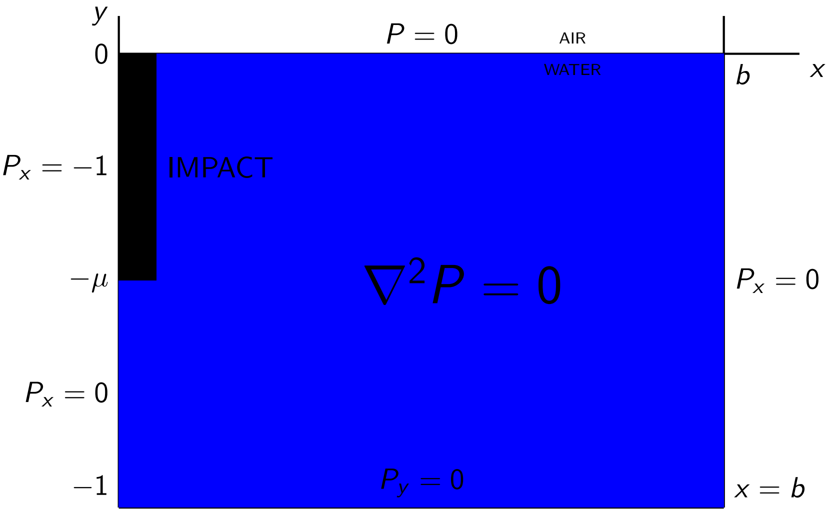


Fig. 4. Pressure impulse theory: A two-dimensional boundary-value problem for P, for a semi-infinite rectangular wave. This is an idealization of a realistic wave profile which is shown as a dashed line. The pressure impulse, P is defined to be the time-integral of fluid pressure over the short time-interval of impact.

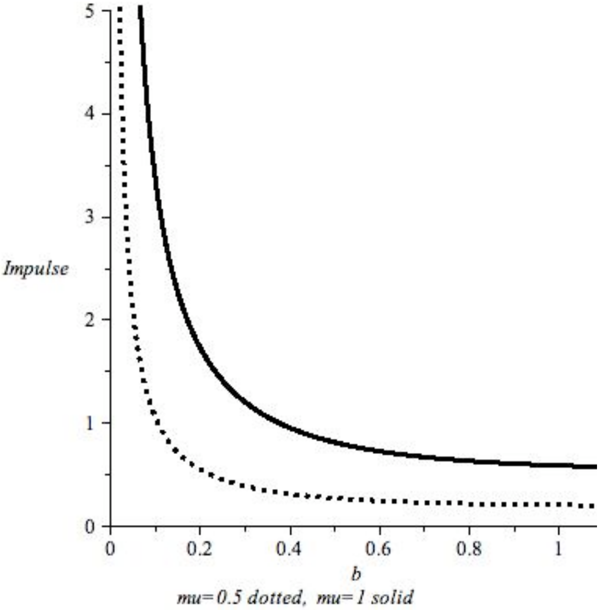

y
0
P = 0
AIR
WATER
b
x
Px = −1
IMPACT
−μ
∇²P = 0
Px = 0
Px = 0
Py = 0
−1
x = b

$$P(x,y) = -2\sum_{n=1}^{\infty} \frac{(1-\cos[\mu\lambda_n])}{{\lambda_n}^2}\sin(\lambda_n y)\frac{\cosh(\lambda_n[x-b])}{\sinh(b\lambda_n)}$$

where $\lambda_n = \pi(n-1/2)$. A measure of violence of impact is the total impulse $I(b,\mu)$ on left wall at $x=0$:

$$I(b,\mu) = \int_{-1}^{0} P(0,y)\,\mathrm{d}y = 2\sum_{n=1}^{\infty} \frac{1-\cos(\mu\lambda_n)}{{\lambda_n}^3\tanh(b\lambda_n)}.$$

As $b \to \infty$ we find $I(b,\mu)$ tends quickly to the wide-domain limit, but I is singular as $b \to 0$:


Impulse
0
1
2
3
4
5
0
0.2
0.4
0.6
0.8
1
b
mu=0.5 dotted, mu=1 solid

$$I(b,\mu) = 2\sum_{n=1}^{\infty} \frac{1-\cos(\mu\lambda_n)}{{\lambda_n}^3 \tanh(b\lambda_n)}.$$

