The rolling sphere and quantum spin

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with A. Rojo

- **Rolling on a plane**
- **Quantum Spin**
- **Cornu Spiral and Landau Zener**
- **Rolling on a curved surface**
- **Adiabatic approximation**

- **Work with Harris and Mahmut Reyhanoglu – IEEE Transactions on Automatic Control**

- **Distinguished between dynamic and kinematic nonholonomic systems**

- **Geometric Phases**

- **Quantum control**

Consider a question similar to that posed by Montgomery: how much does a rigid body rotate?

Here we consider a related but different problem: a sphere is made to roll without slipping on a given curve Γ on a surface. The question is, if the sphere completes a circuit, what is the rotation matrix connecting the initial and final configuration of the sphere?

In particular, we address a nice question posed by Brockett and Dai a sphere lies on a table and is made to rotate by a flat plane on top of it, parallel to the table. The question is: if every point of the plane describes a circle, what is the trajectory and motion of the sphere?

We treat the problem by exploiting its isomorphism to the precession of a spin 1/2 in a time-dependent magnetic field. In the mapping, the arc length of the curve plays the role of time.

For rolling on a plane the magnitude of the magnetic field is $1/R$ **with** R **the radius of the sphere, and the direction of the magnetic field is that of the instantaneous angular velocity of the rolling sphere.**

For a curved surface the normal curvature and the torsion of the curve affect the value of the effective magnetic field.

Closely related to the present paper is the use of the isomorphism between classical dynamics and that of a spin $1/2$ by Berry and Robbins, especially their classical view of the Landau-Zener proble

As an application we show that the Landau-Zener problem corresponds to the rolling of a sphere on a Cornu spiral, and derive the probability of a non-adiabatic transition using the rolling language. We do so by a qualitative argument and by an exact computation of the rotation matrix in the non-adiabatic approximation.

•The Physics of Spin

The idea of an internal, magnetic degree of freedom of the electron that can have only two values (and an angular momentum of $\hbar/2$) was proposed in the early 1920's by Sommerfeld, Landé, Pauli and Goudsmit and Uhlenbeck. This "duplexity" was invoked to account for the observed splitting of levels observed in the so called "anomalous Zeeman effect", that showed twice as many states as the ones predicted by the then current model of the quantum atom. After the development in 1926 of wave mechanics by Schrödinger, in 1927, Pauli formalized this concept as follows.

In Schrödinger's picture, the electron is described by a wave function $\Psi(\mathbf{x})$ of the spatial coordinates x. In this formalism, the linear momentum p (the variable "conjugate" to x) corresponds to the operator $-i\hbar\nabla$.

Pauli opts to incorporate a discrete degree of freedom that can take only two values, corresponding to the two possible projections $\pm\hbar$ of the angular momentum in an arbitrary direction. Now, what are the angular momentum operators that act on this states? In order to answer this question Pauli borrows the algebra of the components of angular momentum operator $\mathbf{L} = \mathbf{r} \times \mathbf{p}$ in x space: $L_x L_y - L_y L_z = i\hbar L_z$ (and cyclical permutations of the coordinates). A representation of operators that satisfy this algebra and that act on a space of two states is the set of celebrated Pauli matrices:

$$S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

and satisfy $S_x S_y - S_y S_z = i\hbar L_z$ etc.

Most relevant here is that when Pauli considers the rotations of the spin (or the "spinor") he points out that the 2×2 rotational matrices correspond to a possible representation of rotations of a rigid body as had been presented previously by Klein in his treatment of the top. In other words, although the spin is a "classically non-describable" quantity, there is a formal equivalence with the rotation of a rigid body that was pointed out by Pauli in his original paper. We will use this equivalence to discuss the rolling of a sphere on a surface and interpret its motion vis-a-vis spin precession.

•Rolling on a plane and quantum precession

Consider a sphere of radius R rolling on a curve Γ on a plane. We define a local triad of unit vectors at the contact point (the so called Darboux frame): the tangent t to Γ, the normal n to the surface, and $\mathbf{u} = \mathbf{n} \times \mathbf{t}$, the tangent normal. For rolling on a plane n is a constant vector, and the velocity of the center of the sphere is along the tangent to the curve.

The translational velocity of the sphere is $\mathbf{V} = \mathbf{t}V(t)$ and the rolling constraint means that the instantaneous velocity at the contact point is zero.

$$\overrightarrow{\omega} \times (\mathbf{n}R) \;=\; \mathbf{V} = \mathbf{t}V(t) \qquad (2)$$

with $\overrightarrow{\omega}$ the angular velocity and R the radius of the sphere.

Taking the cross product with $\mathbf{n}$ on both sides of the above equation we have

$$\overrightarrow{\omega} = \frac{V(t)}{R}\mathbf{n} \times \mathbf{t} \equiv \frac{V(t)}{R}\mathbf{u}. \tag{3}$$

Notice that in the above equation we have used the "no twist" condition $\overrightarrow{\omega} \cdot \mathbf{n} = 0$, that is, we are consider rolling without an instantaneous rotation along the normal.

