Open Problems in Noisy Quantum Metrology.

Newton Institute
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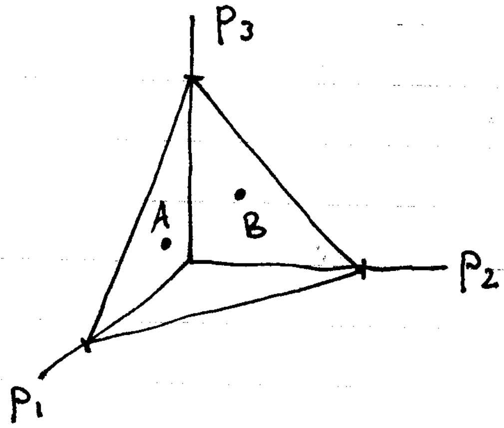
Some background (classical case)

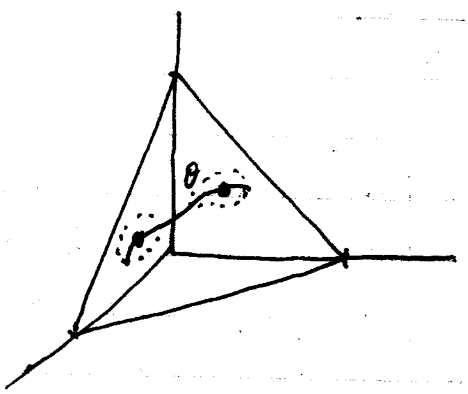
Suppose we want to tell the difference between two probability distributions.

Every probability distribution is part of a simplex:



"How many times do I have to sample the system to tell the difference between A and B?"

Special case: probability distribution is parameterised by θ.



"How big must θ be in order to move from one distribution to another that is distinguishable?"

We need some distance measure (metric) on the simplex that fits naturally with statistics:

$$ds^2 = \sum_{jk} g_{jk} dp^j dp^k$$

p^j are contravariant.

with g_{jk} the metric tensor obeying $g_{jk} g^{kl} = \delta_{kl}$.

We now derive a natural form for g_{jk}.

expectation value of a random variable A:

$$\langle A \rangle = \sum_j A_j \, p^j$$

natural inner product

$A_j =$ ~~contra~~covariant because points of constant $\langle A \rangle$ form surfaces in the dual space to dp...

Metric is determined by a quadratic form of covariants:

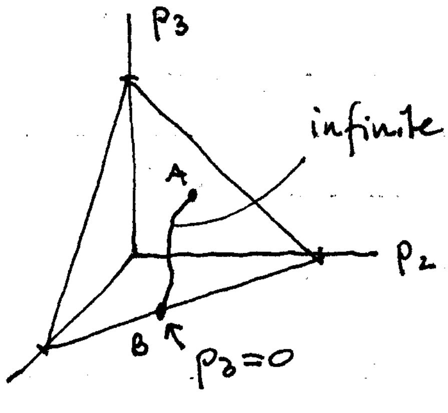
$$\langle AB \rangle = \sum_{jk} A_j B_k \, g^{jk} = \sum_j A_j B_j \, p^j \quad \leftarrow \text{correlation}$$

$\uparrow$ key step!

Then $g^{jk} = p^j \delta_{jk}$ (no summation convention)

and $\boxed{g_{jk} = \dfrac{\delta_{jk}}{p^j}}$. (strong curvature)

Infinitesimal distance: $$ds^2 = \sum_j \frac{(dp^j)^2}{p^j}$$



infinite distance: obtaining result x_3 in an experiment immediately implies $p_3 \neq 0 \Rightarrow A$ with certainty.

Neat aside: if $dp^j = (dr^j)^2$, then $ds^2 = 4 \sum_j (dr^j)^2$.

Euclidean space.

No QM!

Now let's look at displacement ds along $d\theta$:

$$\frac{ds^2}{d\theta^2} = \sum_j \frac{1}{p^j} \frac{(dp^j)^2}{d\theta^2} = \sum_j p_j \left(\frac{d \ln p^j}{d\theta} \right)^2 \equiv F$$

"logarithmic derivative"

F is the Fisher information : velocity^2 through the probability simplex along θ

= average amount of information about θ in a single measurement

For small, finite distances starting at $\theta = 0$:

$$s(\delta\theta) = s(0) + \delta\theta \left.\frac{ds}{d\theta}\right|_{\theta=0} + \mathcal{O}(\delta\theta^2)$$

$$\delta s \equiv s(\delta\theta) - s(0) = \delta\theta \sqrt{F(\theta=0)} \,. \qquad (*)$$

Short cut to Cramér-Rao bound:

Two prob. dist. are distinguishable after N independent samples if

$$N\, \delta s^2 \geq 1 \qquad \text{(additivity of indep. samples)}$$

$$\therefore \quad N \frac{\delta s^2}{\delta\theta^2} \geqslant \frac{1}{\delta\theta^2} \quad \Rightarrow \quad \delta\theta^2 \geqslant \frac{1}{NF} \,.$$

[A proper proof will show that] $\delta\theta^2 = \text{MSE}$.