As $b \to 0$ we find that $I(b,\mu)$ is singular:

$$I(b,\mu) \sim b^{-1}F(\mu) + O(b) \qquad \text{where} \qquad F(\mu) = 2\sum_{n=1}^{\infty} \frac{1-\cos(\mu\lambda_n)}{{\lambda_n}^4},$$

and notice $0 \leq F(\mu) \leq F(1)$, and $F(1)$ is

$$F(1) = 2.\frac{2^4}{\pi^4}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} = \frac{32}{\pi^4}\frac{\pi^4}{96} = \frac{1}{3}.$$

Velocity field after impact: $\mathbf{v}_a = \mathbf{v}_b - \rho^{-1}\nabla P$. The vertical velocity component at the free surface $y = 0$ is:

$$v_a = 2\sum_{n=1}^{\infty} \frac{(1 - \cos[\mu\lambda_n])}{\lambda_n} \frac{\cosh(\lambda_n[x - b])}{\sinh(b\lambda_n)}$$

where $\lambda_n = \pi(n - 1/2)$ and $0 < x \leq b$. The velocity $v_a(x, 0)$ has a log-singularity at $x = 0$. So the above is the outer solution to an inner region at the root of the splash jet (Wagner Theory, [H. Wagner, 1932], supplies the details).

Pressure-Impulse distribution on the wall $x = 0, \quad -1 \le y \le 0$ is:

$$P(0, y) = -2 \sum_{n=1}^{\infty} \frac{1 - \cos(\mu\lambda_n)}{\lambda_n{}^2 \tanh(b\lambda_n)} \sin(\lambda_n y)$$

where $\lambda_n = \pi(n - 1/2)$. The temporal peak in pressure is $p_{pk} = 2P/(t_a - t_b)$. In space the maximum in P (and hence in p_{pk}) at the wall, occurs about mid-way down the impact zone. This tallies with many experimental measurements of wave impact pressures (Bredmose et al., 2010). Also damage to seawall occurs at elevated positions on the wall (not near the sea bed). The magnitude of the maximum pressure-impulse is $O(\rho U_b H)$ where H is the width of the impact zone. So $p_p k = O(\rho U_b H/(t_a - t_b))$.

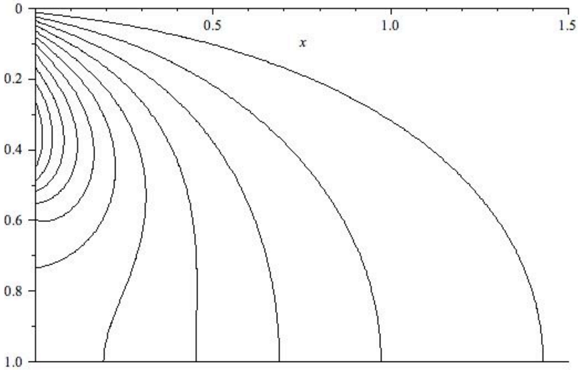


Figure: Contours of *Pressure Impulse*, P for $\mu = \frac{1}{2}$ $b = \infty$.

Conclusions:

1. Short time-scale of wave impact suggests sudden change in velocity & impulsive pressures;
2. Time-stepping – acceleration and *tt*-derivatives;
3. The pressure-impulse $P(\mathbf{x}) = \int_{t_b}^{t_a} \mathrm{pressure}(\mathbf{x}, t)\,\mathrm{d}t$;
4. $P(\mathbf{x})$ models 3D sudden change in velocity (e.g. free surface) & total impulse on structure due to wave-impact.

Thank you: Any questions?

1. M.J. Cooker & D.H. Peregrine (1995) Pressure impulse theory for liquid impact problems. *J. Fluid Mech.*, **297**, 193–214.

2. H.Bredmose, A.Hunt-Ruby, R.Jayaratne & G.N.Bullock (2010) The ideal flip-through impact: Experimental and numerical investigation. *J. Eng. Math.* **67** (1–2), 115–136.

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4. Nor Aida Md Noar (2012) PhD Thesis, Brunel University, (supervised by Martin Greenhow).

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Definitely the end: Stop!