The instantaneous velocity $\dot{\mathbf{X}}$ of a point of coordinate $\mathbf{X}$ (with respect to the center of the sphere) on the surface of the sphere is

$$\dot{\mathbf{X}} = \overrightarrow{\omega} \times \mathbf{X} = \frac{V(t)}{R}\mathbf{u} \times \mathbf{X}. \tag{4}$$

Now we rewrite $V(t) = ds/dt$ where s is the arc length of the curve $\Gamma(t)$, and (4) becomes

$$\frac{d\mathbf{X}}{ds} = \frac{\mathbf{u}}{R} \times \mathbf{X}. \tag{5}$$

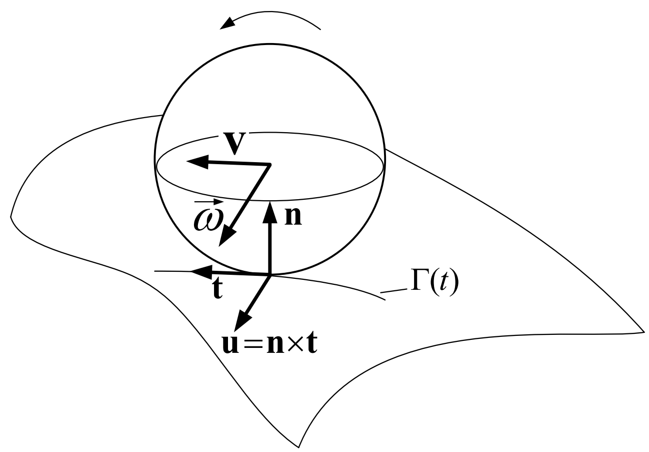

Figure 1: Sphere rolling along a curve Γ of zero torsion (meaning that the velocity of the center of the sphere is parallel to the tangent of the curve at the contact point).

If we regard X $= (x, y, z)$ **as a magnetic moment, the above equation describes its precession in the presence of a magnetic field B** $= -\frac{1}{R}(u_x, u_y, u_z) = -\overrightarrow{\omega}$ **of constant magnitude** $1/R$**. The direction of B is** $-$**u, and varies with** s**, the arc length, which plays the role of time. If the rolling is on a horizontal plane, then** B_z**=0, but we keep this notation to make contact with the rolling on an arbitrary surface.**

There is an isomorphism between the rolling sphere written in this way with a spin $1/2$ precessing in this magnetic field. This isomorphism can be seen clearly if, (using B $= -\overrightarrow{\omega}$) we rewrite Equation (5) in the form

$$\frac{d}{ds}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{6}$$

which is the same as the following equation of motion for two complex numbers a and b (we write s instead of t for time in order to keep the analogy)

$$i\frac{d}{ds}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{7}$$

with the identification

$$\begin{aligned} x &\equiv ab^* + ba^* \\ y &\equiv i\left(ab^* - ba^*\right) \\ z &\equiv aa^* - bb^*. \end{aligned} \tag{8}$$

The real numbers (x, y, z) represent the coordinates of a point on the surface of the sphere referred to a coordinate system fixed in space (that is, not rotating), and whose origin is in the center of the sphere.

Equation (7) is Schrödinger's equation for the spinor $\chi = (a, b)$ in the presence of a magnetic field B:

$$i\frac{d}{ds}\chi = -\mathbf{B} \cdot \mathbf{S}\chi \equiv H\chi, \tag{9}$$

where $\hbar = 1$ and H the Hamiltonian. Also, the vector $\mathbf{S} = \frac{1}{2}(\sigma_x, \sigma_y, \sigma_z)$ is the spin operator, and σ_i are Pauli's matrices.

Equation (8) implies that we can extract the behavior of the rolling sphere as a function of arc length by solving the motion of a spin $1/2$ in a time-varying magnetic field. To our knowledge the equivalence between the motion of rigid body and a two-level system (a spin $1/2$), in the form of the mapping of Eq. (8) was first pointed out by Feynman, Vernon and Hellwarth. Earlier, Bloch had derived the precession equation for the density matrix of spin 1/2 and therefore the points (x, y, z) that result from the mapping from spinors are called the Bloch sphere.

• Constant magnetic field

Consider the simplest case of constant magnetic field. We choose B $= B_0\hat{\mathbf{k}}$**, constant in the** $+z$ **direction. This corresponds to the sphere rolling on a vertical plane. Eq (7) becomes:**

$$i\frac{d}{ds}\begin{pmatrix} a \\ b \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} B_0 & 0 \\ 0 & -B_0 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix}, \tag{10}$$

with solutions:

$$\begin{pmatrix} a(s) \\ b(s) \end{pmatrix} = \begin{pmatrix} e^{isB_0/2}a(0) \\ e^{-isB_0/2}b(0) \end{pmatrix}. \tag{11}$$

Replacing (11) in (8) we obtain:

$$\begin{aligned} x(s) &= x(0)\cos(B_0 s) + y(0)\sin(B_0 s) \\ y(s) &= y(0)\cos(B_0 s) - x(0)\sin(B_0 s) \\ z(s) &= z(0), \end{aligned} \tag{12}$$

which means that the sphere is rotating clockwise around a constant axis in the z **direction. This corresponds to** $\overrightarrow{\omega}$ **in the** $-z$ **direction.**

In other words, a constant magnetic field in the z direction corresponds to the sphere moving in a straight line in the xy plane, rolling on a vertical wall. The same situation applies if a constant field is directed in any other orientation.

• The lollipop and the planar field

Consider a magnetic field varying on the xy **plane as B** $=$ $B(\cos\alpha s, \sin\alpha s, 0)$**. This corresponds to u rotating with the same frequency in the same plane, and the rolling problem becomes that of a sphere of radius** $R = 1/B$ **rolling counterclockwise on a circle of radius** $r = 1/\alpha$ **(see Figure 2).**

In turn, this corresponds to a time (or arc length) dependent Hamiltonian $H = -\mathbf{B}\cdot\mathbf{S}$**, which can be solved by noting that**

$$2\mathbf{B}\cdot\mathbf{S} \;=\; \begin{pmatrix} 0 & Be^{-i\alpha s} \\ Be^{i\alpha s} & 0 \end{pmatrix} = U^* \begin{pmatrix} 0 & B \\ B & 0 \end{pmatrix} U, \qquad (13)$$

with

$$U \;=\; \begin{pmatrix} e^{i\alpha s/2} & 0 \\ 0 & e^{-i\alpha s/2} \end{pmatrix}. \qquad (14)$$