<u>Quantum case</u>

$$\langle \hat{A} \rangle = \mathrm{Tr}\left[\rho \hat{A}\right]$$

self-adjoint operator (observable)

density operator

Correlation $\langle AB \rangle$ does not work, because not observable:

$$(AB)^\dagger = B^\dagger A^\dagger = BA \neq AB \text{ in general.}$$

Natural fix: use $\frac{1}{2}\{A, B\}$ (anti-commutator)

$$\langle \tfrac{1}{2}\{A,B\} \rangle = \tfrac{1}{2} \mathrm{Tr}\left[\rho \{A,B\}\right] = \tfrac{1}{2} \mathrm{Tr}\left[A \{\rho, B\}\right]$$

$$\equiv \mathrm{Tr}\left[A \, \mathcal{R}_\rho(B)\right]$$

with $\mathcal{R}_\rho(\cdot) = \frac{1}{2}\{\rho, \cdot\}$ a superoperator

"R" for "raising" operator analogous to g^{jk} before.

We need $\mathcal{L}_\rho = \mathcal{R}_\rho^{-1}$.

Check that $\mathcal{L}_\rho(B) = \sum_{jk} \frac{2 B_{jk}}{p_j + p_k} |j\rangle\langle k|$ in eigenbasis of ρ.

same

Others use directly $B = \frac{d\rho}{d\theta}$ and $\frac{\partial \rho}{\partial \theta} = \frac{1}{2}\left(\rho \mathcal{L}_\rho + \mathcal{L}_\rho \rho\right)$.

Then $ds_\rho^2 = \mathrm{Tr}\left[d\rho \, \mathcal{L}_\rho(d\rho)\right]$

Again: $\frac{ds_\rho^2}{d\theta^2} = \mathrm{Tr}\left[\rho' \mathcal{L}_\rho(\rho')\right] \equiv F_Q(\theta)$

Bures metric on quantum state space. (Riemannian)

Note: F depends on prob. dist, which includes measurement
F_Q depends on state ρ and its derivative

F_Q is F for best possible measurement choice.

The quantum Fisher information gives a bound on how well you can do in estimating θ for <u>any</u> measurement.

So the quantum Cramèr-Rao bound (QCRB) is given by

$$\delta\theta \geq \frac{1}{\sqrt{N F_Q}} .$$

This is a <u>fundamental</u> bound on the precision of θ.

F_Q can itself be bounded by noting that

$$\frac{d\rho}{d\theta} = -\frac{i}{\hbar}\left[\hat{G},\rho\right] \qquad \text{with} \quad U_\theta = e^{-i\theta\hat{G}}$$

$$= -\frac{i}{\hbar}\left[\Delta\hat{G},\rho\right] \qquad \text{since} \quad \Delta G = \hat{G} - \underset{\uparrow \text{number commutes with everything.}}{\langle G\rangle}$$

Sub. into F_Q:

$$F_Q = \mathrm{Tr}\left[\rho' \mathcal{L}_\rho(\rho')\right] = \overset{4!}{\frac{2}{\hbar^2}} \sum_{jk} \frac{(p_j - p_k)^2}{p_j + p_k} |\Delta G_{jk}|^2 \underset{\uparrow \text{ "=" for pure states.}}{\leq} \frac{4}{\hbar^2}(\Delta G)^2 .$$

People often call $(\Delta G)^2$ the QFI

<u>Two types of metrology</u>:

* N independent measurements with $F \sim 1$

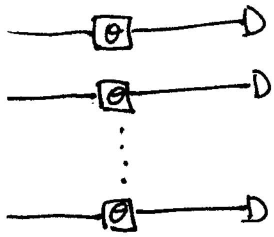
$$\delta\theta \geq \frac{1}{\sqrt{N}} \qquad \text{"standard quantum limit"},$$

* 1 measurement on M entangled systems: $F \propto M^2$

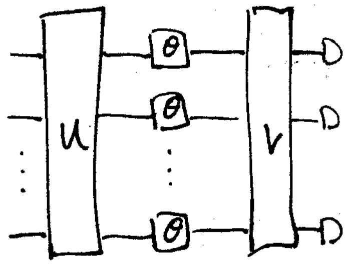
$$\delta\theta \geq \frac{1}{M} \qquad \text{"Heisenberg limit"}.$$

Resources

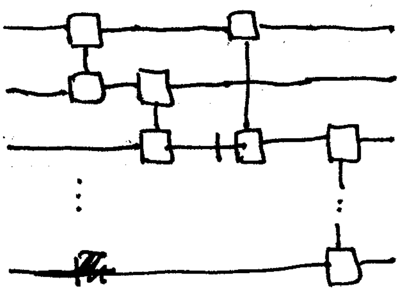
Note the different N and M.
Often they can be related to each other ($N = M$)
but not always! → source of confusion and false claims of "beating the Heisenberg limit".



N photons
independently: $F \cong 1$



N entangled photons
$N = 1 \quad F = N^2$



etc.

generator is not $\sum_j G_j$

but $\sum_{j,k} G_j \otimes G_k \equiv H$.

N photons, but how many probes? $\frac{1}{2}N(N-1) = M$; $F \sim M^2$.

Resource count is given by numbers of probes, or $\langle H \rangle$.

Heisenberg limit is $\propto \frac{1}{\langle H \rangle}$. Never beaten!