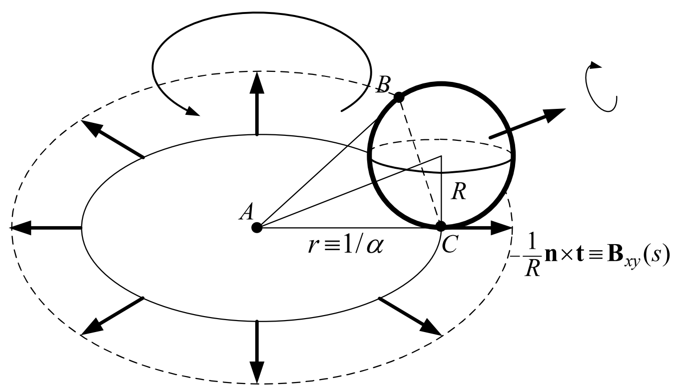

Figure 2: The lollipop, or a sphere rolling counterclockwise on a circle of radius r corresponds to a spin 1/2 precessing on a magnetic field that rotates in the xy plane.

Substituting the above relations in (7) we obtain a time independent equation for the coefficients $\tilde{\chi}(s) = (\tilde{a}, \tilde{b}) = (e^{i\alpha s/2}a, e^{-i\alpha s/2}b)$

$$i\frac{d}{ds}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \alpha & B \\ B & -\alpha \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \equiv \tilde{H}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}. \tag{15}$$

Transformations (13) and (14) correspond to transforming to a frame that rotates with angular velocity α**. When transforming to the rotating frame, the angular velocity acquires a component** $\alpha = 1/r$ **in the** z **direction and the frequency of rotation in the rotating frame is**

$$\Omega = \sqrt{B^2 + \alpha^2} = \frac{1}{rR}\sqrt{r^2 + R^2} \tag{16}$$

This can be seen in the spinor language by noting that, since $\tilde{H}$ in Eq. (15) is time-independent , the solutions are

$$\begin{aligned}\tilde{\chi}(s) &= e^{\frac{i}{2}s\begin{pmatrix}\alpha & B \\ B & -\alpha\end{pmatrix}}\tilde{\chi}(0) \\ &= \left[\cos\left(\Omega s/2\right) + i\vec{\sigma}\cdot\mathbf{m}\sin\left(\Omega s/2\right)\right]\tilde{\chi}(0),\end{aligned} \tag{17}$$

with $\pm\Omega = \sqrt{B^2+\alpha^2}$ **(twice) the eigenvalues of** $\tilde{H}$ **and** $\mathbf{m} = (r,0,R)/\sqrt{r^2+R^2}$ **a unit vector with its** x **axis in the direction AC. Notice the presence of the factor of** 2 **in the relation between the eigenvalues of** H **and the corresponding rotational frequencies of the rolling problem. This comes from the factor** $1/2$ **that emerges naturally in the mapping to the spin problem of Eq. (7).**

Equation (17) describes a rotation at a rate Ω with respect to an axis in the direction of the "stick" of the lollipop (the direction joining A to the center of the sphere (see Fig. (2)). Notice that solving for the evolution by exponentiating $\tilde{H}$ is possible because $\tilde{H}$ does not depend on s. If there is an s-dependence and the matrices $\tilde{H}$ at different s do not commute the solution is a "time ordered" exponential that in general is not exactly solvable.

After the lollipop completes a circle, the angle δ of rotation is

$$\delta = \frac{2\pi}{\alpha}\Omega = 2\pi\sqrt{1 + \left(\frac{r}{R}\right)^2}. \tag{18}$$

Notice that, when $R \ll r$ the angle of rotation is $\delta \approx 2\pi r/R$, corresponding to rolling in a line of length equal to the perimeter of the circle.

We see that, after traveling on a circle the sphere is rotated by $2\pi\Omega/\alpha$ with respect to an axis tilted with respect to the plane.

When the sphere rolls on a plane, and on a circle of radius r much larger than its radius R, it comes back rotated around an axis that lies on the plane, by an angle given only by the dynamical phase. The extra term that originates in the curvature of the surface is what we call the geometric phase.

The angle of rotation δ (of both the spin and the lollipop) has a simple geometric interpretation: when the lollipop rolls, the point of contact C moves on the circular rim of the cone ABC (see Figure 2). At the same time, the point C "paints" on the sphere a circle of diameter $BC = 2rR/\sqrt{r^2 + R^2}$. (This is easily calculated with simple geometrical considerations from Figure 2.) This means that after a revolution of length $2\pi r$ the angle rotated is $2\pi r/(BC/2)$ from which Eq. (18) follows immediately.

At this point we consider Brockett's question.

Notice first that, as the sphere rolls on a circle, the velocity at the top of the sphere is twice the velocity V at the center of the sphere. Since each point of the plane on top of the sphere describes a circle of radius R_1, the velocity $\mathbf{V}_P$ of the plane also describes a circle. Therefore, since the sphere has a rolling condition with the upper plane, then $\mathbf{V}_P = 2\mathbf{V}$, meaning that, as the plane describes a circle of radius R_1 the sphere describes a circle of radius $R_1/2$.

Demo: on a piece of paper draw a circle of radius 5 inches (twice that of a tennis ball). Orient the label of the tennis ball at 45 degrees with the vertical (the sphere is going to roll on a circle of radius $r = R$, and therefore the axis of rotation is going to be at 45 degrees and the precession frequency will be, from (16), $\sqrt{2}/R$).

Paint a mark on a transparent glass, which in turn will serve as the upper plane. Also mark three points on the circle separated by $\beta = 127$ degrees ($\pi/\sqrt{2}$). Looking through the glass, guide the mark on the glass over the circle on the paper, and notice that, each time the glass rotates by β, the tennis ball rotates by π with respect to a moving axis at 45 degrees.

Here

$$\Delta\theta = \frac{\sqrt{2}}{R}\Delta s\,, \qquad \Delta s = R\Delta\theta = R\frac{\pi}{\sqrt{2}}$$

Notice also that for $s = 2\pi/\alpha$ the spinor χ changes sign due to the $1/2$ factor in the transformation. Nevertheless, since the mapping of (8) is quadratic in a and b, changing their signs corresponds to the same values (x, y, z) for the orientations. More specifically, the quantities a and b determine univocally x, y and z, but the reverse is not valid: the quantum evolution determines univocally the classical evolution but there is some ambiguity in going from the classical to the quantum case. For example if we perform the "gauge transformation" $(a, b) \to e^{i\phi(s)}(a, b)$ the mapping to the X coordinate remains unchanged.

•Rolling on a Cornu spiral and the Landau-Zener problem

In this section we consider a "magnetic field" of constant magnitude B_0 varying on the xy plane as

$$\mathbf{B} = B_0(\cos\phi(s), \sin\phi(s), 0) = -\overrightarrow{\omega}, \tag{19}$$

which corresponds to the angular frequency rotating with the (varying) frequency $\phi(s)$ in the plane. The rolling problem becomes that of a sphere of radius $R = 1/B_0$ rolling on a planar curve of local curvature given by

$$\kappa(s) = \dot{\phi} \equiv \frac{d\phi}{ds}. \tag{20}$$

This corresponds to a time (or arc length) dependent Hamiltonian $H = -\mathbf{B} \cdot \mathbf{S}$, which can be solved by noting that

$$\mathbf{B} \cdot \mathbf{S} = \begin{pmatrix} 0 & B_0 e^{-i\phi(s)} \\ B_0 e^{i\phi(s)} & 0 \end{pmatrix}$$
$$= U^* \begin{pmatrix} 0 & B_0 \\ B_0 & 0 \end{pmatrix} U, \qquad (21)$$

with

$$U = \begin{pmatrix} e^{i\phi(s)/2} & 0 \\ 0 & e^{-i\phi(s)/2} \end{pmatrix}. \qquad (22)$$

Substituting the above relations in (7), and using (20) we obtain a time independent equation for the coefficients $\tilde{\chi}(s) = (\tilde{a}, \tilde{b}) = (e^{i\phi/2}a, e^{-i\phi/2}b)$

$$i\frac{d}{ds}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \kappa(s) & B_0 \\ B_0 & -\kappa(s) \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} \equiv \tilde{H}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}. \qquad (23)$$

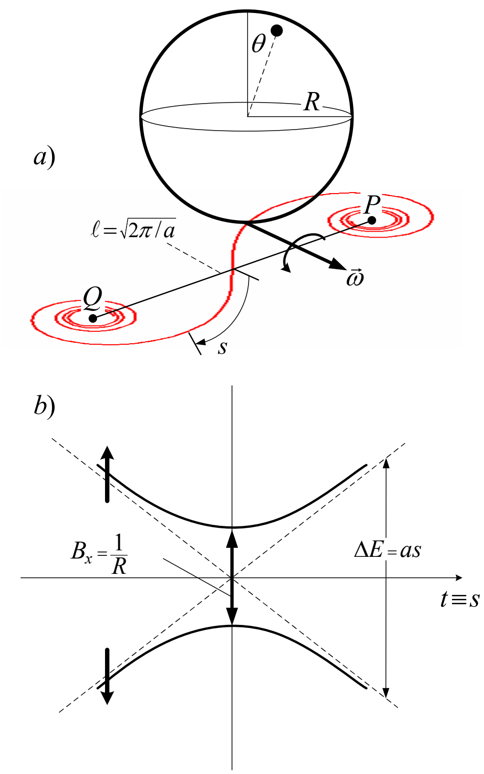

Figure 3: Equivalence between a) rolling on a Cornu spiral and b) the Landau-Zener problem of the spin flip probability on a time dependent field.

We have obtained the result that rolling on a planar curve is isomorphic to spin precession in a magnetic field that is constant in (some direction of) the xy plane, and with a z component that varies in time according to the local curvature. For example, this means that rolling on a Cornu spiral (a curve whose curvature is proportional to the arc length), defined as

$$\phi(s) = as^2/2, \tag{24}$$

with a a constant, corresponds to the Landau-Zener problem of a spin in a magnetic field whose z component varies linearly with time $B_z = as$ and a constant xy coupling of magnitude $1/R$.

The prototypical question in the Landau-Zener problem is the flipping of a spin that starts, for $t \to -\infty$, in a well defined orientation in the z direction. This means that $|a(-\infty)| = 1$, and $|b(-\infty)| = 0$. In the sphere isomorphism, the level splitting ΔE (See Figure 3) increases in time (arc length s) as $\Delta E = as$ and the level coupling is $B_0 = 1/R$, constant in time. The problem is also called the avoided level crossing. The name comes from the fact that, when $B_0 = 0$ the levels for spin up and down cross at $s = 0$. The remarkable result obtained by Zener is that the probability of the spin remaining up after the evolution is, in our notation

$$|a(\infty)| = e^{-\pi/2aR^2}. \tag{25}$$

We now show that the non-adiabatic limit when the levels are crossed very fast (which for rolling corresponds to a sphere much larger than the size of the Cornu spiral) can be obtained in a simple way using the rolling picture.

- **Landau-Zener expression in rolling language**

We start by a qualitative derivation of the non–adiabatic limit which reveals the power of the rolling picture. Consider the rolling from P to Q (See Figure 3) of a sphere of radius R much larger than the "size" $\ell = PQ$ of the spiral. We want to estimate the angle θ of rotation of the North pole when the sphere rolls from P to Q, and, from this, obtain the change in the probability of finding the spin up $|a(\infty)|^2$ sing the equivalence stated in Equation (8).

Qualitatively, since the sphere is very large, the rotation following the Cornu spiral is roughly that of a rolling on a straight line from P to Q. After the rolling the z_N coordinate of the North pole changes from R to

$$
\begin{aligned}
z_N(\infty) &\simeq R\left(1-\frac{1}{2}\theta^2\right) \\
&\simeq R\left(1-\frac{1}{2}\left(\frac{\ell}{R}\right)^2\right)
\end{aligned}
$$

The length ℓ of the segment PQ is:

$$
\ell = \sqrt{2}\int_{\infty}^{\infty} ds \sin\frac{as^2}{2} = \sqrt{\frac{2\pi}{a}}. \tag{26}
$$

Noting that $|a|^2+|b|^2=1$, we have (see Eq (8))

$$
|a(\infty)|^2 = \frac{z_N(\infty)/R+1}{2} = 1-\frac{\pi}{2aR^2}, \tag{27}
$$

which [see Eq.(25)] is the exact expression for the Landau-Zener effect in the non-adiabatic limit.

Next we use our isomorphism to re-derive this result calculating the rotation matrix exactly to the same order in $1/aR^2$. The Landau-Zener problem is described by the following evolution:

$$i\frac{d}{ds}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} as & 1/R \\ 1/R & -as \end{pmatrix}\begin{pmatrix} \tilde{a} \\ \tilde{b} \end{pmatrix}. \tag{28}$$

From our previous discussion, in the rolling language this matrix corresponds to the following evolution for a point in the sphere of radius R:

$$\begin{aligned} \frac{d}{ds}\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{1}{R}\begin{pmatrix} 0 & 0 & -\sin\frac{as^2}{2} \\ 0 & 0 & \cos\frac{as^2}{2} \\ \sin\frac{as^2}{2} & -\cos\frac{as^2}{2} & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \\ &\equiv M(s)\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \end{aligned} \tag{29}$$

or equivalently $\dot{\mathbf{X}} = M(s)\mathbf{X}$.

Since the matrices do not commute at different values of s, the formal solution of this equation is

$$\mathbf{X}(s) = Te^{\int_{s_0}^{s} ds' M(s')}\mathbf{X}(s_0), \tag{30}$$

with T the normal ordering operator.

We are interested in $s_0 = -\infty$ and $s_0 = \infty$. Also, we are interested in large values of R compared to the size $1/\sqrt{a}$ of the spiral (or the non-adiabatic limit for spins) we consider the lowest orders of the expansion of the time ordered exponential

$$T e^{\int_{-\infty}^{\infty} ds M(s)} \simeq 1 + \int_{-\infty}^{\infty} ds M(s) + \int_{-\infty}^{\infty} ds \int_{s}^{\infty} ds' M(s) M(s'). \tag{31}$$

We are interested in the "permanence", after the evolution, in a particular initial state (spin up) which corresponds, in the rolling language, to $\mathbf{X}(-\infty) = (0, 0, 1)$. This means that we only need to consider the $(3, 3)$ element of the matrix of the time ordered exponential: $U_{33} = \left[T \exp \int_{-\infty}^{\infty} ds M(s)\right]_{33} = z(\infty)$.

Multiplying the two M matrices we get

$$\begin{aligned} z(\infty) &\simeq 1 - \frac{1}{R^2} \int_{-\infty}^{\infty} ds \int_{s}^{\infty} ds' \left(\cos \frac{as^2}{2} \cos \frac{as'^2}{2} \right. \\ &\quad \left. + \sin \frac{as^2}{2} \sin \frac{as'^2}{2} \right) \\ &= 1 - \frac{\pi}{aR^2}. \end{aligned} \tag{32}$$

Now we connect to the spin problem using the equivalence Equation (8) and we obtain:

$$|a(\infty)|^2 = \frac{z(\infty) + 1}{2} = 1 - \frac{\pi}{2aR^2}, \tag{33}$$

which, again coincides with the exact expression for the Landau-Zener effect in the non-adiabatic limit.

•Rolling on a curved surface

We extend the treatment of rolling on a plane to rolling on a curved surface (See Figure 4). If we call $\mathbf{X}_P$ the coordinate of the contact point, the coordinate $\mathbf{X}_c$ of the center of the sphere is:

$$\mathbf{X}_c = \mathbf{X}_P + R\mathbf{n}, \tag{34}$$

and its velocity is given by

$$\begin{aligned} \dot{\mathbf{X}}_c &= \dot{\mathbf{X}}_P + R\dot{\mathbf{n}}, \\ &= \left(\mathbf{t} + R\frac{d\mathbf{n}}{ds}\right)\frac{ds}{dt}. \end{aligned} \tag{35}$$

The rolling condition is that the velocity of a point of the sphere in contact with the surface is zero (See Eq.(2)):

$$\overrightarrow{\omega} \times (\mathbf{n}R) = \dot{\mathbf{X}}_c. \tag{36}$$

Again, taking the cross product with n on both sides of the equation above we obtain

$$\overrightarrow{\omega} = \frac{1}{R}\mathbf{n} \times \dot{\mathbf{X}}_c. \tag{37}$$

We now replace (35) in (37), and use the fact that, for a curved surface, the variation of the normal is given by

$$\frac{d\mathbf{n}}{ds} = -\kappa_n \mathbf{t} - \tau_r \mathbf{u}, \tag{38}$$

with κ_n **the normal curvature and** τ_r **the torsion of the curve, both evaluated at the contact point. We obtain**

$$\overrightarrow{\omega} = \left[\frac{1}{R}\left(1 - \kappa_n R\right)\mathbf{u} + \tau_r \mathbf{t}\right]\frac{ds}{dt}. \tag{39}$$

The discussion for the planar case extends to the curved surface, and the rolling of the sphere is equivalent to a spin $1/2$ precessing on a magnetic field B(s) given by

$$\mathbf{B}(s) = -\left[\frac{1}{R}\left(1-\kappa_n R\right)\mathbf{u} + \tau_r \mathbf{t}\right], \tag{40}$$

with the arc length s playing the role of time. In the following section, as an example of this formulation we consider rolling on a spherical surface.

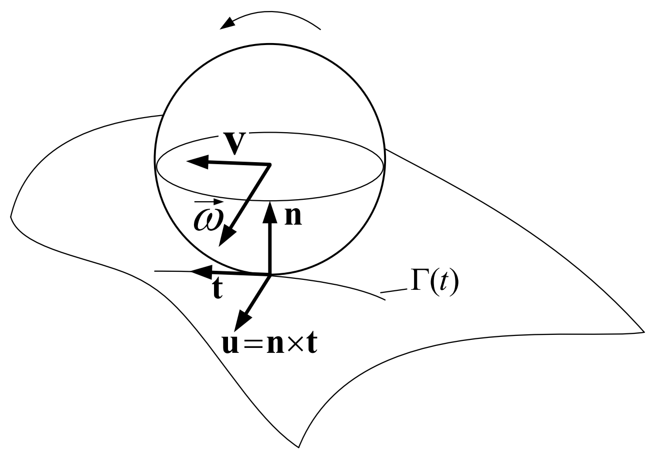

Figure 4: Sphere rolling along a curve Γ of zero torsion (meaning that the velocity of the center of the sphere is parallel to the tangent of the curve at the contact point).

• **Sphere rolling on a spherical surface We consider a sphere of radius R rolling on a second sphere of radius r. The rolling line will be a parallel of latitude $\pi/2-\theta$ (see Figure 5). This means that the normal curvature is constant $1/r$, and also that the torsion is zero. The magnetic field for the corresponding spin problem is therefore:**

$$\mathbf{B}_{\pm}(s) = -(\pm)\left[\frac{1}{R}\left(1 \pm \frac{R}{r}\right)\mathbf{u}\right] = -(\pm)\frac{1}{\widetilde{R}_{\pm}}\mathbf{u}, \tag{41}$$

with $\widetilde{R}_{\pm} = rR/(r \pm R)$ a reduced radius and the plus and minus signs refer to the rolling outside and inside of the sphere of radius r respectively.

For a sphere rolling on a parallel, the instantaneous angular velocity (and the magnetic field) describes a cone forming an angle θ with the vertical. The total arc length of the parallel is $r\sin\theta$ meaning that the vector u rotates with angular frequency α given by $\alpha = 1/(r\sin\theta)$. The corresponding magnetic field is

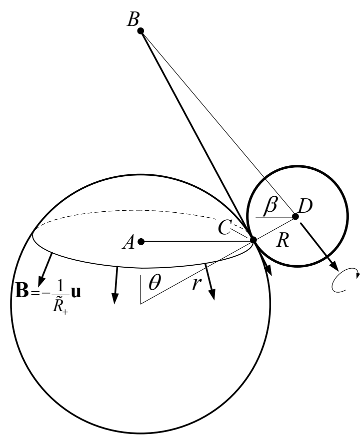

Figure 5: Sphere rolling on a sphere.

therefore

$$\begin{aligned}\mathbf{B}_{\pm}(s) &= (B_x, B_y, B_z)_{\pm} \\ &= (\pm)\frac{1}{\widetilde{R}_{\pm}}(\cos\theta\cos\alpha s, \cos\theta\sin\alpha s, -\sin\theta) \end{aligned} \tag{42}$$

with the term $\mathbf{B}\cdot\mathbf{S}$ in the corresponding Hamiltonian given in this case by

$$\mathbf{B}\cdot\mathbf{S} = \pm\frac{1}{2}\frac{1}{\widetilde{R}_{\pm}}\begin{pmatrix} -\sin\theta & \cos\theta e^{-i\alpha s} \\ \cos\theta e^{i\alpha s} & \sin\theta \end{pmatrix}. \tag{43}$$

Again exactly solvable Hamiltonian studied by Rabi.

Using the same transformation matrix of Eq. (14) the above Hamiltonian can be rendered time independent. We write it in the following form

$$\tilde{H} = -\frac{1}{2}\begin{pmatrix} -B_{\pm}\sin\theta + \alpha & B_{\pm}\cos\theta \\ B_{\pm}\cos\theta & B_{\pm}\sin\theta - \alpha \end{pmatrix}, \tag{44}$$

with $B_{\pm} = 1/\widetilde{R}_{\pm}$.

The eigenvalues of $\tilde{H}$ are $E_{\pm} = \Omega_{\pm}/2$ with

$$\Omega_{\pm} \; = \; (\pm)\frac{1}{\widetilde{R}_{\pm}}\sqrt{1 - \frac{2\widetilde{R}_{\pm}}{r} + \left(\frac{\widetilde{R}_{\pm}}{r\sin\theta}\right)^2}, \tag{45}$$

with the spinor precessing, in the rotating frame, around an axis that forms an angle β (see Figure 5) with the xy plane, with

$$\tan\beta = \tan\theta - \frac{R}{r+R}\frac{1}{\sin\theta\cos\theta} \tag{46}$$

The second term in (46) reflects the fact that the small sphere rotates instantaneously on the tangent plane that contains BC (see Figure 5). Equation (46) can be easily derived by simple geometric considerations from Figure (5).

After a complete revolution the angle or rotation δ is

$$\delta_{\pm} = 2\pi r\sin\theta\Omega_{\pm}. \tag{47}$$

After a little algebra we obtain

$$\delta_{\pm} = \pm 2\pi \cos\theta \sqrt{1 + \left(\frac{r \tan\theta}{R}\right)^2} \tag{48}$$

Notice that, if we compare with the rotation in a plane from Eq. (16), the rotation corresponds to rolling on a circle of radius equal to that of the unfolded cone tangent to the parallel (See Fig. 5). The angle of rotation along that circle is not 2π but $2\pi \cos\theta$. This geometric factor is the same that appears in Foucault's pendulum and in Berry's phase for a spin precessing on a cone (we will come back to this point below). Also, notice that when $r = R$ the angle of rotation is always 2π independent of latitude.

Consider now the differences and similarities between the Berry phase for a precessing spin $1/2$ in the adiabatic approximation and the rolling of two spheres.

The Hamiltonian for a spin in a magnetic field that precesses along the z axis at frequency α is given by (44), where in principle α and B_0 are independent parameters. If $\alpha \ll B_0$ (the adiabatic approximation) the eigenvalues (eigenfrequencies) of $\widetilde{H}$ are

$$\Omega \simeq \sqrt{B_\pm^2 - 2\alpha B_\pm \sin\theta} \simeq B_\pm - (\pm)\alpha \sin\theta. \tag{49}$$

After a period of time $2\pi/\alpha$ the change $\Delta\phi$ in the phase of the spin is

$$\Delta\phi = 2\pi \frac{B_\pm}{\alpha} - (\pm) 2\pi \sin\theta. \tag{50}$$

The first term is the dynamical phase and the second is a purely geometrical one, independent of the parameters B_0 and α, and given by (half) the solid angle described by the field.

For the rolling sphere we can also study an "adiabatic approximation" since $\alpha \ll B_0$ corresponds to $r \gg R$. In other words, in general the adiabatic approximation will correspond to the radius of the rolling sphere much smaller than the radius of curvature of the surface. On the other hand, in contrast with the spin case, the frequency of rotation $\alpha = 1/r \sin\theta$ "knows" about the latitude and the curvature. So we expect some differences and some similarities. Replacing the values of $B_{\pm} = \pm 1/\widetilde{R}_{\pm} \equiv (\pm)1/R \pm 1/r$ in (50) we obtain the angle of rotation of the sphere in each case (in the adiabatic approximation)

$$\begin{aligned} \Delta\phi_{\pm} &= \pm 2\pi r \sin\theta \left(\frac{1}{R} \pm \frac{1}{r} \right) - (\pm) 2\pi \sin\theta. \\ &= \pm 2\pi \frac{r \sin\theta}{R} \end{aligned} \tag{51}$$

Notice that there is a cancelation of the geometric phase for rolling. In the spin problem, the frequency ω of rotation of the field (α for rolling) and the magnitude of the field $B_{\pm}$ are independent and therefore the total angle of rotation is given by Eq.(50), with the second term a purely geometric term independent of the parameters of the problem. In the rolling case the frequency and the field are not independent, and the "dynamical" phase contains a term that cancels the geometric one. As a result, the total rotation is given by a magnitude that depends on the parameters of the problem, which, in the spin language corresponds to the dynamical phase only. This cancelation is a general result that we will visit in the next section.

•The adiabatic approximation and rolling on a curved surface

Compare the equivalence between the adiabatic approximation for a spin precessing in a magnetic field that changes direction at a slow rate and rolling on a surface. In the spin case, the dimensionless parameter controlling the approximation is the ratio of the instantaneous frequency (proportional to the instantaneous magnitude of the field) with the rate at which it's direction is changing.

In the rolling case the instantaneous frequency corresponds to the magnitude of B(s) and the rate of change in its direction is related to the normal curvature and to the curve's torsion.

In the adiabatic approximation for spins [?], one works in an "instantaneous" basis, treating first s (time) as a parameter and solving the eigenvalue equation as though the problem were static:

$$H(s)\chi(s) = E(s)\chi(s) \equiv \frac{\Omega(s)}{2}\chi(s). \tag{52}$$

Then the general solution is written as linear combinations of the instantaneous eigenstates. As a result, in the adiabatic approximation, the spinor at time s is given by

$$\chi(s) = e^{i\gamma(s)} e^{i\int_0^s ds' E(s')}\chi(0). \tag{53}$$

The argument of the second exponential above represents the dynamic phase, which involves the integral of (half) the following angular frequency:

$$\begin{aligned} \Omega(s) &= |\mathbf{B}(s)| = \frac{1}{R}\sqrt{[1-\kappa_n(s)R]^2 + [\tau(s)R]^2} \\ &\simeq \frac{1}{R} - \kappa_n(s) \end{aligned} \tag{54}$$

This can be seen, for example from Equation (43): the eigenvalues of $\mathbf{B}\cdot\mathbf{S}$ with s treated as a parameter are $\pm|\mathbf{B}(s)|/2$.

The (instantaneous) direction of the field is in the direction $\mathbf{u}_B$ given by

$$\mathbf{u}_B = \frac{\mathbf{B}(s)}{|\mathbf{B}(s)|} = -\frac{(1-\kappa_n R)\,\mathbf{u} + \tau_r \mathbf{t}}{\sqrt{(1-\kappa_n R)^2 + \tau_r^2}} \tag{55}$$

In general, the eigenvalues of a Pauli matrix in an arbitrary direction $\mathbf{u}_B\cdot\vec{\sigma}$ given by the unit vector $\mathbf{u}_B = (u_x, u_y, u_z)$ are ± 1. This is verified by noting that (defining $u_x + iu_y = \rho e^{i\phi}$)

$$(\mathbf{u}_B\cdot\vec{\sigma})\,\chi_\pm(\mathbf{u}_B) = \begin{pmatrix} u_z & \rho e^{-i\phi} \\ \rho e^{i\phi} & -u_z \end{pmatrix} \chi_\pm(\mathbf{u}_B) = \pm\chi_\pm(\mathbf{u}_B), \tag{56}$$

with $\chi_\pm(\mathbf{u}_B) = (1, \pm(1-u_z)e^{\pm i\phi}/\rho)$. Notice that the dependence of χ on s is through the orientation of $\mathbf{u}$.

The first term of (53), the geometric phase γ, is the Berry phase, and is given by

$$\dot{\gamma}(s) = i\chi(\mathbf{u}_B(s))^\dagger \frac{d}{ds}\chi(\mathbf{u}_B(s)). \tag{57}$$

Without loss of generality we express $\mathbf{u}_B$ in polar coordinates $\mathbf{u}_B = (\cos\theta\cos\phi, \cos\theta\cos\phi, \sin\theta)$, where the quantization axis z is perpendicular to the instantaneous plane of motion of the center of mass of the rolling sphere. This means that the normalized spinor is:

$$\chi(\mathbf{u}_B(s)) = \begin{pmatrix} \frac{\cos\theta(s)}{\sqrt{1+\sin\theta(s)}} \\ \sqrt{1+\sin\theta(s)}e^{-i\phi(s)} \end{pmatrix}. \tag{58}$$

From the above expression and (57) we can compute the geometric phase:

$$\dot{\chi} = i\chi^\dagger \frac{d}{ds}\chi = \frac{1+\sin\theta}{2}\frac{d\phi}{ds}. \tag{59}$$

Here $d\phi$ is the angle of rotation of the center of mass of the sphere with respect to an instantaneous axis of rotation. The first term of the right hand side is 2π after integration on a closed circuit. And the second term cancels the curvature term from Eq. (54). This results from the identity[?]

$$\frac{d\phi}{ds} = \frac{\kappa_n}{\sin\theta}. \tag{60}$$

Our final result is that, as anticipated in the two spheres case, in general there is no Berry phase for rolling as a result of the above cancelation:

$$\begin{aligned} \delta_{\pm} &= \pm\left(\int_0^s ds'\Omega(s') + 2\gamma(s)\right) \\ &= \pm\frac{L}{R} + 2\pi \end{aligned} \tag{61}$$

Note that, if we specify this result to the sphere rolling on the parallel of a sphere of radius r, we have $L = 2\pi r \sin\theta$. Replacing

these in Eq. (54) we obtain the result of Eq. (51) as expected. The discrepancy of the (unimportant) factor 2π results from the fact that the treatment in the present section is in the rest frame and that of section is in the rotating frame. The plus and minus signs correspond both to the two senses of traveling the circuit and the two sides of the surface on which the sphere can roll.

•Kinematics of Rolling on a sphere Consider the rolling of a sphere of radius R on a sphere of radius r along a parallel of latitude θ.

In Figure (6) we show the instantaneous motion in the xz plane, where the two sphere (inner and outer) are moving into the plane.

The speed of the center of mass of each sphere is constant along the rolling and given by

$$V_{o,i} = \frac{1}{T} 2\pi(r \pm R)\sin\theta,$$

where T is the time to complete a full rolling. For the picture shown in Figure 6, V is directed into the page.

The rolling condition for the angular velocity in each case is

$$\omega_{o,i} R = V_{o,i},$$

where $\omega_{o,i}$ are the magnitudes if the instantaneous rotational angular velocity. Notice that the directions are opposed.

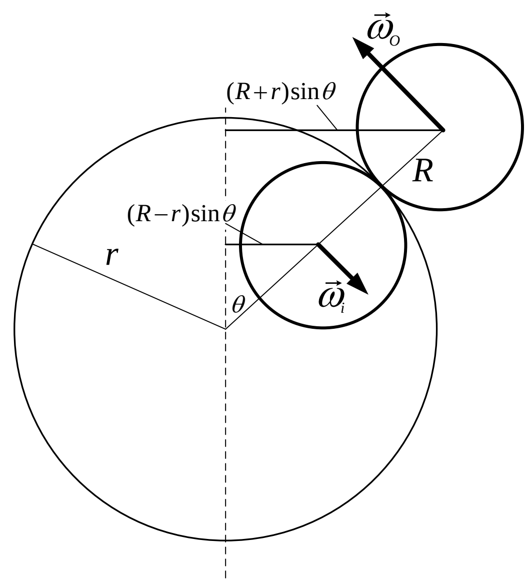

Figure 6: Kinematics of rolling spheres

The instantaneous components of the angular velocity are

$$\overrightarrow{\omega}_o = \frac{2\pi}{T}\left(\frac{r}{R}+1\right)\sin\theta(-\cos\theta, 0, \sin\theta)$$

$$\overrightarrow{\omega}_i = \frac{2\pi}{T}\left(\frac{r}{R}-1\right)\sin\theta(\cos\theta, 0, -\sin\theta)$$

Since this angular velocities are time dependent, we trans-

form to a moving frame, where they are constant. The moving frame is rotating at angular frequency $2\pi/T$ around the vertical (z) axis. This means that, in the moving frame M the corresponding angular frequencies are

$$\overrightarrow{\omega}_{M,o} = \frac{2\pi}{T}\left[-\left(\frac{r}{R}+1\right)\sin\theta\cos\theta, 0, \left(\frac{r}{R}+1\right)\sin^2\theta - 1\right]$$

$$\overrightarrow{\omega}_{M,i} = \frac{2\pi}{T}\left[\left(\frac{r}{R}-1\right)\sin\theta\cos\theta, 0, -\left(\frac{r}{R}-1\right)\sin^2\theta - 1\right]$$

The corresponding angle of rotation $\delta = |\overrightarrow{\omega}_{M,o}|T| = \overrightarrow{\omega}_{M,i}|T$, after a complete circle is

$$\delta = 2\pi\cos\theta\sqrt{\left(\frac{r}{R}\right)^2\tan^2\theta + 1},$$

which corresponds to the result obtained using the spin language.